Math 431 Homework 7
Due 11/6

1. Prove the Crossbar theorem: If ray $\overrightarrow{AD}$ is between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ then $\overrightarrow{AD}$ intersects segment BC.

Proof. Suppose the contrary, that BC does not meet ray $\overrightarrow{AD}$. Either BC meets line $\overleftrightarrow{AD}$ or it does not. If it meets $\overleftrightarrow{AD}$, by Proposition 3.4 (line separation property) it must meet the ray opposite to $\overrightarrow{AD}$ at a point $E \neq A$. According to Proposition 3.8(b), E is not in the interior of $\sphericalangle CAB$. Point B does not lie on $\overleftrightarrow{AD}$; this is because $D \in$int$\sphericalangle CAB$, so D and C lie on the same side of $\overleftrightarrow{AB}$, so $D \notin \{\overleftrightarrow{AB}\}$, so $B \notin \{\overleftrightarrow{AD}\}$. Thus, since $E \in \{\overleftrightarrow{AD}\}$, we have $E \neq B$. By the same reasoning with C and B interchanged, we have $E \neq C$. Since $E \in BC$ and E is not an endpoint, we have $B * E * C$. Thus by Proposition 3.7, E is in the interior of $\sphericalangle CAB$, a contradiction. Thus $\overleftrightarrow{AD}$ does not meet BC at all; that is, B and C are on the same side of $\overleftrightarrow{AD}$. By B-2, we have a point E such that $C * A * E$. By Lemma 3.2.2, C and E are on opposite sides of $\overleftrightarrow{AD}$. Thus, by B-4(iii), E and B are on opposite sides of $\overleftrightarrow{AD}$. But by Proposition 3.8(c), B is on the interior of $\sphericalangle DAE$, so E and B are on the same side of $\overleftrightarrow{AD}$. This is a contradiction. Thus, $\overrightarrow{AD}$ meets BC. □

2. Prove the best theorem you can come up with that roughly corresponds to Pasch's theorem where line that intersects one of the sides is replaced by a ray. Make sure to define all the terms you are using.

3. Prove Proposition 3.11: If $A*B*C$, $D*E*F$, $AB \cong DE$, and $AC \cong DF$, then $BC \cong EF$.

Assume to the contrary that BC is not congruent to EF.

By C-1 there exists a point G on $\overrightarrow{EF}$ such that $BC \cong EG$. From here and our assumption that $BC \ncong EF$, we have $G \neq F$. By C-3 we get $AC \cong DG$, and by hypothesis $AC \cong DF$, hence, by C-2, $DG \cong DF$. By the uniqueness part of C-1, $G = F$. But this is a contradiction to the above statement that $G \neq F$.