

Solutions to Homework 9  
As presented by your peers

I am only giving the first problem because the last two were done fairly well by everybody.

**1. – by Julia, Whitney and Laura. Consider the following statement: If we replaced axiom B4 by Pasch’s theorem, the geometry with that set of axioms is the same as neutral geometry. What do you think ‘geometries are the same’ means? Whether you think they are the same or not, explain how you reached your conclusion.**

**AWESOME!!!!**

Note: Pasch’s Theorem: If  $A$ ,  $B$ , and  $C$  are three distinct point, and  $l$  is any line intersecting segment  $AB$  at a point between  $A$  and  $B$ , then  $l$  also intersects either segment  $AC$  or segment  $BC$ . If  $C$  does not lie on  $l$ , then  $l$  does not intersect both  $AC$  and  $BC$ .

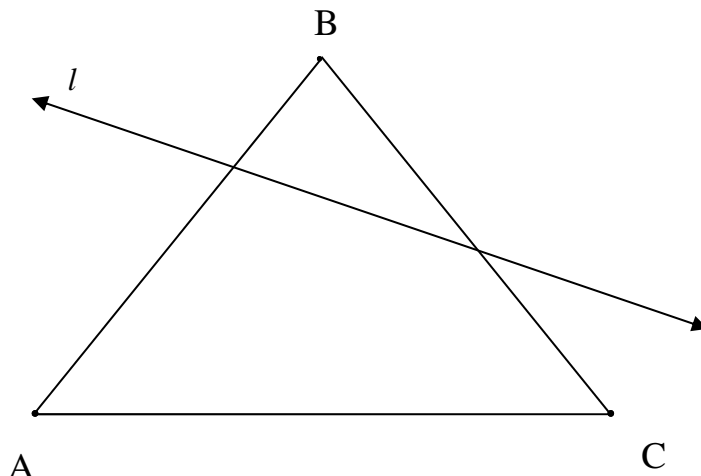
For geometries to be the same, all of the same theorems and propositions must hold. With this in mind, the new geometry would be the same as neutral geometry if B4 can be proved from the new set of axioms, B4 replaced by Pasch’s theorem.

Claim: For every line  $l$  and any three points  $A$ ,  $B$ , and  $C$  not lying on  $l$ :

- (i) If  $A$  and  $B$  are on opposite sides of  $l$  and  $B$  and  $C$  are on opposite sides of  $l$ , then  $A$  and  $C$  are on the same side of  $l$ .
- (ii) If  $A$  and  $B$  are on the same side of  $l$  and  $B$  and  $C$  are on the same side of  $l$ , then  $A$  and  $C$  are on the same side of  $l$ .

Let  $l$  be a line and let  $A$ ,  $B$  and  $C$  be three distinct points that do not lie on  $l$ .

- (i) By the given in Pasch’s Thm, and definition of opposite sides,  $A$  and  $B$  are on opposite sides of  $l$  and  $B$  and  $C$  are on opposite sides of  $l$ . There are two cases: the three points are either noncollinear or collinear, but not both.
  - a. First, assume  $A$ ,  $B$  and  $C$  are distinct noncollinear points. By hypothesis,  $A$  and  $B$  are on opposite sides of  $l$ . So, segment  $AB$  and  $l$  intersect at a point between  $A$  and  $B$  by the definition of opposite sides, and since  $C$  does not lie on  $l$ ,  $l$  must intersect either  $AC$  or  $BC$ , but not both by Pasch’s Theorem. Since  $B$  and  $C$  are on opposite sides of  $l$ ,  $BC$  and  $l$  intersect at a point between  $B$  and  $C$  by the definition of opposite sides, therefore  $l$  does not intersect  $AC$ , and by definition  $A$  and  $C$  are on the same side of  $l$ . In conclusion, if  $A$ ,  $B$ , and  $C$  are three distinct noncollinear points, with  $A$  and  $B$  on opposite sides of  $l$  and  $B$  and  $C$  on opposite sides of  $l$ , then  $A$  and  $C$  are on the same side of  $l$ .

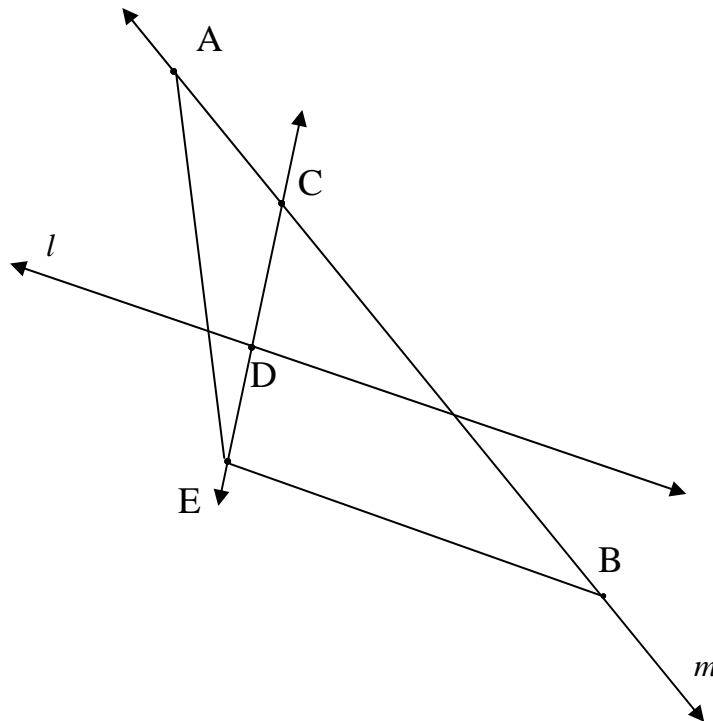


b. A, B and C are distinct collinear points on line  $m$ . By I-2 there exist two distinct points on line  $l$ , let D be a point on  $l$  that is not the intersection of BC and  $l$ . Since by Proposition 2.1 the point of intersection between two lines is unique, CD does not lie on line  $m$ . Since C and D are distinct points, by I-1 there is a unique line CD passing through both. By B-2, let E be a point such that  $C^*D^*E$  on line CD. Since line CD and  $m$  are distinct lines, and C lies on both lines by proposition 2.1, C is the unique-point of intersection, therefore E does not lie on  $m$ . We now have 3 distinct noncollinear points B, C and E. Since  $C^*D^*E$ , by the definition of opposite sides, C and E are on opposite sides of  $l$ , and by hypothesis B and C are on opposite sides of  $l$ , so from the previous step (i.a), B and E are on the same side of  $l$ .

By hypothesis, A and B are on opposite sides of  $l$ , and so  $l$  and AB intersect at a point between A and B, by the definition. Since A, B and E are distinct noncollinear points and E does not lie on  $l$ ,  $l$  intersects AE or BE, but not both by Pasch's Theorem. We have shown that B and E are on the same side of  $l$ , therefore,  $l$  must intersect AE by Pasch's Theorem, therefore A and E are on opposite sides of  $l$ .

Now, we have that A and E are on opposite sides of  $l$ , and C and E are on opposite sides of  $l$ , therefore A and C are on the same side of  $l$ , from step (i.a).

Therefore, whether A, B and C are collinear or noncollinear, if A and B are on opposite sides of  $l$  and B and C are on opposite sides of  $l$ , then A and C are on the same side of  $l$ .



- (ii) By hypothesis A and B are on the same side of  $l$ , and B and C are on the same side of  $l$ , they may be collinear or noncollinear. By I-2, there is a point D on  $l$ . Let E be a point such that  $A^*D^*E$ , therefore A and E are on opposite sides of  $l$ , and  $l$  and AE intersect at point D between A and E. Since B does not lie on  $l$ , by Pasch's Theorem (applied to triangle ABE)  $l$  must intersect AB or BE, but not both. A and B are on the same side of  $l$ , so AB does not intersect  $l$  by definition, therefore BE must intersect  $l$ .

Since BE intersects  $l$  at a point between B and E, and C does not lie on  $l$ , (By Pasch's Theorem applied to triangle BCE)  $l$  must intersect either CE or BC, but not both. B and C are on the same side of  $l$ , so  $l$  does not intersect BC by definition, therefore  $l$  intersect CE and C and E are on opposite sides of  $l$  by definition of opposite sides.

Now we have that A and E are on opposite sides of  $l$ , and C and E are on opposite sides of  $l$ , therefore, A and C must be on the same side of  $l$  by step (i.a). Therefore given that A and B are on the same side of  $l$ , and B and C are on the same side of  $l$ , A and C are also on the same side of  $l$ .

