
Proposition 2.0. In incidence geometry, there is a line.

Proposition 2.1. If l and m are distinct lines that are not parallel, then l and m have a unique point in common.

Proposition 2.2. There exist three distinct lines that are not concurrent.

Proposition 2.3. For every line there is at least one point not lying on it.

Proposition 2.4. For every point there is at least one line not passing through it.

Proposition 2.5. For every point P there exist at least two distinct lines throught P.

Proposition 3.0. For any two distinct points A, B the following holds:

(i) $AB = BA$

(ii) $AB \subsetneq \overrightarrow{AB}$

Proposition 3.1. For any two distinct points A, B the following holds:

(i) $\overrightarrow{AB} \cap \overrightarrow{BA} = AB$

(ii) $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$

Lemma 3.1.5 Points A and B are on the same side of l iff $\text{side}(A, l) = \text{side}(B, l)$

Proposition 3.2. Every line has exactly two sides and they are disjoint.

Lemma 3.2.2. If $A * B * C$ and l is any line passing through C that is distinct from $\overleftrightarrow{AC}$, then A and B are on the same side of l.

Lemma 3.2.3. If A and B are distinct points on the same side of a line l and $\overleftrightarrow{AB}$ intersects l at a point C, then $A * B * C$ or $B * A * C$.

Lemma 3.2.4. If $A * B * C$ and l is a line passing through B that is distinct from $\overleftrightarrow{AC}$, then A and C are on opposite sides of l.

Lemma 3.2.5. If A and C are on opposite sides of a line l, then there exists a unique point B such that l passes through B and $A * B * C$.

Proposition 3.3. If $A * B * C$ and $A * C * D$ then $B * C * D$ and $A * B * D$.

Proposition 3.4. LSP: If $C * A * B$ and l is the line through A, B and C, then any point P lying on l lies either on $\overrightarrow{AB}$ or on $\overrightarrow{AC}$.

Pasch's Theorem If A, B, C are distinct, noncollinear points and l is any line intersecting AB in a point between A and B, then l also intersects AC or BC. If C does not lie on l, then l does not intersect both AC and BC.

Proposition 3.5. If $A * B * C$, then $AC = AB \cup BC$ and B is the only point in common to segments AB and BC.

Proposition 3.6. If $A * B * C$, then B is the only point in common to rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and $\overrightarrow{AB} = \overrightarrow{AC}$.

Proposition 3.7. Let $\sphericalangle BAC$ be an angle and D a point lying on the line $\overleftrightarrow{BC}$. Then D is in the interior of $\sphericalangle BAC$ iff $B * D * C$.

Proposition 3.7.5 Let l be a line, A a point on l, and B a point not on l. Then every point on $\overrightarrow{AB}$, except A, lies on the same side of l as B.

Proposition 3.8 If D is in the interior of an $\sphericalangle CAB$ then:

1. so is every point on $\overrightarrow{AD}$ except A,
2. no point on the opposite ray to $\overrightarrow{AD}$ is in the interior of $\sphericalangle BAC$
3. if $C * A * E$, then B is in the interior of $\sphericalangle DAE$

Crossbar Theorem If $\overrightarrow{AD}$ is between rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then $\overrightarrow{AD}$ intersects segment BC.

Proposition 3.10 If in a $\triangle ABC$ we have $AB \cong AC$ then $\sphericalangle B \cong \sphericalangle C$.

Proposition 3.11 – Segment subtraction If $A * B * C$ and $D * E * F$ and $AC \cong DF$ and $AB \cong DE$, then $BC \cong EF$.

Proposition 3.12 If $AC \cong DF$, then for any point B between A and C, there is a unique point E between D and F such that $AB \cong DE$.

Proposition 3.13 – Segment ordering

- Exactly one of the following holds: $AB < CD$, $AB \cong CD$, or $AB > CD$.
- If $AB < CD$ and $CD \cong EF$, then $AB < EF$.
- If $AB < CD$ and $AB \cong EF$, then $EF < CD$.
- If $AB < CD$ and $CD < EF$, then $AB < EF$.

Proposition 3.14 Supplements of congruent angles are congruent.

Proposition 3.15

- Vertical angles are congruent.
- An angle congruent to a right angle is a right angle.

Proposition 3.16 For every line l and every point P not on l there is a line through P perpendicular to l.

Proposition 3.17 – ASA If $\triangle ABC$ and $\triangle DEF$ are two triangles with $\sphericalangle A \cong \sphericalangle D$, $\sphericalangle C \cong \sphericalangle F$, and $AC \cong DF$, then $\triangle ABC \cong \triangle DEF$.

Proposition 3.18 If in a $\triangle ABC$ we have $\sphericalangle B \cong \sphericalangle C$, then $AB \cong AC$.

Proposition 3.19 – Angle addition If $\overrightarrow{BG}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$, $\sphericalangle CBG \cong \sphericalangle FEH$, and $\sphericalangle GBA \cong \sphericalangle HED$, then $\sphericalangle ABC \cong \sphericalangle DEF$.

Proposition 3.20 – Angle subtraction If $\overrightarrow{BG}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$, $\sphericalangle CBG \cong \sphericalangle FEH$, and $\sphericalangle ABC \cong \sphericalangle DEF$, then $\sphericalangle GBA \cong \sphericalangle HED$.

Proposition 3.21 – Angle ordering

- Exactly one of the following holds: $\sphericalangle P < \sphericalangle Q$, $\sphericalangle P \cong \sphericalangle Q$, or $\sphericalangle P > \sphericalangle Q$.
- If $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle Q \cong \sphericalangle R$, then $\sphericalangle P < \sphericalangle R$.
- If $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle P \cong \sphericalangle R$, then $\sphericalangle R < \sphericalangle Q$.
- If $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle Q < \sphericalangle R$, then $\sphericalangle P < \sphericalangle R$.

Proposition 3.21 – SSS If $\triangle ABC$ and $\triangle DEF$ are two triangles with $AB \cong DE$, $CA \cong FD$, and $BC \cong EF$, then $\triangle ABC \cong \triangle DEF$.

Proposition 3.22 All right angles are congruent.

Theorem 4.1 – AIA theorem – Alternate interior angle theorem If two lines cut by a transversal have a pair of congruent interior angles then the two lines are parallel.

Corollary 4.1.1 Two lines perpendicular to the same line are parallel.

Corollary 4.1.2 If l is any line and P is any point not on l, there exists at least one line m through P parallel to l.

Theorem 4.2 – Exterior angle theorem An exterior angle of a triangle is greater than either remote interior angle.

Proposition 4.2.1 – **AAS** If $AC \cong DF$, $\sphericalangle A \cong \sphericalangle D$ and $\sphericalangle B \cong \sphericalangle C$, then $\triangle ABC \cong \triangle DEF$.

Proposition 4.2.2 Two right triangles are congruent if the hypothenuse and a leg of one are congruent respectively to the hypothenuse and a leg of the other.

Proposition 4.2.3 Every segment has a unique midpoint.

Theorem 4.3 There is a unique way of assigning a degree measurement to each angle so that 1)-7) hold. Further given a segment OI called unit segment there is a unique way of assigning a length $l(AB)$ to each segment AB so that 8)-12):

1. $m(\sphericalangle A)$ is a real number such that $0 < m(\sphericalangle A) < 180°$
2. $m(\sphericalangle A) = 90°$ iff $\sphericalangle A$ is a right angle.
3. $m(\sphericalangle A) = m(\sphericalangle B)$ iff $\sphericalangle A \cong \sphericalangle B$.
4. If C is in the interior of $\sphericalangle DAB$ then $m(\sphericalangle DAB) = m(\sphericalangle DAC) + m(\sphericalangle CAB)$
5. For every real number x between 0 and 180 there is an angle $\sphericalangle A$ such that $m(\sphericalangle A) = x°$
6. If $\sphericalangle B$ is supplementary to $\sphericalangle A$, then $m(\sphericalangle A) + m(\sphericalangle B) = 180°$
7. $m(\sphericalangle A) > m(\sphericalangle B)$ iff $\sphericalangle A > \sphericalangle B$
8. $l(AB)$ is a positive real number and $l(OI) = 1$
9. $l(AB) = l(CD)$ iff $AB \cong CD$
10. $A * B * C$ iff $l(AC) = l(AB) + l(BC)$
11. $l(AB) < l(CD)$ iff $AB < CD$
12. For every positive real number x, there exists a segment AB such that $l(AB) = x$.

Theorem 4.4.0 Every angle has a unique bisector.

Theorem 4.4 Angle bisectors of a triangle meet at a point.

Theorem 4.5 A point P lies on the angle bisector of $\sphericalangle BAC$ iff it is equidistant from the sides of $\sphericalangle BAC$.

In next five propositions $\triangle ABC$ is a triangle and D a point on line $\overleftrightarrow{AB}$.

Proposition 4.5.1 If $AC \cong BC$ and CD is a median then $\overrightarrow{CD}$ is the angle bisector of $\sphericalangle ACB$.

Proposition 4.5.2 If $AC \cong BC$ and CD is a median then CD is the altitude.

Proposition 4.5.3 If $AC \cong BC$ and $\overrightarrow{CD}$ is the angle bisector of $\sphericalangle ACB$ then CD is the altitude.

Proposition 4.5.4 If CD is a median and $\overrightarrow{CD}$ is the angle bisector of $\sphericalangle ACB$ then the triangle $\triangle ABC$ is isosceles.

Proposition 4.5.5 If CD is a median and the altitude then the triangle $\triangle ABC$ is isosceles.

Theorem 4.5 Diagonals of a convex quadrilateral meet at a point.

Theorem 4.6 The angle bisectors of a square meet at a point.

Theorem 5.2: If there is a triangle whose angle sum is not 180° then no triangle has angle sum 180°.

Theorem 5.3: No triangle in neutral geometry can have angle sum greater than 180°.

Theorem 5.4: If there is a triangle with angle sum 180°, then all triangles have angle sum 180°.

Theorem 5.5: A rectangle exists iff EPP holds.

Theorem 5.6: If a rectangle does not exist, there is a triangle with angle sum less than 180°.

Theorem 5.7: If a rectangle exists, then there are arbitrarily large rectangles.

Theorem 5.8: If a rectangle exists, then for any right triangle $\triangle XYZ$ (with right angle at X), there is a rectangle $\square DEFG$ such that $DE > XY$ and $DG > XZ$.

Theorem 5.9: If a rectangle exists, then every right triangle has angle sum of 180°.

Theorem 5.10: If every right triangle has angle sum 180°, then every triangle has angle sum 180°.

Theorem 5.11: If there is a right triangle with angle sum 180°, then a rectangle exists.

Theorem 5.12: In hyperbolic geometry every triangle has angle sum less than 180°.

Theorem 5.13: In Euclidean geometry every triangle has angle sum of 180°.