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Pseudospectra of isospectrally reduced matrices

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SUMMARY

An isospectral matrix reduction is a procedure that reduces the size of a matrix while maintaining its eigenvalues up to a known set. As to not violate the fundamental theorem of algebra, the reduced matrices have rational functions as entries. Because isospectral reductions can preserve the spectrum of a matrix, they are fundamentally different from say the restriction of a matrix to an invariant subspace. We show that the notion of pseudospectrum can be extended to a wide class of matrices with rational function entries and that the pseudospectrum of such matrices shrinks with isospectral reductions. Hence, the eigenvalues of a reduced matrix are more robust to entry-wise perturbations than the eigenvalues of the original matrix. Moreover, the isospectral reductions considered here are more general than those considered elsewhere. We also introduce the notion of an inverse pseudospectrum (or pseudoresonances), which indicates how stable the poles of a rational function valued matrix are to entry-wise perturbations. Illustrations of these concepts are given for mass-spring networks. Copyright © 2014 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Isospectral matrix reductions were first considered in [1], where it was shown that a weighted digraph could be reduced in size while maintaining the eigenvalues of the graph's weighted adjacency matrix, up to a known set. The motivation, in this setting, was to simplify the structure of a complicated network (graph) while preserving its spectral properties.

The method of reduction developed in [1] allows for a matrix A to be reduced over any of its principal submatrices B under the condition that B is similar to an upper triangular matrix with non-zero diagonal via some permutation matrix. Although this puts restrictions on the amount a matrix can be reduced by a single reduction, from a graph theoretic point of view, such reductions have a very natural interpretation in terms of a graph's path and cycle structure.

In this paper, we generalize this previous work by first showing that a matrix can be isospectrally reduced over any of its principal submatrices, under mild conditions. This is a fundamental improvement over the isospectral reduction method in [1–3], as it forgoes the sequence of reductions that was previously necessary for certain matrix reductions. This greatly simplifies the computational and algorithmic complexity needed to reduce a matrix.

Matrices with rational function entries, or rational matrix functions, have a well-developed spectral theory dating back to McMillan's initial work on the subject [4, 5]. This work, motivated by the characterization and synthesis of electrical networks, introduces a number concepts that are

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important here. This includes the spectrum and inverse spectrum (poles) of matrix with rational function entries, which has continued to be an area of interest [6–8].

When a square matrix is isospectrally reduced, the result is a smaller matrix that again has a spectrum and an inverse spectrum [2]. The relation between the spectrum and inverse spectrum of the reduced and unreduced matrices is dictated by the specific submatrix over which the matrix is reduced and is described in Theorem 2.1.

Because isospectral reductions give rise to rational function valued matrices, it is natural to ask how stable is the spectrum (and inverse spectrum) of such matrices to perturbations. For scalar valued matrices, this stability question can be answered with the *pseudospectrum*, that is, the scalars that behave similar to eigenvalues to within a certain tolerance. This concept is particularly useful in analyzing the properties of matrices that are non-normal. The pseudospectrum of a complex valued matrix has been introduced independently many times (see [9] for details). It has also been studied in the case of matrix polynomials [10, 11] and more generally in the case of nonlinear matrix functions that are entire [12, 13]. Here, we extend the definition of pseudospectrum to *certain matrices with rational function entries* (matrices in $\mathbb{W}_\pi^{n\times n}$, see Definition 2.3). This class of matrices does not contain matrix polynomials, so our results cannot be used to prove those in [10, 11]. We do show that the pseudospectra of a reduced matrix are always contained in the pseudospectra of the original matrix for a given tolerance. In other words, the eigenvalues of a reduced matrix are less susceptible to perturbations than the original unreduced matrix.

To study the inverse eigenvalues (or poles) of a matrix with the same tools we have used for its eigenvalues, we introduce the concept of a *spectral inverse*. The spectral inverse of a matrix is the matrix in which the eigenvalues are the inverse eigenvalues of the original matrix and vice versa. Using the spectral inverse of a matrix, we are able to define its *inverse pseudoeigenvalues* or *pseudoresonances*. The inverse pseudoeigenvalues of a matrix are the set of scalars that act as its inverse eigenvalues (or resonances) for a given tolerance.

The paper is organized as follows. Section 2 introduces and extends the theory of isospectral matrix reductions. The pseudospectrum of scalar matrices and its extension to certain matrices with rational functions is given in Section 3. Then, we show in Section 4 that the pseudospectrum of a matrix shrinks in size as the matrix is reduced (one of the main results in this paper). We introduce the spectral inverse, which exchanges the roles of eigenvalues and inverse eigenvalues in Section 5. This too is used to define and study pseudoresonances in Section 6. Although pseudospectrum and pseudoresonances are intimately related, we show that pseudoresonances do contain information that is not included in the pseudospectrum. Then, in Section 7, we show that if isospectral reductions are applied sequentially, the particular steps to reach a certain reduction is not important. Finally, we include in Section 8 our strategy for computing the pseudospectra and pseudoresonances of matrices and also a large-scale example. Throughout the paper we consider numerous examples, some of them involving networks of masses and springs (introduced below) to give a physical interpretation.

1.1. Mass-spring networks: a preview of results

Consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where the set of vertices (or nodes) is $\mathcal{V}$ and the set of edges is $\mathcal{E} \subset \{\{i,j\}; i,j \in \mathcal{V}\}$. A mass-spring network is specified by giving the masses $m \in (0,\infty)^{\mathcal{V}}$ and reference positions $x \in (^d)^{\mathcal{V}}$ of the vertices and by assigning the spring constants $k \in (0,\infty)^{\mathcal{E}}$ to the edges. Here, we are using the set theory notation, for example, m is a function that to each vertex $i \in \mathcal{V}$ associates its mass $m(i) > 0$. For simplicity, we restrict ourselves to either discrete strings ($d = 1$) or planar trusses ($d = 2$), and we ignore damping. If the nodes are subject to a time-harmonic displacement $u(\cdot,\omega)e^{\jmath\omega t}$ with frequency ω and $\jmath = \sqrt{-1}$ (i.e., the position at time t of vertex $i \in \mathcal{V}$ is $x(i) + u(i,\omega)e^{\jmath\omega t}$), and then, the resulting forces are of the form $f(\cdot,\omega)e^{\jmath\omega t}$ and are given in terms of the displacements by Newton's second law

$$f(\cdot,\omega) = (K - \omega^2 M)u(\cdot,\omega). \tag{1}$$

By fixing the ordering of the nodes, we may identify $(\mathbb{R}^d)^{\mathcal{V}}$ to $(\mathbb{R}^d)^n$, with $n = |\mathcal{V}|$ and think of the *mass matrix* M as a $nd \times nd$ diagonal matrix such that for $u, v \in (^d)^n$,

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$$u^T M v = \sum_{i \in \mathcal{V}} m(i) u(i)^T v(i), \tag{2}$$

where $\delta_{i,j}$ is the usual Kronecker delta. The *stiffness matrix* $K \in \mathbb{R}^{nd \times nd}$ can be defined for $u, v \in (^d)^{\mathcal{V}}$ by

$$u^T K v = \sum_{\{i,j\} \in \mathcal{E}} k(\{i,j\})(u(i) - u(j))^T \; n(\{i,j\}) n(\{i,j\})^T \; (v(i) - v(j)), \tag{3}$$

where $n(\{i,j\})$ is the unit (Euclidean) length vector with direction $x(i) - x(j)$. If all the masses are equal to one (i.e., $M = I$), an eigenpair (λ, u) of K satisfies $(K - \lambda I)u = 0$ and thus corresponds to frequencies $\omega^2 = \lambda$ at which the non-zero displacement u does not generate forces. For instance, in the static case ($\lambda = \omega = 0$), there are eigenvectors that correspond to displacing each node by the same amount $v \in^d$ (i.e., of the form $u(i) = v, i \in \mathcal{V}$). Indeed, a translation of every node of the network by v does not generate any net forces at the nodes.

Now, suppose we only have access to a few terminal (or boundary) nodes $\mathcal{B} \subset \mathcal{V}$. Then, we can write the equilibrium of forces at the interior nodes $\mathcal{I} = \mathcal{V} - \mathcal{B}$ and conclude that the net forces $f_{\mathcal{B}}(\cdot, \omega)$ at the terminal nodes depend linearly on the displacements $u_{\mathcal{B}}(\cdot, \omega)$ at the terminal nodes according to the equation

$$f_{\mathcal{B}}(\cdot, \omega) = (R_{\omega^2}(K; \mathcal{B}) - \omega^2 I) u_{\mathcal{B}}(\cdot, \omega), \tag{4}$$

where

$$R_{\omega^2}(K; \mathcal{B}) = K_{\mathcal{B}\mathcal{B}} - K_{\mathcal{B}\mathcal{I}}(K_{\mathcal{I}\mathcal{I}} - \omega^2 I)^{-1} K_{\mathcal{I}\mathcal{B}}) \tag{5}$$

and subscripts with sets indicating appropriate submatrices; that is, $K_{\mathcal{B}\mathcal{I}}$ is the submatrix of K with rows in $\mathcal{B}$ and columns in $\mathcal{I}$ (as a matrix of $d \times d$ blocks).

The eigenvalues of $R_{\omega^2}(K; \mathcal{B})$ are the frequencies $\omega^2 = \lambda$ for which there is a non-zero displacement of the boundary nodes $\mathcal{B}$ that generate no forces. The inverse eigenvalues (resonances or poles) of R_{ω^2} are the frequencies at which there is a displacement of the boundary nodes for which the forces are infinitely large (i.e., the poles of $f_{\mathcal{B}}(\cdot, \omega)$). The typical time domain manifestation of a resonance is that the amplitude of the displacements of the nodes increases with time, which may lead to catastrophic failure of the structure, hence the importance of knowing which frequencies are resonances or are close to resonances.

The main results that we introduce here are as follows:

(1) The eigenvalues of $R_{\omega^2}(K; \mathcal{B})$ are a subset of the eigenvalues of K (Theorem 2.1; hence, we call the matrix $R_{\omega^2}(K; \mathcal{B})$ an isospectral reduction.
(2) The inverse eigenvalues of $R_{\omega^2}(K; \mathcal{B})$ are a subset of the eigenvalues of $K_{\mathcal{I}\mathcal{I}}$ (Theorem 2.1).
(3) The frequencies ω that are, within some tolerance, eigenvalues of $R_{\omega^2}(K; \mathcal{B})$ are eigenvalues, within the same tolerance, of K. This result is formulated by inclusion of pseudospectra (Theorem 4.1).
(4) Frequencies that are not, within a certain tolerance, eigenvalues of $R_{\omega^2}(K; \mathcal{B})$ are frequencies that are close to being resonances. The converse is in general not true (Theorem 6.2).

2. ISOSPECTRAL MATRIX REDUCTIONS

We start by recalling some properties of rational function valued matrices, as such matrices arise naturally if we wish or need to reduce the size of a matrix (or system) while maintaining its spectral properties. Then, we present isospectral reductions in a more general setting than in [1–3]. The main result of this section is Theorem 2.1, which gives the relation between the spectrum (or inverse spectrum) of a reduced matrix and that of the unreduced matrix.

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2.1. Matrices with rational function entries

We consider square matrices whose entries are rational functions of a parameter λ. Specifically, let $\mathbb{C}[\lambda]$ be the set of polynomials in the complex variable λ with complex coefficients. We denote by $\mathbb{W}$ the set of rational functions of the form

$$w(\lambda) = p(\lambda)/q(\lambda)$$

where $p(\lambda), q(\lambda) \in \mathbb{C}[\lambda]$ are polynomials having no common linear factors and where $q(\lambda)$ is not identically zero.

More generally, each rational function $w(\lambda) \in \mathbb{W}$ is expressible in the form

$$w(\lambda) = \frac{a_i\lambda^i + a_{i-1}\lambda^{i-1} + \cdots + a_0}{b_j\lambda^j + b_{j-1}\lambda^{j-1} + \cdots + b_0}$$

where, without loss in generality, we can take $b_j = 1$. The domain of $w(\lambda)$ consists of all but a finite number of complex numbers for which $q(\lambda) = b_j\lambda^j + b_{j-1}\lambda^{j-1} + \cdots + b_0$ is zero.

Addition and multiplication on the set $\mathbb{W}$ are defined as follows. For $p(\lambda)/q(\lambda)$ and $r(\lambda)/s(\lambda)$ in $\mathbb{W}$, let

$$\left(\frac{p}{q} + \frac{r}{s}\right)(\lambda) = \frac{p(\lambda)s(\lambda) + q(\lambda)r(\lambda)}{q(\lambda)s(\lambda)}; \text{ and} \tag{6}$$

$$\left(\frac{p}{q} \cdot \frac{r}{s}\right)(\lambda) = \frac{p(\lambda)r(\lambda)}{q(\lambda)s(\lambda)} \tag{7}$$

where the common linear factors in the right-hand side of equations (6) and (7) are canceled. The set $\mathbb{W}$ is then a field under addition and multiplication.

Because we are primarily concerned with the eigenvalues of a matrix, which is a set that includes multiplicities, we note the following. The element α of the set A that includes multiplicities has *multiplicity* m if there are m elements of A equal to α. If $\alpha \in A$ with multiplicity m and $\alpha \in B$ with multiplicity n, then

(i) the *union* $A \cup B$ is the set in which α has multiplicity $m + n$ and
(ii) the *difference* $A - B$ is the set in which α has multiplicity $m - n$ if $m - n > 0$ and where $\alpha \notin A - B$ otherwise.

Definition 2.1
Let $\mathbb{W}^{n\times n}$ denote the set of $n \times n$ matrices with entries in $\mathbb{W}$. For a matrix $M(\lambda) \in \mathbb{W}^{n\times n}$, the determinant

$$\det\big(M(\lambda) - \lambda I\big) = p(\lambda)/q(\lambda)$$

for some $p(\lambda)/q(\lambda) \in \mathbb{W}$. The *spectrum* (or eigenvalues) of $M(\lambda)$ is defined as

$$\sigma(M) = \{\lambda \in \mathbb{C} : p(\lambda) = 0\}.$$

The inverse spectrum (or resonances) of $M(\lambda)$ is defined as

$$\sigma^{-1}(M) = \{\lambda \in \mathbb{C} : q(\lambda) = 0\}.$$

Both $\sigma(M)$ and $\sigma^{-1}(M)$ are understood to be sets that include multiplicities. For example, if the polynomial $p(\lambda) \in \mathbb{C}[\lambda]$ factors as

$$p(\lambda) = \prod_{i=1}^{m}(\lambda - \alpha_i)^{n_i} \quad \text{for } \alpha_i \in \mathbb{C} \text{ and } n_i \in \mathbb{N}$$

where the α_i are distinct, then $\{\lambda \in \mathbb{C} : p(\lambda) = 0\}$ is the set in which α_i has multiplicity n_i.

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Remark 2.1
The definition of eigenvalues we use (Definition 2.1 and [1–3]) requires that the spectral parameter appears as λI when taking the determinant and as such is in some sense limited compared with the more commonly [12, 14] used definition $\sigma(M) = \{\lambda \in \mathbb{C} : \det M(\lambda) = 0\}$ for the spectrum of matrices with polynomial and rational function entries. The main reason we define the eigenvalues of a matrix via Definition 2.1 is that it allows us to easily guarantee that isospectral reductions exist for a subset of $\mathbb{W}^{n\times n}$ that includes $\mathbb{C}^{n\times n}$ (Lemma 2.1). Of course, because $\mathbb{C} \subset \mathbb{W}$, Definition 2.1 is an extension of the standard definition of the eigenvalues for a matrix in $\mathbb{C}^{n\times n}$ to the larger class of matrices $\mathbb{W}^{n\times n}$: if $M \in \mathbb{C}^{n\times n}$, then $\sigma(M)$ are the standard eigenvalues of M. It is an open question whether our isospectral reduction results extend to all matrices in $\mathbb{W}^{n\times n}$.

In what follows, we may, for convenience, suppress the dependence of the matrix $M(\lambda) \in \mathbb{W}^{n\times n}$ on λ and simply write M. One reason for this is that for much of what we do in this paper, we do not evaluate $M(\lambda)$ at any particular point $\lambda \in \mathbb{C}$. Rather, we consider M formally as a matrix with rational function entries.

However, when we do consider the matrix $M(\lambda) \in \mathbb{W}^{n\times n}$ to be a function of λ, we mean M is the function

$$M : \mathrm{dom}\,(M) \to \mathbb{C}^{n\times n},$$

where $\mathrm{dom}\,(M)$ are the complex numbers λ for which every entry of $M(\lambda)$ is defined. Surprisingly, it may be the case that $\sigma(M) \nsubseteq \mathrm{dom}(M)$ as the following example shows.

Example 2.1
Consider the matrix $M(\lambda) \in \mathbb{W}^{2\times 2}$ given by

$$M(\lambda) = \begin{bmatrix} 0 & \frac{1}{\lambda} \\ 0 & 0 \end{bmatrix}.$$

As one can compute, $\det(M(\lambda) - \lambda I) = \lambda^2$ implying $\sigma(M) = \{0, 0\}$. Therefore, $\sigma(M) \nsubseteq \mathrm{dom}(M)$.

2.2. *Isospectral matrix reductions*

We can now describe an isospectral reduction of a matrix $M \in \mathbb{W}^{n\times n}$. We then compare the spectrum of M with the spectrum of its isospectral reduction.

Let $M \in \mathbb{W}^{n\times n}$ and $N = \{1, \ldots, n\}$. If the sets $\mathcal{R}, \mathcal{C} \subseteq N$ are non-empty, we denote by $M_{\mathcal{RC}}$ the $|\mathcal{R}| \times |\mathcal{C}|$ *submatrix* of M with rows indexed by $\mathcal{R}$ and columns by $\mathcal{C}$. Suppose the non-empty sets $\mathcal{B}$ and $\mathcal{I}$ form a partition of N. The *Schur complement* of $M_{\mathcal{II}}$ in M is the matrix

$$M/M_{\mathcal{II}} = M_{\mathcal{BB}} - M_{\mathcal{BI}} M_{\mathcal{II}}^{-1} M_{\mathcal{IB}}, \tag{8}$$

assuming $M_{\mathcal{II}}$ is invertible. We say a matrix in $A \in \mathbb{W}^{n\times n}$ is invertible if there is a $B \in \mathbb{W}^{n\times n}$ such that $AB = I$ the $n \times n$ identity matrix. As with scalar valued matrices, we write $A^{-1} = B$.

The Schur complement arises in many applications. For example, if the matrix M is the Kirchhoff matrix of a network of resistors with n nodes, then its Schur complement is the Dirichlet to Neumann (or voltage to currents) map of the network given by considering the nodes in $\mathcal{B}$ as terminal or *boundary* nodes and the nodes in $\mathcal{I}$ as *interior* nodes (see, e.g., [15]). A physical interpretation of an isospectral reduction is given in Example 2.4.

We are now ready to define the isospectral reduction of a matrix $M \in \mathbb{W}^{n\times n}$.

Definition 2.2
For $M(\lambda) \in \mathbb{W}^{n\times n}$, let $\mathcal{B}$ and $\mathcal{I}$ form a non-empty partition of N. The *isospectral reduction* of M over the set $\mathcal{B}$ is the matrix

$$R_\lambda(M; \mathcal{B}) = M_{\mathcal{BB}} - M_{\mathcal{BI}}(M_{\mathcal{II}} - \lambda I)^{-1} M_{\mathcal{IB}} \in \mathbb{W}^{|\mathcal{B}|\times|\mathcal{B}|}. \tag{9}$$

if the matrix $M_{\mathcal{II}} - \lambda I$ is invertible.

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It follows from Definition 2.2 that a matrix $M \in \mathbb{W}^{n\times n}$ can be reduced over $\mathcal{B}$ if and only if $\det(M_{\mathcal{I}\mathcal{I}} - \lambda I)$ is not identically zero.

Remark 2.2
For a matrix to be reduced in each of [1–3] the matrix $M_{\mathcal{I}\mathcal{I}} - \lambda I$ is required to be similar via a permutation matrix to a non-singular upper triangular matrix. Hence, isospectral reductions defined by Definition 2.2 are more general than those previously considered.

Note that the reduced matrix $R_\lambda(M;\mathcal{B})$ is a Schur complement plus a multiple of the identity:

$$R_\lambda(M;\mathcal{B}) = (M - \lambda I)/(M_{\mathcal{I}\mathcal{I}} - \lambda I) + \lambda I. \tag{10}$$

More often than not, we suppress the dependence of $R_\lambda(M;\mathcal{B})$ on λ and instead write this as $R(M;\mathcal{B})$.

Example 2.2
Consider the matrix $M \in \mathbb{W}^{6\times 6}$ with (0, 1) entries given by

$$M = \begin{bmatrix} 0&0&1&1&0&0\\ 0&1&0&0&1&1\\ 1&0&1&0&0&0\\ 0&1&0&1&0&0\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0 \end{bmatrix}.$$

For $\mathcal{B} = \{1,2\}$ and $\mathcal{I} = \{3,4,5,6\}$, we have

$$(M_{\mathcal{I}\mathcal{I}} - \lambda I)^{-1} = \begin{bmatrix} \frac{1}{1-\lambda} & 0 & 0 & 0\\ 0 & \frac{1}{1-\lambda} & 0 & 0\\ 0 & 0 & -\frac{1}{\lambda} & 0\\ 0 & 0 & 0 & -\frac{1}{\lambda} \end{bmatrix}.$$

The isospectral reduction of M over $\mathcal{B} = \{1,2\}$ is then defined as

$$\begin{aligned} R(M;\mathcal{B}) &= \begin{bmatrix} 0&0\\ 0&1 \end{bmatrix} - \begin{bmatrix} 1&1&0&0\\ 0&0&1&1 \end{bmatrix} \begin{bmatrix} \frac{1}{1-\lambda} & 0 & 0 & 0\\ 0 & \frac{1}{1-\lambda} & 0 & 0\\ 0 & 0 & -\frac{1}{\lambda} & 0\\ 0 & 0 & 0 & -\frac{1}{\lambda} \end{bmatrix} \begin{bmatrix} 1&0\\ 0&1\\ 1&0\\ 0&1 \end{bmatrix}\\ &= \begin{bmatrix} \frac{1}{\lambda-1} & \frac{1}{\lambda-1}\\ \frac{1}{\lambda} & \frac{\lambda+1}{\lambda} \end{bmatrix} \in \mathbb{W}^{2\times 2}. \end{aligned}$$

If a matrix has an isospectral reduction, the spectrum and inverse spectrum of the isospectral reduction and the original matrix are related as follows.

Theorem 2.1 (Spectrum and Inverse Spectrum of Isospectral Reductions)
For $M(\lambda) \in \mathbb{W}^{n\times n}$, let $\mathcal{B}$ and $\mathcal{I}$ form a non-empty partition of N. If $R_\lambda(M;\mathcal{B})$ exists, then its spectrum and inverse spectrum are given by

$$\begin{aligned} \sigma\big(R(M;\mathcal{B})\big) &= \big(\sigma(M) \cup \sigma^{-1}(M_{\mathcal{I}\mathcal{I}})\big) - \big(\sigma(M_{\mathcal{I}\mathcal{I}}) \cup \sigma^{-1}(M)\big)\,; \text{ and}\\ \sigma^{-1}\big(R(M;\mathcal{B})\big) &= \big(\sigma(M_{\mathcal{I}\mathcal{I}}) \cup \sigma^{-1}(M)\big) - \big(\sigma(M) \cup \sigma^{-1}(M_{\mathcal{I}\mathcal{I}})\big). \end{aligned}$$

Proof
The proof of Theorem 2.1 is included in Appendix A. □

Because a matrix $M \in \mathbb{C}^{n\times n}$ has no inverse spectrum (i.e., $\sigma^{-1}(M) = \emptyset$), Theorem 2.1 applied to complex valued matrices has the following corollary.

Corollary 2.1
For $M \in \mathbb{C}^{n\times n}$ let $\mathcal{B}$ and $\mathcal{I}$ form a non-empty partition of N. Then,

$$\sigma\big(R(M;\mathcal{B})\big) = \sigma(M) - \sigma(M_{\mathcal{I}\mathcal{I}}) \text{ and } \sigma^{-1}\big(R(M;\mathcal{B})\big) = \sigma(M_{\mathcal{I}\mathcal{I}}) - \sigma(M).$$

Example 2.3
Let M, $\mathcal{B}$ and $\mathcal{I}$ be as in Example 2.2. As one can compute $\sigma(M) = \{2,-1,1,1,0,0\}$ and $\sigma(M_{\mathcal{I}\mathcal{I}}) = \{1,1,0,0\}$. By Corollary 2.1, we then have

$$\sigma\big(R(M;\mathcal{B})\big) = \{2,-1,1,1,0,0\} - \{1,1,0,0\} = \{2,-1\}; \; and$$
$$\sigma^{-1}\big(R(M;\mathcal{B})\big) = \{1,1,0,0\} - \{2,-1,1,1,0,0\} = \emptyset.$$

Observe that, by reducing M over $\mathcal{B}$, we lose the eigenvalues corresponding to the 'interior' degrees of freedom $\sigma(M_{\mathcal{I}\mathcal{I}}) = \{1,1,0,0\}$. That is, if we knew both $\sigma(M_{\mathcal{I}\mathcal{I}})$ and $\sigma(R(M;\mathcal{B}))$ but not $\sigma(M)$, then Corollary 2.1 states that the set $\sigma(M_{\mathcal{I}\mathcal{I}})$ is the most by which $\sigma(R(M;\mathcal{B}))$ and $\sigma(M)$ can differ.

Theorem 2.1 therefore describes exactly which eigenvalues we may gain from an isospectral reduction and which we may lose. In this way, an isospectral reduction of a matrix preserves the spectral information of the original matrix. However, it may not always be possible to find an isospectral reduction of a matrix $M \in \mathbb{W}^{n\times n}$.

For instance, consider the matrix $M \in \mathbb{W}^{2\times 2}$ given by

$$M = \begin{bmatrix} 1 & 1 \\ 1 & \lambda \end{bmatrix}. \tag{11}$$

For $\mathcal{B} = \{1\}$ and $\mathcal{I} = \{2\}$, note that $M_{\mathcal{I}\mathcal{I}} - \lambda I = [0]$, which is not invertible. Therefore, M cannot be isospectrally reduced over $\mathcal{B}$.

In general, there is no way to know beforehand if the isospectral reduction $R(M;\mathcal{B})$ exists without computing $(M_{\mathcal{I}\mathcal{I}} - \lambda I)^{-1}$. However, the following subset of $\mathbb{W}^{n\times n}$ can always be reduced over any non-empty subset $\mathcal{B} \subset N$.

For any polynomial $p(\lambda) \in \mathbb{C}[\lambda]$, let $\deg(p)$ denote its degree. If $w(\lambda) = p(\lambda)/q(\lambda)$ where both $p(\lambda), q(\lambda) \in \mathbb{C}[\lambda]$ are not identically zero, we define the degree of the rational function $w(\lambda)$ by

$$\pi(w) = \deg(p) - \deg(q).$$

In the case where $p(\lambda) = 0$, we let $\pi(w) = 0$.

Definition 2.3
We denote by $\mathbb{W}_\pi$ the set of rational functions

$$\mathbb{W}_\pi = \{w \in \mathbb{W} : \pi(w) \leqslant 0\}$$

and let $\mathbb{W}_\pi^{n\times n}$ be the set of $n \times n$ matrices with entries in $\mathbb{W}_\pi$.

Lemma 2.1
If $M(\lambda) \in \mathbb{W}_\pi^{n\times n}$ and $\mathcal{B} \subset N$ is non-empty, then $R_\lambda(M;\mathcal{B}) \in \mathbb{W}_\pi^{|\mathcal{B}|\times|\mathcal{B}|}$.

Proof
The proof of Lemma 2.1 is in Appendix C. □

Note that Lemma 2.1 implies the existence of any isospectral reduction $R(M;\mathcal{B})$ if $M \in \mathbb{W}_\pi^{n\times n}$ and $\mathcal{B} \subset N$. In particular, any complex valued matrix can be reduced over any index set. Because the matrix M given in (11) does not belong to $\mathbb{W}_\pi^{2\times 2}$, Lemma 2.1 does not apply in this particular case.

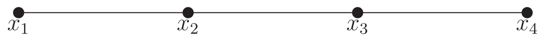

Figure 1. The mass-spring network of Example 2.4 with boundary nodes $\mathcal{B} = \{1, 4\}$ and interior nodes $\mathcal{I} = \{2, 3\}$.

In the following example, we demonstrate how one can use an isospectral reduction to study the dynamics of a mass-spring network in which access is limited.

Example 2.4
Consider the mass-spring network illustrated in Figure 1, with nodes at locations x_i, $i = 1, 2, 3, 4$ lying on a line and edges representing springs between nodes. For simplicity, we assume that all the springs have the same spring constant ($k = 1$) and that all the nodes have unit mass. (The precise position of the nodes on the line does not matter for this discussion; see Section 1.1). The stiffness matrix is then

$$K = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix}.$$

If we let $\lambda \equiv \omega^2$, we see that the eigenmodes of the stiffness matrix K correspond to non-zero displacements that do not generate forces. For instance, the eigenmode corresponding to the zero frequency is $\mathbf{u} = [1, 1, 1, 1]^T$; that is, by displacing all nodes by the same amount, there are no net forces at the nodes.

Suppose we only have access to certain terminal (or boundary) nodes of this network, say $\mathcal{B} = \{1, 4\}$. Then, the net forces $\mathbf{f}_\mathcal{B}$ at the terminal nodes depend linearly on the displacements $\mathbf{u}_\mathcal{B}$ at the terminal nodes according to the equation

$$\mathbf{f}_\mathcal{B}(\omega) = (R_{\omega^2}(K; \mathcal{B}) - \omega^2 I)\mathbf{u}_\mathcal{B}(\omega). \tag{12}$$

The spectrum and inverse spectrum of the response are

$$\sigma(R_{\omega^2}(K; \mathcal{B})) = \{2 \pm \sqrt{2}, 2, 0\} \text{ and } \sigma^{-1}(R_{\omega^2}(K; \mathcal{B})) = \{3, 1\}.$$

The eigenvalues of $R_{\omega^2}(K; \mathcal{B})$ correspond to frequencies for which there is a displacement of the boundary nodes $\mathcal{B}$ that generate no forces at these nodes. Conversely, the resonances (or inverse eigenvalues) of $R_{\omega^2}(K; \mathcal{B})$ correspond to frequencies at which there is a displacement of the boundary nodes for which the resulting forces are infinitely large (i.e., the poles of $\mathbf{f}_\mathcal{B}(\omega)$).

Importantly, we note that although $R_\lambda(K; \mathcal{B})$ is the 2×2 matrix,

$$R_\lambda(K; \mathcal{B}) = \frac{1}{\lambda^2 - 4\lambda + 3} \begin{bmatrix} \lambda^2 - 3\lambda + 1 & -1 \\ -1 & \lambda^2 - 3\lambda + 1 \end{bmatrix},$$

it has four eigenvalues. That is, an $n \times n$ matrix $M(\lambda) \in \mathbb{W}^{n \times n}$ may have more than n eigenvalues.

3. PSEUDOSPECTRA

A pseudospectrum of a matrix $M \in \mathbb{C}^{n \times n}$ is essentially the collection of scalars that behave, within a given tolerance, as an eigenvalue of M. These values indicate to what extent the eigenvalues of the matrix M are stable under perturbation of the matrix entries (Section 3.1). For a review of pseudospectra including their history and applications, see, for example, [9]. We extend the notion of pseudospectra to matrices in $\mathbb{W}_\pi^{n \times n}$ in Section 3.2. Throughout this discussion, we consider the simple mass-spring network introduced in Section 2.2 to give a physical interpretation to these concepts.

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3.1. Pseudospectra of scalar valued matrices

For completeness, we begin by recalling three equivalent definitions for the pseudospectrum of a matrix $A \in \mathbb{C}^{n \times n}$ (see, e.g., [9] for more details).

Definition 3.1
Let $\epsilon > 0$. The *ϵ-pseudospectrum* of $A \in \mathbb{C}^{n \times n}$ is defined equivalently by the following:

(a) *eigenvalue perturbation*:

$$\sigma_\epsilon(A) = \{\lambda \in \mathbb{C} : \|(A - \lambda I)\mathbf{v}\| < \epsilon \text{ for some } \mathbf{v} \in \mathbb{C}^n \text{ with } \|\mathbf{v}\| = 1\};$$

(b) *the resolvent*:

$$\sigma_\epsilon(A) = \{\lambda \in \mathbb{C} : \|(A - \lambda I)^{-1}\| > \epsilon^{-1}\} \cup \sigma(A); \text{ and}$$

(c) *perturbation of the matrix*:

$$\sigma_\epsilon(A) = \{\lambda \in \mathbb{C} : \lambda \in \sigma(A + E) \text{ for some } E \in \mathbb{C}^{n \times n} \text{ with } \|E\| < \epsilon\}.$$

The matrix and vector norms in (a)–(c) are assumed to be consistent in the sense that

$$\|A\mathbf{v}\| \leqslant \|A\|\|\mathbf{v}\| \text{ for all } A \in \mathbb{C}^{n \times n} \text{ and } \mathbf{v} \in \mathbb{C}^n.$$

The proof that parts (a)–(c) of Definition 3.1 are equivalent for any $A \in \mathbb{C}^{n \times n}$, and $\epsilon > 0$ can be found in, for example, [9].

We note that the assumption $\|\mathbf{v}\| = 1$ in part (a) of Definition 3.1 is necessary because any $\lambda \in \mathbb{C}$ could be in $\sigma_\epsilon(A)$ by choosing $\|\mathbf{v}\|$ small enough. Also, including $\sigma(A)$ on the right-hand side of part (b) is necessary because the pseudospectrum $\sigma_\epsilon(A)$ in parts (a) and (c) contain $\sigma(A)$, but the matrix $A - \lambda I$ in part (b) is non-invertible for $\lambda \in \sigma(A)$.

3.2. Pseudospectra of rational function valued matrices

For a matrix $A \in \mathbb{C}^{n \times n}$, if $\lambda \in \sigma(A)$, then there is always at least one eigenvector $\mathbf{v} \in \mathbb{C}^n$ of A associated with λ. However, recall from Section 2.1 that a matrix $M(\lambda) \in \mathbb{W}_\pi^{n \times n}$ may have an eigenvalue λ_0 for which $M(\lambda_0)$ is undefined. This may seem problematic especially if we would like to find an eigenvector associated with λ_0. In fact, it is still possible to do so.

Assuming λ_0 is a solution to the equation $\det(M(\lambda) - \lambda I) = 0$, the standard theory of linear algebra implies that there is a vector $\mathbf{v}$ such that when the product $(M(\lambda) - \lambda I)\mathbf{v}$ is evaluated at $\lambda = \lambda_0$, the result is the zero vector. Keeping this sequence in mind, we define the product of a matrix $M(\lambda) \in \mathbb{W}_\pi^{n \times n}$ and a vector $\mathbf{v} \in \mathbb{C}^n$ as follows

$$(M(\lambda) - \lambda I)\mathbf{v} \equiv (M(s) - sI)\mathbf{v}|_{s=\lambda}.$$

This definition allows us to associate an eigenvector to each eigenvalue of a matrix $M(\lambda) \in \mathbb{W}_\pi^{n \times n}$. To demonstrate this idea, we give the following example.

Example 3.1
Consider the matrix $M(\lambda) \in \mathbb{W}_\pi^{2 \times 2}$ given by

$$M(\lambda) = \begin{bmatrix} 1 & \frac{1}{\lambda - 1} \\ 0 & 1 \end{bmatrix}.$$

Here, one can readily see that $\sigma(M) = \{1, 1\}$. Although $M(1)$ is undefined, the vector $\mathbf{v} = [1\ 0]^T$ has the property

$$(M(1) - 1I)\mathbf{v} = \begin{bmatrix} 1 - s & \frac{1}{s-1} \\ 0 & 1 - s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}\Big|_{s=1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

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By definition, the vector $\mathbf{v}$ is an eigenvector associated with the eigenvalue 1 despite the fact that $M(\lambda)$ is not defined for $\lambda = 1$.

Importantly, for the vector norm $\|\cdot\|$, we have

$$\|(M(\lambda) - \lambda I)\mathbf{v}\| = \left\| \begin{bmatrix} 1-\lambda \\ 0 \end{bmatrix} \right\|.$$

Hence, the size of $(M(\lambda) - \lambda I)\mathbf{v}$ varies continuously with respect to λ even where $M(\lambda)$ is undefined. This is useful because we study values of λ that act almost like eigenvalues of $M(\lambda)$.

Suppose that for a given tolerance $\epsilon > 0$, there is a scalar $\lambda \in \mathbb{C}$ and a unit vector $\mathbf{v} \in \mathbb{C}^n$ for which $\|(M(\lambda)-\lambda I)\mathbf{v}\| < \epsilon$. If this is the case, then the vector $\mathbf{v}$ is said to be an ϵ*-pseudoeigenvector* of the matrix $M(\lambda)$ corresponding to the ϵ*-pseudoeigenvalue* λ. The $\epsilon-$pseudospectrum of $M(\lambda)$ is defined as the set of all such λ. We state this and two other equivalent definitions of the $\epsilon-$pseudospectrum in the succeeding text. For $\Omega \subset \mathbb{C}$, let $cl(\Omega)$ be the closure of Ω in $\mathbb{C}$.

Definition 3.2
Let $\epsilon > 0$. The ϵ-pseudospectrum of $M(\lambda) \in \mathbb{W}_\pi^{n\times n}$ is defined equivalently by

(a) *eigenvalue perturbation*:

$$\sigma_\epsilon(M) = cl\big(\{\lambda \in \mathbb{C} : \|(M(\lambda) - \lambda I)\mathbf{v}\| < \epsilon \text{ for some } \mathbf{v} \in \mathbb{C}^n \text{ with } \|\mathbf{v}\| = 1\}\big);$$

(b) *the resolvent*:

$$\sigma_\epsilon(M) = cl\big(\{\lambda \in \mathbb{C} : \|(M(\lambda) - \lambda I)^{-1}\| > \epsilon^{-1}\}\big); and$$

(c) *perturbation of the matrix*:

$$\sigma_\epsilon(M) = cl\big(\{\lambda \in \mathbb{C} : \lambda \in \sigma(M(\lambda) + E) \text{ for some } E \in \mathbb{C}^{n\times n} \text{ with } \|E\| < \epsilon\}\big).$$

The matrix and vector norms in (a)–(c) are assumed to be consistent as in Definition 3.1. The spectrum in part (c) is understood as the spectrum of the matrix $M(\lambda) + E$, which is in $\mathbb{C}^{n\times n}$ for a *fixed* $\lambda \in \mathbb{C}$ (and in $\text{dom}(M)$).

As a consequence of Definition 3.2, the eigenvalues of a matrix $M \in \mathbb{W}_\pi^{n\times n}$ belong to all its pseudospectra:

$$\sigma(M) \subset \sigma_\epsilon(M) \text{ for each } \epsilon > 0.$$

The proof that Definition 3.2 parts (a)–(c) are equivalent relies on the proof that Definition 3.2 parts (a)–(c) are equivalent for scalar valued matrices. For completeness, the proofs are included in Appendix D.

Remark 3.1
In Definition 3.2(c), we could have perturbed $M(\lambda)$ with a matrix with rational function entries $E(\lambda) \in \mathbb{W}_\pi^{n\times n}$ (or even in $\mathbb{W}^{n\times n}$). For a fixed λ, the assumption $\|E(\lambda)\| < \epsilon$ implies that $\lambda \in \text{dom}(E)$. Moreover, the spectrum in part (c) is understood as the spectrum of the matrix $M(\lambda) + E(\lambda)$ for a *fixed* $\lambda \in \mathbb{C}$. Thus, this alternative definition is equivalent to part (c), which is simpler.

We now compare the pseudospectra of a matrix and its reduction for a few examples.

Example 3.2
Consider the matrices M and $R(M;\mathcal{B})$ given in Example 2.2 where $\mathcal{B} = \{1, 2\}$. The pseudospectra of both matrices are displayed in Figure 2 for $\epsilon = 1$, $10^{-1/2}$, 10^{-1} using the matrix 2-norm. Notice that although $0, 1 \in \sigma(M)$ these values do not belong to $\sigma(R(M;\mathcal{B}))$ because of cancellations resulting from the matrix reduction, that is, $\sigma(M_{\mathcal{I}\mathcal{I}}) = \{0, 0, 1, 1\}$. However, for the ϵ, we consider $0, 1 \in \sigma_\epsilon(R(M;\mathcal{B}))$ meaning that these eigenvalues remain as pseudoeigenvalues of the reduced matrix.

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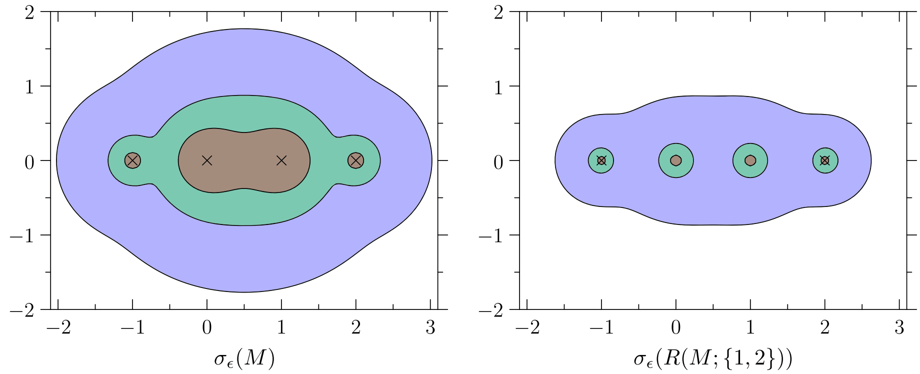

Figure 2. Pseudospectra of the matrices given in Example 2.2 for $\epsilon = 1$ (blue), $\epsilon = 10^{-1/2}$ (green), and $\epsilon = 10^{-1}$ (red), obtained with the matrix 2-norm. The respective spectra are indicated by '×'.

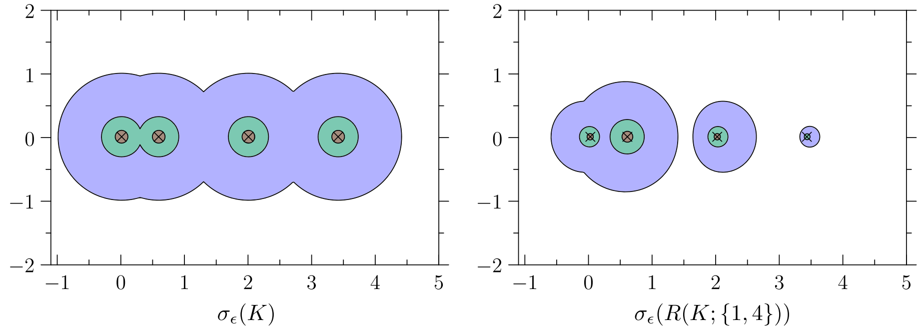

Figure 3. Pseudospectra of the stiffness matrix K for the mass-spring system in Example 2.4 and of its reduction $R_\lambda(K, \{1, 4\})$. The latter corresponds to the effective stiffness of the mass-spring system when we only have access to nodes $\{1, 4\}$. The tolerances shown are $\epsilon = 1$ (blue), $\epsilon = 10^{-1/2}$ (green), and $\epsilon = 10^{-1}$ (red), using the matrix 2-norm. The '×' correspond to spectra of the respective matrices.

To give a possible physical interpretation of pseudospectra, we again consider a mass-spring network.

Example 3.3

For the mass-spring network considered in Example 2.4, recall that the eigenvalues of K correspond to frequencies for which there exists a non-zero displacement that generates no forces on these nodes. The pseudoeigenvalues of this system have a similar physical interpretation. Namely, the pseudospectra indicate the frequencies for which there is a displacement that generates 'small' forces relative to the (norm of the) displacement.

For example, as the frequency $\omega^2 = 2.1$ in Figure 3 (right) is within the green tolerance region, there is a non-zero vector of displacements such that the forces generated from this displacement have norm $\epsilon = 10^{-1/2}$ times less than the norm of this displacement vector. That is, if we only have access to the boundary nodes $\mathcal{B} = \{1, 4\}$, then the pseudoeigenvalues of $R_{\omega^2}(K; \mathcal{B})$ correspond to frequencies for which there is a displacement at the boundary nodes $\mathcal{B}$ that generates very small forces on these nodes. The pseudospectra regions of $R_\lambda(K; \mathcal{B})$ are shown in Figure 3 (right) for $\epsilon = 1, 10^{-1/2}, 10^{-1}$.

Observe that the pseudospectra of $R_\lambda(K; \mathcal{B})$ are included in the pseudospectra of K for a given tolerance ϵ. That is, less access to network nodes means there are fewer frequencies for which displacements generate relatively small forces. Phrased less formally, the more a network is reduced, the less susceptible to perturbations its eigenvalues are.

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Remark 3.2
In the standard theory of pseudospectra, if $M \in \mathbb{C}^{n\times n}$ has complex entries, then its pseudospectrum $\sigma_\epsilon(M)$ can have at most n connected components each corresponding to at least one eigenvalue of M (see, e.g., theorem 2.4 in [9]). In contrast, a matrix $M(\lambda) \in \mathbb{W}^{n\times n}$ can have more than n eigenvalues. Hence, $\sigma_\epsilon(M)$ can have more than n connected components. This is the reason why the region $\sigma_\epsilon(R(K;\{1,4\}))$ shown in Figure 3 (right) can have more than two connected components, although $R(K;\{1,4\}) \in \mathbb{W}^{2\times 2}$.

Remark 3.3
If a matrix $M \in \mathbb{C}^{n\times n}$ is normal, then its ϵ-pseudospectrum $\sigma_\epsilon(M)$, using the matrix 2-norm, is the union of closed balls of radius ϵ centered at its eigenvalues (see, e.g., theorem 2.2 in [9]). As the stiffness matrix $K \in \mathbb{C}^{4\times 4}$ is normal, then $\sigma_\epsilon(K)$ has this property as can be seen in Figure 3 (left). However, the ϵ-pseudospectrum of the reduced matrix $R(K;\{1,4\}) \in \mathbb{W}^{2\times 2}$, which is contained in the ϵ-pseudospectrum of K, does not have this property for the ϵ we consider. This is even more surprising when we note that the matrix $R_\lambda(K;\{1,4\})$ is normal for each λ in its domain.

Note that in Examples 3.2 and 3.3, we have $\sigma(M) \subset \sigma_\epsilon(R(M;\mathcal{B}))$ for the ϵ we consider. It seems that even under reduction, the ϵ-pseudospectrum remembers where the eigenvalues of the original matrix are. However, this is not always the case, as the following example shows.

Example 3.4
Consider the matrix $M \in \mathbb{C}^{3\times 3}$ given by

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

with $\sigma(M) = \{0, \pm 1\}$. By reducing M over $\mathcal{B} = \{1\}$, we obtain the matrix $R(M;\mathcal{B}) = [1/\lambda]$ for which

$$\|(R(M;\mathcal{B}) - \lambda I)^{-1}\| = \left|\frac{\lambda}{1-\lambda^2}\right|.$$

Hence, $0 \notin \sigma_\epsilon(R(M;\mathcal{B}))$ for any ϵ. Moreover, as $\sigma(M_{\mathcal{I}\mathcal{I}}) = \{0,0\}$ for $\mathcal{I} = \{2,3\}$, it is not always the case that either $\sigma(M)$ or $\sigma(M_{\mathcal{I}\mathcal{I}})$ is contained in $\sigma_\epsilon(R(M;\mathcal{B}))$.

4. PSEUDOSPECTRA UNDER ISOSPECTRAL REDUCTION

One of the major goals of this paper is to understand how the pseudospectra of a matrix $M \in \mathbb{W}_\pi^{n\times n}$ are affected by an isospectral reduction. In order to study this change in pseudospectra, we need to consider two vector norms. Specifically, we need one norm $\|\cdot\|$ defined on $\mathbb{C}^n$ for the pseudospectrum of M and another norm $\|\cdot\|'$ defined on $\mathbb{C}^m$ $(m < n)$ for the pseudospectrum of $R(M;\mathcal{B})$. Our comparison of the pseudospectra of the original and reduced matrices assumes that for $\mathbf{v} = (\mathbf{v}_\mathcal{B}^T, \mathbf{v}_\mathcal{I}^T)^T \in \mathbb{C}^n$, these two norms are related by

$$\|\mathbf{v}\| = \left\|\begin{bmatrix}\mathbf{v}_\mathcal{B}\\ \mathbf{v}_\mathcal{I}\end{bmatrix}\right\| \geqslant \left\|\begin{bmatrix}\mathbf{v}_\mathcal{B}\\ 0\end{bmatrix}\right\| = \|\mathbf{v}_\mathcal{B}\|'. \tag{13}$$

Examples of norms satisfying property (13) are the p−norms for $1 \leqslant p \leqslant \infty$. For the sake of simplicity, we use the same notation for both of these $\mathbb{C}^n$ and $\mathbb{C}^m$ norms.

The following theorem describes how the ϵ-pseudospectrum of a matrix $M(\lambda)$ is related to the ϵ-pseudospectrum of the isospectral reduction $R_\lambda(M;\mathcal{B})$. It says that the ϵ-pseudospectra of the reduced matrix are contained in the ϵ-pseudospectra of the original matrix for each $\epsilon > 0$.

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Theorem 4.1
For $M(\lambda) \in \mathbb{W}_{\pi}^{n\times n}$ let $\mathcal{B} \subset N$. Then $\sigma_\epsilon(R(M;\mathcal{B})) \subseteq \sigma_\epsilon(M)$ for any $\epsilon > 0$ provided the $\mathbb{C}^n$ and $\mathbb{C}^{|\mathcal{B}|}$ norms in the pseudospectra definitions satisfy (13).

Proof
For $M(\lambda) \in \mathbb{W}_{\pi}^{n\times n}$ let $\mathcal{B}$ and $\mathcal{I}$ form a non-empty partition of N. We assume, without loss of generality, that for a vector $\mathbf{v} \in \mathbb{C}^n$, we have $\mathbf{v} = (\mathbf{v}_{\mathcal{B}}^T, \mathbf{v}_{\mathcal{I}}^T)^T$.

For $\tilde{\lambda}_0 \in \mathbb{C}$ and $\epsilon > 0$ suppose there is a unit vector $\mathbf{v}_{\mathcal{B}} \in \mathbb{C}^{|\mathcal{B}|}$ such that

$$\|(R(M;\mathcal{B}) - \tilde{\lambda}_0 I)\mathbf{v}_{\mathcal{B}}\| < \epsilon. \tag{14}$$

As $\sigma(M_{\mathcal{I}\mathcal{I}})$ and $\mathbb{C} - \mathrm{dom}\,(M)$ are finite sets, then by continuity there is a neighborhood U of $\tilde{\lambda}_0$ such that

(i) $M(\lambda) \in \mathbb{C}^{n\times n}$ for $\lambda \in U - \{\tilde{\lambda}_0\}$;
(ii) $\sigma(M_{\mathcal{I}\mathcal{I}}) \cap (U - \{\tilde{\lambda}_0\}) = \emptyset$; and
(iii) $\|(R(M;\mathcal{B}) - \lambda I)\mathbf{v}_{\mathcal{B}}\| < \epsilon$ for $\lambda \in U - \{\tilde{\lambda}_0\}$.

Observe that, for each $\lambda_0 \in U - \{\tilde{\lambda}_0\}$ it follows that the vector

$$\mathbf{v}_{\mathcal{I}} = -(M(\lambda_0)_{\mathcal{I}\mathcal{I}} - \lambda_0 I)^{-1} M(\lambda_0)_{\mathcal{I}\mathcal{B}}\mathbf{v}_{\mathcal{B}}$$

is defined. Let $\mathbf{v} = (\mathbf{v}_{\mathcal{B}}^T, \mathbf{v}_{\mathcal{I}}^T)^T$ and note that

$$\begin{aligned}
(M(\lambda_0) - \lambda_0 I)\mathbf{v} &= \begin{bmatrix} (M - \lambda I)_{\mathcal{B}\mathcal{B}}\mathbf{v}_{\mathcal{B}} + (M - \lambda I)_{\mathcal{B}\mathcal{I}}\mathbf{v}_{\mathcal{I}} \\ (M - \lambda I)_{\mathcal{I}\mathcal{B}}\mathbf{v}_{\mathcal{B}} + (M - \lambda I)_{\mathcal{I}\mathcal{I}}\mathbf{v}_{\mathcal{I}} \end{bmatrix}\Bigg|_{\lambda=\lambda_0} \\
&= \begin{bmatrix} M_{\mathcal{B}\mathcal{B}}\mathbf{v}_{\mathcal{B}} - M_{\mathcal{B}\mathcal{I}}(M_{\mathcal{I}\mathcal{I}} - \lambda I)^{-1} M_{\mathcal{I}\mathcal{B}}\mathbf{v}_{\mathcal{B}} \\ M_{\mathcal{I}\mathcal{B}}\mathbf{v}_{\mathcal{B}} - (M_{\mathcal{I}\mathcal{I}} - \lambda I)(M_{\mathcal{I}\mathcal{I}} - \lambda I)^{-1} M_{\mathcal{I}\mathcal{B}}\mathbf{v}_{\mathcal{B}} \end{bmatrix}\Bigg|_{\lambda=\lambda_0} \\
&= \begin{bmatrix} (R(M;\mathcal{B}) - \lambda I)\mathbf{v}_{\mathcal{B}} \\ 0 \end{bmatrix}\Bigg|_{\lambda=\lambda_0}.
\end{aligned}$$

By the property (13) of the norms in $\mathbb{C}^n$ and $\mathbb{C}^{|\mathcal{B}|}$ we must have

$$\|(M(\lambda_0) - \lambda_0 I)\mathbf{v}\| = \|(R(M(\lambda_0);\mathcal{B}) - \lambda_0 I)\mathbf{v}_{\mathcal{B}}\| < \epsilon. \tag{15}$$

As $\mathbf{v}_{\mathcal{B}} \neq \mathbf{0}$, consider the unit vector $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\| \in \mathbb{C}^n$. Again by (13), we have $\|\mathbf{v}\| \geqslant \|\mathbf{v}_{\mathcal{B}}\| = 1$. Hence, we obtain the bound

$$\|(M(\lambda_0) - \lambda_0 I)\mathbf{u}\| = \frac{\|(M(\lambda_0) - \lambda_0 I)\mathbf{v}\|}{\|\mathbf{v}\|} \leqslant \|(M(\lambda_0) - \lambda_0 I)\mathbf{v}\| < \epsilon,$$

where the last inequality comes from (15). This implies $\lambda_0 \in \sigma_\epsilon(M)$.

As this holds for any $\lambda_0 \in U - \{\tilde{\lambda}_0\}$ then $\tilde{\lambda}_0 \in cl(\sigma_\epsilon(M))$. Because $\sigma_\epsilon(M)$ is a closed set, then in fact $\tilde{\lambda}_0 \in \sigma_\epsilon(M)$. Because $\tilde{\lambda}_0$ is an arbitrary point in $\sigma_\epsilon(R(M;\mathcal{B}))$, the result follows by inequality (14). □

Remark 4.1
Let us contrast Theorem 4.1 with the more common projection onto an invariant subspace, which also preserves certain eigenvalues of a matrix and for which there is a similar pseudospectrum inclusion result. When $M \in \mathbb{C}^{n\times n}$ consider the projection Q^*MQ, where Q has orthonormal columns. In general $\sigma_\epsilon(Q^*MQ)$ is not a subset of $\sigma_\epsilon(M)$. However if range (Q) is an invariant subspace of M, we do have $\sigma_\epsilon(Q^*MQ) \subseteq \sigma_\epsilon(M)$ (see, e.g., [9, §40]). By Theorem 4.1, the pseudospectra of the reduced matrix are *always* included in those of the original matrix.

Example 4.1
In the mass-spring system of Example 2.4, we consider four different sets of boundary nodes $\{1,2,3,4\} \supset \{1,2,4\} \supset \{1,4\} \supset \{1\}$. Note that Theorem 4.1 implies that the corresponding pseudospectra for a given ϵ obey the same inclusions. This is shown in Figure 4 for $\epsilon = 1$, $10^{-1/2}$, and $\epsilon = 10^{-1}$.

In physical terms, this means that as we increase the number of internal degrees of freedom (or decrease the number of boundary nodes), it becomes harder to find frequencies for which there is a displacement that generates forces of magnitude below a certain fixed level. Hence, the fewer boundary nodes we have, the more robust to perturbations are the frequencies that generate small forces. Another way of looking at this result comes when we think of the mass-spring network as a plate and we assume we can induce time-harmonic vibrations at certain portions of the plate boundary. The more boundary portions we control, the more frequencies we will be able to find at which the forces are small. This phenomenon is also illustrated in Section 8.3 for a 2D truss.

Notice that the inclusion given in Theorem 4.1 is not a strict inclusion. In fact, it may be the case that a matrix M and its reduction $R(M;\mathcal{B})$ have the same pseudospectra as the following example demonstrates.

Example 4.2
Consider the matrix $M \in \mathbb{C}^{4\times 4}$ given by

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \text{ and its reduction } R(M;\mathcal{B}) = \begin{bmatrix} \frac{\lambda}{\lambda-1} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

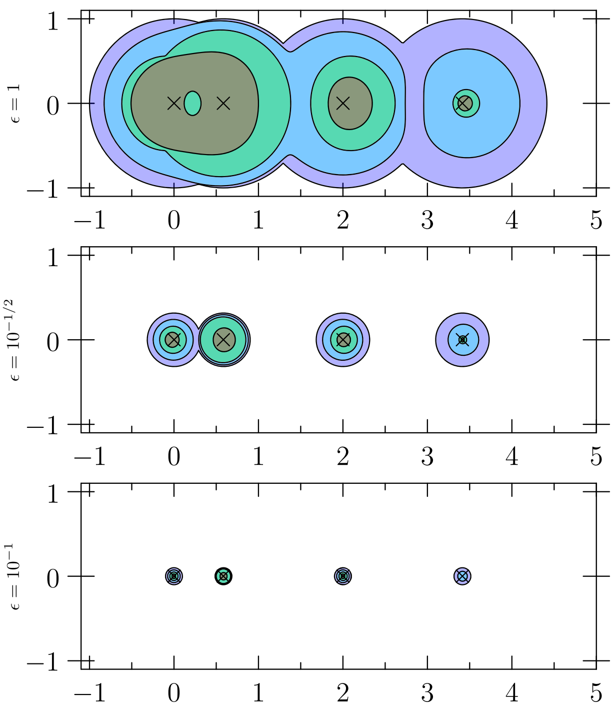

Figure 4. Pseudospectra of the matrix K from the mass-spring system of Example 2.4(blue) together with the pseudospectra for the reduced matrices where the terminal nodes are $\mathcal{B} = \{1,2,4\}$ (cyan), $\mathcal{B} = \{1,4\}$ (green), and $\mathcal{B} = \{1\}$ (red). Note how the pseudospectra shrink as the number of boundary nodes decreases.

where $\mathcal{B} = \{2, 3, 4\}$. Computing the Euclidean induced matrix norm of the resolvents, we obtain

$$\|(M - \lambda I)^{-1}\| = \max(|\lambda|^{-1}, |\lambda - 2|^{-1}) \text{ and}$$
$$\|(R(M; \mathcal{B}) - \lambda I)^{-1}\| = \max(|\lambda|^{-1}, |\lambda - 2|^{-1}, |\lambda - 1||\lambda|^{-1}|\lambda - 2|^{-1}).$$

To show that the pseudospectra of M and $R(M; \mathcal{B})$ are the same, we only need to demonstrate that the norms above are equal. This happens if we can show the inequality

$$|\lambda - 1||\lambda|^{-1}|\lambda - 2|^{-1} \leqslant \max(|\lambda|^{-1}, |\lambda - 2|^{-1}). \tag{16}$$

Notice that the triangle inequality implies

$$|\lambda - 1| \leqslant \frac{1}{2}|\lambda - 2| + \frac{1}{2}|\lambda| \leqslant \max(|\lambda|, |\lambda - 2|). \tag{17}$$

Inequality (16) follows for $\lambda \notin \{0, 2\}$ by dividing (17) by $|\lambda||\lambda - 2|$. As $\{0, 2\} \subset \sigma(M), \sigma(R(M; \mathcal{B}))$, then both 0 and 2 are included in the pseudospectra of these matrices. We conclude that $\sigma_\epsilon(M) = \sigma_\epsilon(R(M; \mathcal{B}))$ for all $\epsilon > 0$.

5. EXCHANGING EIGENVALUES AND INVERSE EIGENVALUES: THE SPECTRAL INVERSE

Although a matrix $M \in \mathbb{W}(\lambda)^{n \times n}$ has both a spectrum and an inverse spectrum, the techniques that have been developed to analyze its spectral properties have been restricted to its spectrum [1–3]. The goal in this section is to introduce a new matrix transformation that exchanges a matrix' spectrum and inverse spectrum. In Section 6, we use this transformation to define the inverse pseudospectrum (or pseudoresonances) of a matrix from the pseudospectrum of a matrix.

Definition 5.1
For $M(\lambda) \in \mathbb{W}^{n \times n}$ let $S_\lambda(M) \in \mathbb{W}^{n \times n}$ be the matrix

$$S_\lambda(M) = (M(\lambda) - \lambda I)^{-1} + \lambda I \in \mathbb{W}^{n \times n},$$

if the inverse $(M(\lambda) - \lambda I)^{-1}$ exists. The matrix $S_\lambda(M)$ is called the *spectral inverse* of the matrix $M(\lambda)$.

We typically write the spectral inverse of $M \in \mathbb{W}^{n \times n}$ as $S(M)$ unless otherwise needed. Observe that a necessary condition for a spectral inverse $S(M)$ to exist is that the matrix $M - \lambda I$ be invertible. Not every matrix $M \in \mathbb{W}^{n \times n}$ has a spectral inverse. For instance, the matrix

$$M = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

cannot be spectrally inverted. However, if M has a spectral inverse then the following holds.

Theorem 5.1
Suppose $M(\lambda) \in \mathbb{W}^{n \times n}$ has a spectral inverse $S(M)$. Then

$$\sigma\big(S(M)\big) = \sigma^{-1}(M) \text{ and } \sigma^{-1}\big(S(M)\big) = \sigma(M),$$

where the spectrum $\sigma(M)$ and inverse spectrum $\sigma^{-1}(M)$ are as in Definition 2.1.

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Proof
Let $M(\lambda) \in \mathbb{W}^{n\times n}$ with spectral inverse $S(M)$. Note that

$$\det\big((S(M) - \lambda I)(M - \lambda I)\big) = \det\big((M - \lambda I)^{-1}(M - \lambda I)\big) = \det(I) = 1.$$

As the determinant is multiplicative then

$$\det(S(M) - \lambda I) = \det(M - \lambda I)^{-1},$$

and the result follows. □

A matrix $M \in \mathbb{W}^{n\times n}$ may or may not have a spectral inverse. However, if $M \in \mathbb{W}_\pi^{n\times n}$ then the proof of Lemma 2.1 implies that $M - \lambda I$ is invertible. Therefore, $S(M)$ exists. This result is stated in the following lemma.

Lemma 5.1
If $M(\lambda) \in \mathbb{W}_\pi^{n\times n}$, then $M(\lambda)$ has a spectral inverse.

Example 5.1
Let $M \in \mathbb{W}_\pi^{4\times 4}$ be the matrix given by

$$M = \begin{bmatrix} \frac{1}{\lambda} & \frac{1}{\lambda} & 0 & 0 \\ 0 & \frac{1}{\lambda} & 1 & 0 \\ 0 & 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda} \end{bmatrix}$$

for which we have

$$\det\big(M(\lambda) - \lambda I\big) = \frac{\lambda^8 - 4\lambda^6 + 6\lambda^4 - 4\lambda^2 + 1}{\lambda^4}.$$

As one can calculate, the spectral inverse $S(M)$ is the matrix

$$S(M) = \begin{bmatrix} \frac{-\lambda}{\lambda^2-1} & \frac{-\lambda}{(\lambda^2-1)^2} & \frac{-\lambda^2}{(\lambda^2-1)^3} & \frac{-\lambda^3}{(\lambda^2-1)^4} \\ 0 & \frac{-\lambda}{\lambda^2-1} & \frac{-\lambda^2}{(\lambda^2-1)^2} & \frac{-\lambda^3}{(\lambda^2-1)^3} \\ 0 & 0 & \frac{-\lambda}{\lambda^2-1} & \frac{-\lambda^2}{(\lambda^2-1)^2} \\ 0 & 0 & 0 & \frac{-\lambda}{\lambda^2-1} \end{bmatrix} + \lambda I.$$

Taking the determinant of $S(M) - \lambda I$ one has

$$\det\big(S(M) - \lambda I\big) = \frac{\lambda^4}{\lambda^8 - 4\lambda^6 + 6\lambda^4 - 4\lambda^2 + 1}.$$

That is, $\det\big(S(M) - \lambda I\big) = \det(M(\lambda) - \lambda I)^{-1}$.

Observe, that for any $M \in \mathbb{W}_\pi^{n\times n}$ the spectral inverse $S(M) \notin \mathbb{W}_\pi^{n\times n}$. Therefore, we have no guarantee that $S(M)$ can be isospectrally reduced. However, the following holds.

Theorem 5.2 (Reductions of the Spectral Inverse)
For $M(\lambda) \in \mathbb{W}_\pi^{n\times n}$ suppose $\mathcal{B}$ and $\mathcal{I}$ form a non-empty partition of N. Then

(i) $R_\lambda\big(S(M); \mathcal{B}\big)$ exists; and
(ii) $R_\lambda(S(M); \mathcal{B}) = (M - \lambda I)^{-1} / \big[(M - \lambda I)^{-1}\big]_{\mathcal{I}\mathcal{I}} + \lambda I$.

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Proof
For $M \in \mathbb{W}_\pi^{n\times n}$ suppose $\mathcal{B}$ and $\mathcal{I}$ form a non-empty partition of N. By Lemmas 2.1 and 5.1, the matrix $S(M)$ exists and

$$S(M) - \lambda I = (M - \lambda I)^{-1} \in \mathbb{W}_\pi^{n\times n}.$$

Equating blocks in the previous equation implies that each of the matrices $[S(M)]_{\mathcal{BB}} - \lambda I$, $[S(M)]_{\mathcal{BI}}$, $[S(M)]_{\mathcal{IB}}$ and $[S(M)]_{\mathcal{II}} - \lambda I$ all have entries in $\mathbb{W}_\pi$. Moreover $[S(M)]_{\mathcal{II}} - \lambda I$ is not identically zero so its inverse exists. We deduce that the reduction of $S(M)$ exists and is given by

$$\begin{aligned} R_\lambda(S(M);\mathcal{B}) - \lambda I &= ([S(M)]_{\mathcal{BB}} - \lambda I) - [S(M)]_{\mathcal{BI}} \left([S(M)]_{\mathcal{II}} - \lambda I\right)^{-1} [S(M)]_{\mathcal{IB}} \\ &\in \mathbb{W}_\pi^{|\mathcal{B}|\times|\mathcal{B}|}. \end{aligned}$$

To prove (ii), simply notice that $[S(M)]_{\mathcal{BB}} - \lambda I = \left[(M-\lambda I)^{-1}\right]_{\mathcal{BB}}$, $[S(M)]_{\mathcal{IB}} = [S(M) - \lambda I]_{\mathcal{IB}} = \left[(M-\lambda I)^{-1}\right]_{\mathcal{IB}}$, $[S(M)]_{\mathcal{BI}} = \left[(M-\lambda I)^{-1}\right]_{\mathcal{BI}}$ and $[S(M)]_{\mathcal{II}} - \lambda I = \left[(M-\lambda I)^{-1}\right]_{\mathcal{II}}$. These relations imply (ii). □

Theorem 5.2 states that any matrix $M \in \mathbb{W}_\pi^{n\times n}$ has a spectral inverse and that this inverse can be reduced over any index set. Observe the similarity between (10) and part (ii) of Theorem 5.2.

6. PSEUDORESONANCES

Recall that the resonances of a matrix $M(\lambda) \in \mathbb{W}_\pi^{n\times n}$ are the eigenvalues of its spectral inverse. Thus we may think of 'almost resonances' or pseudoresonances of $M(\lambda)$ as pseudoeigenvalues of the spectral inverse $S(M)$. The precise definition is below, together with other equivalent definitions. These are analogous to the pseudospectra Definitions 3.2(a)–(c). Thus it is natural to ask whether we could have gotten the same information from the pseudospectra of the matrix M. We answer this question in Section 6.1: pseudoresonances do provide information that is not contained in the pseudospectrum.

Definition 6.1
Let $\epsilon > 0$. The set of $\epsilon-$pseudoresonances of a matrix $M(\lambda) \in \mathbb{W}_\pi^{n\times n}$ is defined equivalently by the following:

(a) *resonance perturbation:*

$$\sigma_\epsilon^{-1}(M) = cl\left(\{\lambda \in \mathbb{C} : \|(M(\lambda) - \lambda I)^{-1}\mathbf{v}\| < \epsilon \text{ for some } \mathbf{v} \in \mathbb{C}^n \text{ with } \|\mathbf{v}\| = 1\}\right);$$

(b) *the inverse resolvent:*

$$\sigma_\epsilon^{-1}(M) = cl\left(\{\lambda \in \mathbb{C} : \|M(\lambda) - \lambda I\| > \epsilon^{-1}\}\right); and$$

(c) *perturbation of the spectral inverse:*

$$\sigma_\epsilon^{-1}(M) = cl\left(\{\lambda \in \mathbb{C} : \lambda \in \sigma(S(M) + E) \text{ for some } E \in \mathbb{C}^{n\times n} \text{ with } \|E\| < \epsilon\}\right).$$

The matrix and vector norms in (a)–(c) are assumed to be consistent. The spectrum in part (c) is understood as the spectrum of the matrix $S(M) + E$, which is in $\mathbb{C}^{n\times n}$ for a *fixed* $\lambda \in \mathbb{C}$ (and in dom$(S(M))$).

Note that Definition 6.1 is simply Definition 3.2 in which $M(\lambda)$ is replaced by the matrix $S(M)$ on the right-hand side of parts (a)–(c). Hence, the equivalence of Definitions 6.1(a)–(c) follow from arguments similar those in Appendix D. Moreover, if $M(\lambda) \in \mathbb{W}_\pi^{n\times n}$ then

$$\sigma^{-1}(M) \subset \sigma_\epsilon^{-1}(M) \text{ for each } \epsilon > 0.$$

Observe that if $w(\lambda) = p(\lambda)/q(\lambda) \in \mathbb{W}_\pi$ then by definition $\pi(p) \leqslant \pi(q)$. Hence, we have the limit,

$$\lim_{|\lambda|\to\infty} |w(\lambda)| = c,$$

for some constant $c \geqslant 0$. Therefore, $\|M(\lambda) - \lambda I\| = \mathcal{O}(\lambda)$ for large λ, for matrices $M \in \mathbb{W}_\pi^{n\times n}$. This leads to the following remark.

Remark 6.1
If $M \in \mathbb{W}_\pi^{n\times n}$, then the value $\lambda = \infty$ is always a pseudoresonance. This means that for each $\epsilon > 0$ the set $\sigma_\epsilon^{-1}(M)$ contains the complement of a ball centered at the origin with sufficiently large radius. (See Figure 5 for example).

Example 6.1
In Figure 5, we show the pseudoresonance regions of the matrix $R(M;\{1,2\})$ from Example 2.2 for $\epsilon = 1, 10^{-1/2}, 10^{-1}$. As is shown in Example 2.3, the inverse spectrum of $R(M;\{1,2\})$ is empty. However, the pseudoresonance regions reveal that the eigenvalues $\sigma(M_{\mathcal{I}\mathcal{I}}) = \{0,1\}$ act as resonances, that is, $\sigma(M_{\mathcal{I}\mathcal{I}}) \subset \sigma_\epsilon^{-1}(R(M;\{1,2\}))$ holds for these specific values of ϵ.

In Figures 2 (left) and 5, note that for the ϵ, we consider

$$\sigma_\epsilon(R(M;\{1,2\})) \cap \sigma_\epsilon^{-1}(R(M;\{1,2\})) \neq \emptyset.$$

That is, values near the set $\sigma(M_{\mathcal{I}\mathcal{I}}) = \{0,1\}$ are both ϵ-pseudoeigenvalues and ϵ-pseudoresonances of $R(M;\{1,2\})$.

As it turns out, the situation in Example 6.1 does not hold for every matrix reduction. Similar to Example 3.4, if

$$M = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

and we consider the sets $\mathcal{B} = \{1\}$ and $\mathcal{I} = \{2\}$, then one can show the set $\sigma(M_{\mathcal{I}\mathcal{I}}) = \{0\}$ is not contained in $\sigma_\epsilon^{-1}(R(M;\mathcal{B}))$ for small $\epsilon > 0$. That is, the eigenvalues $\sigma(M_{\mathcal{I}\mathcal{I}})$ do not always act as resonances of $R(M;\mathcal{B})$.

As with the pseudospectra studied in Section 3.2, we give a physical interpretation of pseudoresonances using a mass-spring system.

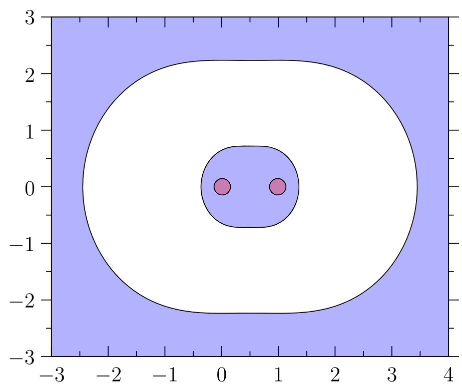

Figure 5. Pseudoresonance regions for the matrix $R(M;\{1,2\})$ given in Example 2.2, with $\epsilon = 10^{-1/2}$ (red) and $\epsilon = 10^{-1}$ (blue). All the points in the display region belong to the pseudoresonance region for $\epsilon = 1$.

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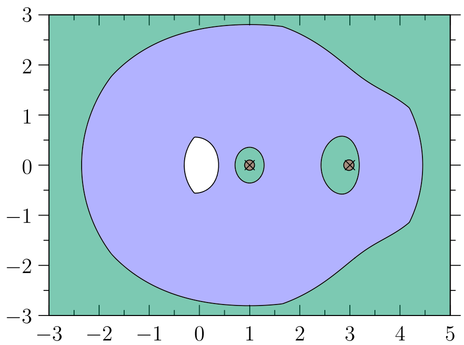

Figure 6. Pseudoresonance regions of the matrix $R_\lambda(K;\{1,4\})$ given in Example 2.4, for $\epsilon = 1$ (blue), $\epsilon = 10^{-1/2}$ (green), and $\epsilon = 10^{-1}$ (red). Resonances are shown with $\times$. All the points in the display region, the white region excepted, belong to the pseudoresonance region for $\epsilon = 1$.

Example 6.2
The mass-spring system considered in Example 2.4 has resonances when restricted to a set of boundary nodes $\mathcal{B} \subset \{1,4\}$. The pseudoresonances of the reduced system correspond to frequencies for which there is a displacement on the boundary that generates relatively large forces at these nodes. In Figure 6, we display some pseudoresonance regions of the mass-spring system restricted to the set $\mathcal{B} = \{1,4\}$.

As we allow ϵ to be any positive value, there is nothing preventing an eigenvalue of a matrix M from also being an ϵ-pseudoresonance of M (or a resonance from being a ϵ-pseudoeigenvalue). In other words, we could have an $\epsilon > 0$ for which

$$\sigma^{-1}(M) \cap \sigma_\epsilon(M) \neq \emptyset \text{ or } \sigma(M) \cap \sigma_\epsilon^{-1}(M) \neq \emptyset$$

as the following example shows.

Example 6.3
Consider the following matrix $M(\lambda) \in \mathbb{W}_\pi^{2\times 2}$ given by

$$M(\lambda) = \begin{bmatrix} \frac{1}{\lambda-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

The spectrum and inverse spectrum of $M(\lambda)$ are respectively

$$\sigma(M) = \{0, (1 \pm \sqrt{5})/2\} \text{ and } \sigma^{-1}(M) = \{1\}.$$

Now, notice that for $0 \in \sigma(M)$, we have

$$\|M(0) - 0I\| = 1,$$

which implies that $0 \in \sigma_\epsilon^{-1}(M)$ for all $\epsilon \geqslant 1$. The resolvent of M is

$$(M(\lambda) - \lambda I)^{-1} = \begin{bmatrix} \frac{\lambda-1}{-\lambda^2+\lambda+1} & 0 \\ 0 & -\frac{1}{\lambda} \end{bmatrix}.$$

Hence, for $\lambda = 1$, we have

$$\|(M(1) - I)^{-1}\| = 1,$$

which means that $1 \in \sigma_\epsilon(M)$ for all $\epsilon \geqslant 1$.

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6.1. Relation between pseudoresonances and pseudospectra

As the pseudoresonances of a matrix $M \in \mathbb{W}_\pi^{n\times n}$ can be defined in terms of the pseudoeigenvalues of the spectral inverse $S(M)$, we can generalize Theorem 5.1 as follows.

Theorem 6.1
Suppose $M(\lambda) \in \mathbb{W}_\pi^{n\times n}$ and $\epsilon > 0$. Then

$$\sigma_\epsilon^{-1}(M) = \sigma_\epsilon(S(M)) \text{ and } \sigma_\epsilon(M) = \sigma_\epsilon^{-1}(S(M)).$$

Proof
Let $M(\lambda) \in \mathbb{W}_\pi^{n\times n}$ and $\epsilon > 0$. Observe that,

$$\sigma_\epsilon(M) = cl\big(\{\lambda \in \mathbb{C} : \|(M(\lambda) - \lambda I)^{-1}\| > \epsilon^{-1}\}\big); \text{ and}$$
$$\sigma_\epsilon^{-1}(S(M)) = cl\big(\{\lambda \in \mathbb{C} : \|S(M) - \lambda I\| > \epsilon^{-1}\}\big)$$

from Definitions 3.2(b) and 6.1(b) respectively. Because $S(M) - \lambda I = (M(\lambda) - \lambda I)^{-1}$, then $\sigma_\epsilon(M) = \sigma_\epsilon^{-1}(S(M))$. The equality $\sigma_\epsilon^{-1}(M) = \sigma_\epsilon(S(M))$ follows similarly. □

Because of the seemingly invertible relationship between pseudospectra and inverse pseudospectra in Theorem 6.1, it is tempting to think that the $\epsilon-$pseudoresonances of a matrix is the complement of its $\epsilon^{-1}-$pseudospectrum. In general, however, the two are not equal as can be seen in the next theorem.

Theorem 6.2
For $M(\lambda) \in \mathbb{W}_\pi^{n\times n}$ let $\epsilon > 0$. Then, $cl\big(\mathbb{C} - \sigma_{1/\epsilon}(M)\big) \subseteq \sigma_\epsilon^{-1}(M)$. However, the reverse inclusion does not hold in general.

This theorem means that, in general, there is not enough information in the pseudospectra of a matrix to reconstruct its pseudoresonances. We now proceed with the proof of the theorem.

Proof
For $M(\lambda) \in \mathbb{W}_\pi^{n\times n}$ and a matrix norm $\|\cdot\|$, the inequality

$$\|M(\lambda) - \lambda I\|^{-1} \leqslant \|(M(\lambda) - \lambda I)^{-1}\| \tag{18}$$

holds for any $\lambda \in \text{dom}\,(M) - \sigma(M)$. Let $int(\Omega)$ denote the *interior* of the set $\Omega \subseteq \mathbb{C}$, that is, the largest open subset of Ω. For $\epsilon > 0$, using Definition 3.2(b),

$$\begin{aligned} cl\big(\mathbb{C} - \sigma_{1/\epsilon}(M)\big) &= cl\big(\mathbb{C} - cl(\{\lambda \in \mathbb{C} : \|(M(\lambda) - \lambda I)^{-1}\| > \epsilon\})\big) \\ &= cl\big(int(\{\lambda \in \mathbb{C} : \|(M(\lambda) - \lambda I)^{-1}\| \leqslant \epsilon\})\big) \\ &= cl\big(\{\lambda \in \mathbb{C} : \|(M(\lambda) - \lambda I)^{-1}\| \leqslant \epsilon\}\big) \end{aligned}$$

Similarly, it follows from Definition 6.1(b) that

$$\begin{aligned} \sigma_\epsilon^{-1}(M) &= cl\big(\{\lambda \in \mathbb{C} : \|M(\lambda) - \lambda I\| > \epsilon^{-1}\}\big) \\ &= cl\big(\{\lambda \in \mathbb{C} : \|M(\lambda) - \lambda I\|^{-1} \leqslant \epsilon\}\big). \end{aligned}$$

By inequality (18), the set

$$\{\lambda \in \mathbb{C} : \|(M(\lambda) - \lambda I)\|^{-1} \leqslant \epsilon\} \subseteq \{\lambda \in \mathbb{C} : \|(M(\lambda) - \lambda I)^{-1}\| \leqslant \epsilon\}$$

implying the first half of the result.

To show that the reverse inclusion does not hold in general, take for instance the matrix $M(\lambda)$ from Example 6.3. It is easy to compute $\|M(2) - 2I\| = 2$ and $\|(M(2) - 2I)^{-1}\| = 1$. Taking $\epsilon = 2/3$, we clearly have $2 \in \sigma_{2/3}^{-1}(M) \cap \sigma_{3/2}(M)$. □

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Remark 6.2
Theorem 4.1 states that the ϵ-pseudospectrum of a matrix becomes a subset of this region as the matrix is reduced. However, for ϵ-pseudoresonances of a matrix, there is no such inclusion result.

7. SEQUENTIAL REDUCTIONS

In the previous section, we observed that the isospectral reduction $R(M;\mathcal{B})$ of $M \in \mathbb{W}_\pi^{n\times n}$ is again a matrix in $\mathbb{W}_\pi^{m\times m}$. It is therefore possible to reduce the matrix $R(M;\mathcal{B})$ again over some subset of $\mathcal{B}$. That is, we may sequentially reduce the matrix M. However, a natural question is to what extent does a sequentially reduced matrix depends on the particular sequence of index sets over which it has been reduced.

As it turns out, if a matrix has been reduced over the index set $\mathcal{B}_1$ and then $\mathcal{B}_2$ up to the index set $\mathcal{B}_m$, then the resulting matrix depends only on the index set $\mathcal{B}_m$. To formalize this, let $M \in \mathbb{W}_\pi^{n\times n}$ and suppose there are non-empty sets $\mathcal{B}_1,\dots,\mathcal{B}_m$ such that $N \supset \mathcal{B}_1 \supset,\dots,\supset \mathcal{B}_m$. Then M can be *sequentially reduced* over the sets $\mathcal{B}_1,\dots,\mathcal{B}_m$ where we write

$$R_\lambda(M;\mathcal{B}_1,\dots,\mathcal{B}_m) = R_\lambda\big(\dots R_\lambda(R_\lambda(M;\mathcal{B}_1);\mathcal{B}_2)\dots;\mathcal{B}_m\big).$$

If M is sequentially reduced over the index sets $\mathcal{B}_1,\dots,\mathcal{B}_m$, we call $\mathcal{B}_m$ the *final index set* of this sequence of reductions.

Theorem 7.1 (Uniqueness of Sequential Reductions)
For $M(\lambda) \in \mathbb{W}_\pi^{n\times n}$ suppose $N \supset \mathcal{B}_1 \supset,\dots,\supset \mathcal{B}_m$ where $\mathcal{B}_m$ is non-empty. Then,

$$R_\lambda(M;\mathcal{B}_1,\dots,\mathcal{B}_m) = R_\lambda(M;\mathcal{B}_m).$$

That is, in a sequence of reductions the resulting matrix is completely specified by the final index set. To prove Theorem 7.1 we first require the following lemma.

Lemma 7.1
Let the non-empty sets $\mathcal{B}$, $\mathcal{I}$, and $\mathcal{J}$ partition N. If $M(\lambda) \in \mathbb{W}_\pi^{n\times n}$, then $R_\lambda(M;\mathcal{B}\cup\mathcal{I},\mathcal{B}) = R_\lambda(M;\mathcal{B})$.

Proof
Assume without loss of generality that $M \in \mathbb{W}_\pi^{n\times n}$ can be written as

$$M(\lambda) = \begin{bmatrix} M_{\mathcal{B}\mathcal{B}} & M_{\mathcal{B}\mathcal{I}} & M_{\mathcal{B}\mathcal{J}} \\ M_{\mathcal{I}\mathcal{B}} & M_{\mathcal{I}\mathcal{I}} & M_{\mathcal{I}\mathcal{J}} \\ M_{\mathcal{J}\mathcal{B}} & M_{\mathcal{J}\mathcal{I}} & M_{\mathcal{J}\mathcal{J}} \end{bmatrix}.$$

Using the definition of isospectral reduction, we have

$$R_\lambda(M;\mathcal{B}) = M_{\mathcal{B}\mathcal{B}} - \begin{bmatrix} M_{\mathcal{B}\mathcal{I}} & M_{\mathcal{B}\mathcal{J}} \end{bmatrix} \begin{bmatrix} M_{\mathcal{I}\mathcal{I}} - \lambda I & M_{\mathcal{I}\mathcal{J}} \\ M_{\mathcal{J}\mathcal{I}} & M_{\mathcal{J}\mathcal{J}} - \lambda I \end{bmatrix}^{-1} \begin{bmatrix} M_{\mathcal{I}\mathcal{B}} \\ M_{\mathcal{J}\mathcal{B}} \end{bmatrix} \text{ and} \tag{19}$$

$$R_\lambda(M;\mathcal{B}\cup\mathcal{I}) = \begin{bmatrix} M_{\mathcal{B}\mathcal{B}} & M_{\mathcal{B}\mathcal{I}} \\ M_{\mathcal{I}\mathcal{B}} & M_{\mathcal{I}\mathcal{I}} \end{bmatrix} - \begin{bmatrix} M_{\mathcal{B}\mathcal{J}} \\ M_{\mathcal{I}\mathcal{J}} \end{bmatrix} (M_{\mathcal{J}\mathcal{J}} - \lambda I)^{-1} \begin{bmatrix} M_{\mathcal{J}\mathcal{B}} & M_{\mathcal{J}\mathcal{I}} \end{bmatrix}. \tag{20}$$

Taking the isospectral reduction of $R_\lambda(M;\mathcal{B}\cup\mathcal{I})$ over $\mathcal{B}$ in (20), we have

$$\begin{aligned} R_\lambda(M;\mathcal{B}\cup\mathcal{I},\mathcal{B}) &= M_{\mathcal{B}\mathcal{B}} - M_{\mathcal{B}\mathcal{J}}K(\lambda)^{-1}M_{\mathcal{J}\mathcal{B}} \\ &\quad - \left[(M_{\mathcal{B}\mathcal{I}} - M_{\mathcal{B}\mathcal{J}}K(\lambda)^{-1}M_{\mathcal{J}\mathcal{I}})T(\lambda)^{-1}(M_{\mathcal{I}\mathcal{B}} - M_{\mathcal{I}\mathcal{J}}K(\lambda)^{-1}M_{\mathcal{J}\mathcal{B}})\right], \end{aligned} \tag{21}$$

where $K(\lambda) \equiv M_{\mathcal{J}\mathcal{J}} - \lambda I$ and $T(\lambda) \equiv M_{\mathcal{I}\mathcal{I}} - \lambda I - M_{\mathcal{I}\mathcal{J}}K(\lambda)^{-1}M_{\mathcal{J}\mathcal{I}}$. Note that both $K(\lambda)^{-1}$ and $T(\lambda)^{-1}$ exist following the proof of Lemma 2.1. To show the desired result we need to verify that expressions (19) and (21) are equal.

Recall the following identity for the inverse of a square matrix M with 2×2 blocks:

$$M^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} E^{-1} & -E^{-1}BD^{-1} \\ -D^{-1}CE^{-1} & D^{-1} + D^{-1}CE^{-1}BD^{-1} \end{bmatrix}, \tag{22}$$

where $E = A - BD^{-1}C$ is the Schur complement of D in M. The determinantal identity (A2) implies that M is invertible if and only if D and E are invertible. Using (22) to find the inverse of the 2×2 block matrix appearing in (19), we obtain

$$\begin{bmatrix} M_{\mathcal{I}\mathcal{I}} - \lambda I & M_{\mathcal{I}\mathcal{J}} \\ M_{\mathcal{J}\mathcal{I}} & M_{\mathcal{J}\mathcal{J}} - \lambda I \end{bmatrix}^{-1} = \tag{23}$$

$$\begin{bmatrix} T(\lambda)^{-1} & -T(\lambda)^{-1}M_{\mathcal{I}\mathcal{J}}K(\lambda)^{-1} \\ -K(\lambda)^{-1}M_{\mathcal{J}\mathcal{I}}T(\lambda)^{-1} & K(\lambda)^{-1} + K(\lambda)^{-1}M_{\mathcal{J}\mathcal{I}}T(\lambda)^{-1}M_{\mathcal{I}\mathcal{J}}K(\lambda)^{-1} \end{bmatrix}.$$

Using (23) in (19), we obtain (21), completing the proof. □

We now give a proof of Theorem 7.1.

Proof
For $M \in \mathbb{W}_{\pi}^{n\times n}$ suppose $N \supset \mathcal{B}_1 \supset \cdots \supset \mathcal{B}_m$ where $\mathcal{B}_m \neq \emptyset$. If $m = 2$, then Lemma 7.1 directly implies that $R_\lambda(M;\mathcal{B}_1,\mathcal{B}_2) = R_\lambda(M;\mathcal{B}_2)$. For $2 \leqslant k < m$ suppose $R_\lambda(M;\mathcal{B}_1,\ldots,\mathcal{B}_k) = R_\lambda(M;\mathcal{B}_k)$. Then

$$R_\lambda(M;\mathcal{B}_1,\ldots,\mathcal{B}_k,\mathcal{B}_{k+1}) = R_\lambda(M;\mathcal{B}_k,\mathcal{B}_{k+1}) = R_\lambda(M;\mathcal{B}_{k+1})$$

where the second equality follows from Lemma 7.1. By induction it then follows that $R_\lambda(M;\mathcal{B}_1,\ldots,\mathcal{B}_m) = R_\lambda(M;\mathcal{B}_m)$. □

Example 7.1
Let $M \in \mathbb{C}^{4\times 4}$ be the matrix given by

$$M = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

and let $\mathcal{B} = \{1,2\}$. Our goal in this example is to illustrate that

$$R_\lambda(M;\mathcal{B}) = R_\lambda(M;\mathcal{B}\cup\{3\},\mathcal{B}) = R_\lambda(M;\mathcal{B}\cup\{4\},\mathcal{B}).$$

As one can compute

$$R_\lambda(M;\mathcal{B}\cup\{3\}) = \begin{bmatrix} 1 & 0 & 1 \\ \frac{1}{\lambda-1} & 1 & \frac{1}{\lambda-1} \\ \frac{1}{\lambda-1} & 1 & \frac{\lambda}{\lambda-1} \end{bmatrix} \text{ and } R_\lambda(M;\mathcal{B}\cup\{4\}) = \begin{bmatrix} 1 & \frac{1}{\lambda-1} & \frac{1}{\lambda-1} \\ 0 & 1 & 1 \\ 1 & \frac{1}{\lambda-1} & \frac{\lambda}{\lambda-1} \end{bmatrix}.$$

Although $R_\lambda(M;\mathcal{B}\cup\{3\}) \neq R_\lambda(M;\mathcal{B}\cup\{4\})$, note that by reducing both of these matrices over $\mathcal{B} = \{1,2\}$ one has

$$R_\lambda(M;B) = R_\lambda(M;\mathcal{B}\cup\{3\},\mathcal{B}) = R_\lambda(M;\mathcal{B}\cup\{4\},\mathcal{B}) = \begin{bmatrix} \frac{\lambda^2-2\lambda+1}{\lambda^2-2\lambda} & \frac{\lambda-1}{\lambda^2-2\lambda} \\ \frac{\lambda-1}{\lambda^2-2\lambda} & \frac{\lambda^2-2\lambda+1}{\lambda^2-2\lambda} \end{bmatrix}.$$

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As a final observation, we note that $\sigma(M) = \{\frac{1}{2}(3 \pm \sqrt{5}), \frac{1}{2}(1 \pm \sqrt{-3})\}$ and $\sigma(M_{II}) = \{0, 2\}$ for $I = \{3, 4\}$. Hence, the matrix M and the reduced matrix $R(M, \mathcal{B})$ have the same eigenvalues by Corollary 2.1. That is, an isospectral reduction need not have any effect on the spectrum of a matrix. (In this example the inverse spectrum does change with the reduction).

As the spectral inverse $S(M) \notin \mathbb{W}_\pi^{n\times n}$ even if $M \in \mathbb{W}_\pi^{n\times n}$ then Theorem 7.1 does not imply that $S(M)$ can be sequentially reduced. However, the following corollary states that this is in fact the case.

Corollary 7.2
For $M(\lambda) \in \mathbb{W}_\pi^{n\times n}$ suppose $N \supset \mathcal{B}_1 \supset, \ldots, \supset \mathcal{B}_m$ where $\mathcal{B}_m$ is non-empty. Then $R_\lambda(S(M); \mathcal{B}_1, \ldots, \mathcal{B}_m) = R_\lambda(S(M); \mathcal{B}_m)$.

Proof
Substituting each submatrix $M_{\mathcal{RC}}$ in the proof of Lemma 7.1 by the matrix

$$S(M)_{\mathcal{RC}} = \begin{cases} (M - \lambda I)^{-1}_{\mathcal{RC}} + \lambda I & \text{if } \mathcal{R} = \mathcal{C}, \\ (M - \lambda I)^{-1}_{\mathcal{RC}} & \text{otherwise} \end{cases}$$

and then following the proof of Theorem 7.1 using $S(M)$ instead of M yields a proof the result. □

8. COMPUTATION OF PSEUDOSPECTRA AND PSEUDORESONANCES

8.1. Pseudospectra

The pseudospectra of a matrix $M(\lambda) \in \mathbb{W}_\pi^{n\times n}$ can be computed by using the resolvent version (part (c)) of Definition 3.2. Indeed it suffices to compute the norm of the resolvent $\|(M(\lambda) - \lambda I)^{-1}\|$ for values of λ on a uniform grid of a region of $\mathbb{C}$. The pseudospectra for different ϵ can then be visualized with a contour routine. We use the Euclidean norm in all of our examples. With this particular norm choice, the norm of the inverse of a matrix (induced by the Euclidean norm) can be quickly computed as the reciprocal of the smallest singular value of the matrix. Thus to plot the pseudospectra we avoid computing the inverse of $M(\lambda) - \lambda I$ and plot instead the reciprocal of the smallest singular value of $M(\lambda) - \lambda I$. With other norm choices (that are not induced by an inner product), the inverse must be computed explicitly.

In either case, the complexity of computing the pseudospectrum in a grid with m^2 points is $\mathcal{O}(m^2 n^3)$ operations, assuming that evaluating $M(\lambda)$ does not take more than $\mathcal{O}(n^3)$ operations. This is comparable to the complexity of computing the pseudospectra of a $\mathbb{C}^{n\times n}$ matrix using the naive approach. There are several optimizations that can be used in the $\mathbb{C}^{n\times n}$ case. One of them is to not compute the full SVD of $M(\lambda) - \lambda I$, but just compute the smallest singular value via inverse iteration or inverse Lanczos (see, e.g., [9, §39]). The other optimization is pre-computing the Schur factorization of M to speed up the algorithm that finds the smallest eigenvalue [16]). Because in our case, $M(\lambda) \in \mathbb{W}_\pi^{n\times n}$ usually changes with the spectral parameter λ, this approach does not appear viable.

In our complexity estimation, we have made the assumption that computing $M(\lambda)$ for one value of λ takes $\mathcal{O}(n^3)$ operations. If $M(\lambda)$ is an isospectral reduction $R_\lambda(A; \mathcal{B})$, with $A \in \mathbb{W}^{n\times n}$ and $|\mathcal{B}| = b$, then each evaluation of $M(\lambda)$ costs as much as computing a Schur complement, that is, $\mathcal{O}(b^3 + (n-b)^3)$ operations. It may be possible to reduce the complexity of these computations, but it is out of the scope of this paper.

We now show a simple MATLAB code to compute pseudospectra for matrices in $\mathbb{W}_\pi^{n\times n}$, based on the 'basic SVD algorithm' [9, §39]. Let us represent the matrix $M(\lambda)$ by a function handle `M` and let `x`, `y` be vectors of length m containing the real and imaginary parts of points in the region of interest, for example, in MATLAB `x = linspace(xmin,xmax,m);`. The pseudospectrum computation can be written in a few lines of MATLAB as follows.

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```
sigmin = @(A,Z) arrayfun(@(z) min(svd(A(z)-z*eye(n))),Z);
[X,Y]=meshgrid(x,y); Z = X + 1i*Y;
contour(x,y,log10(sigmin(M,Z)));
```

Although this is an embarrassingly parallel computation, we did not parallelize the computations in the examples we show. We did use the inverse iteration instead of computing the full SVD. Although the code above is short, it may not be very efficient, as the overhead from using anonymous functions and `arrayfun` instead of a loop may be significant.

8.2. *Pseudoresonances*

To compute the pseudoresonances, we may use Definition 6.1 part (b) that involves $\|M(\lambda) - \lambda I\|$. If the Euclidean norm is used, the pseudospectrum computation code can be modified to compute pseudoresonances by simply calculating the largest singular value of $M(\lambda) - \lambda I$ instead of the smallest and taking its reciprocal (a minus sign when plotting the log-contours). The complexity of this operation is again $\mathcal{O}(m^2 n^3)$ and it is again an embarrassingly parallel computation. The computation can also be sped up by using the power method instead of finding the SVD of $M(\lambda) - \lambda I$. Below is a pseudoresonance computation using the SVD in a few lines of MATLAB (with no claims on its efficiency). Of course, saving the maximum and minimum singular values of $M(\lambda) - \lambda I$ from a single call to `svd` (per value of λ) can give both the pseudospectra and pseudoresonances with essentially the same cost as the code below.

```
sigmax = @(A,Z) arrayfun(@(z) max(svd(A(z)-z*eye(n))),Z);
[X,Y]=meshgrid(x,y); Z = X + 1i*Y;
contour(x,y,-log10(sigmax(M,Z)));
```

8.3. *A 2D truss*

Consider the truss with nodes (ih, jh), $i, j = 0, \dots, n$ and $h = 1/n$. Each node is endowed with a unit mass and two nodes are linked together if their x (or y) coordinates differ by h (Figure 7). The stiffness matrix K is the $2n^2 \times 2n^2$ matrix, defined by (3). Such a truss could be used to model the vibrations of an isotropic and homogeneous square plate. Note that there are two degrees of freedom per node (because the displacement is a vector in ^{2}). Let $\mathcal{B}_1$ denote the degrees of freedom corresponding to the nodes on the four edges of the plate ($|\mathcal{B}_1| = 2(4n-4) = 8(n-1)$) and let $\mathcal{B}_2$ be the degrees of freedom corresponding to the four corners of the plate ($|\mathcal{B}_2| = 8$).

We present in Figure 8 the pseudospectra and pseudoresonances for different reductions of the stiffness matrix K with $n = 10$. The reductions correspond to two different situations. The first is when only the edges of the plate can be controlled (i.e., $\mathcal{B}_1$) and when only the four corners can be controlled (i.e., $\mathcal{B}_2$). Because $\mathcal{B}_2 \subset \mathcal{B}_1$, Theorem 4.1 guarantees that $\sigma_\epsilon(R(K, \mathcal{B}_2)) \subset \sigma_\epsilon(R(K, \mathcal{B}_1))$, and this can be observed in Figure 8. There is no such inclusion result for the pseudoresonances.

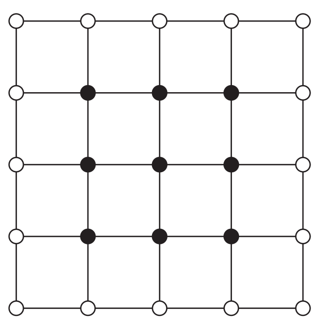

Figure 7. A $n \times n$ 2D truss with $n = 5$. Each edge corresponds to a spring. The white nodes correspond to nodes on the 'boundary' of the plate that this truss models.

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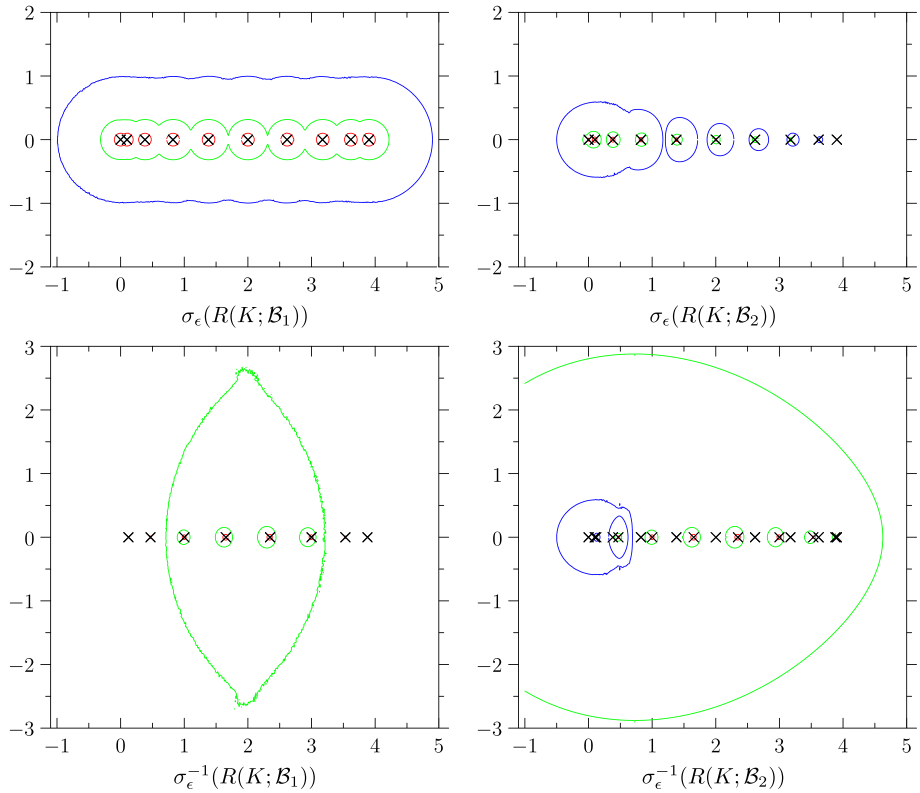

Figure 8. Pseudospectra (top row) and pseudoresonances (bottom row) of the response of a 2D truss with 10×10 nodes at the edges (left column) and at the four corners (right column). The pseudospectra and pseudoresonances are for $\epsilon = 1$ (blue), $\epsilon = 10^{-1/2}$ (green), and $\epsilon = 10^{-1}$ (red) and were obtained with the matrix 2-norm. For the pseudospectra plots, the '$\times$' are $\sigma(K)$. For the pseudoresonance plots, the '$\times$' indicate the spectra of the matrices that are eliminated (i.e., the blocks corresponding to the complement of $\mathcal{B}_i$). Note that by Theorem 2.1, $\sigma^{-1}(R(K;\mathcal{B}_i))$ is only a subset of these eigenvalues.

To give an idea of the complexity of the computations presented in Figure 8, the stiffness matrix K is a 200×200 matrix. To compute $R(K,\mathcal{B}_1)$ (resp. $R(K,\mathcal{B}_2)$) for one value of λ, one needs 72 (resp. 8) solves of a 128×128 (resp. 192×192) system. This is by no means a computationally hard problem, however $R(K,\mathcal{B}_i)$ needs to be generated for all values of λ in a grid. In our case the grid was 500×200, so this operation (plus computing the largest and smallest singular values of $R(K,\mathcal{B}_i) - \lambda I$) needs to be carried out 10,000 times. We did use the power method and the inverse iteration to avoid computing the full SVD, but this seems to introduce some artifacts in the pseudoresonance plots.

9. CONCLUSION

Isospectral reductions were introduced in [1, 3] to simplify a graph's structure while preserving its spectral properties. The classical eigenvalue estimates of Gershgorin, Brauer, Brualdi, and Varga [17–20] can also be improved using isospectral reductions as is shown in [2]. However, the isospectral reductions in [1–3] are limited to reductions over principal submatrices of a specific form. One of the main advances in this paper is to lift this restriction, allowing rational function valued matrices to be reduced over any principal submatrix regardless of its structure. The Schur-complement-based framework that we use here is simpler and computationally more efficient than that used in [1–3].

The other main advance is extending pseudospectra to a wide class of rational function valued matrices. Because such matrices have inverse eigenvalues, we also introduce the notion of inverse pseudospectrum. Moreover, we are able to show that the pseudospectrum of a matrix shrinks under reduction. Therefore, the eigenvalues of a reduced matrix are less susceptible to perturbations than those of the unreduced matrix.

Our theoretical results are illustrated by a mass-spring network. Our pseudospectrum inclusion result means that for such networks, the eigenvalues of the displacement to forces matrix are less susceptible to perturbations when fewer nodes are accessible.

APPENDIX A: PROOF OF THEOREM 2.1 ON THE SPECTRUM AND INVERSE SPECTRUM OF ISOSPECTRAL REDUCTIONS

Proof

For $M \in \mathbb{W}^{n\times n}$, we may assume without loss of generality that M has the block matrix form

$$M = \begin{bmatrix} M_{\mathcal{I}\mathcal{I}} & M_{\mathcal{I}\mathcal{B}} \\ M_{\mathcal{B}\mathcal{I}} & M_{\mathcal{B}\mathcal{B}} \end{bmatrix} \tag{A1}$$

where $M_{\mathcal{I}\mathcal{I}} - \lambda I$ is invertible.

Note that the determinant of a matrix and that of its Schur complement are related by the identity

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A) \cdot \det(D - CA^{-1}B), \tag{A2}$$

provided the submatrix A is invertible. Using this identity on the matrix $M - \lambda I$ yields

$$\det(M - \lambda I) = \det(M_{\mathcal{I}\mathcal{I}} - \lambda I) \cdot \det\big((M_{\mathcal{B}\mathcal{B}} - \lambda I) - M_{\mathcal{B}\mathcal{I}}(M_{\mathcal{I}\mathcal{I}} - \lambda I)^{-1} M_{\mathcal{I}\mathcal{B}}\big).$$

Therefore,

$$\det\big(R(M;\mathcal{B}) - \lambda I\big) = \frac{\det(M - \lambda I)}{\det(M_{\mathcal{I}\mathcal{I}} - \lambda I)}.$$

To compare the eigenvalues of $R(M;\mathcal{B})$, M, and $M_{\mathcal{I}\mathcal{I}}$, write

$$\det(M - \lambda I) = \frac{p(\lambda)}{q(\lambda)} \quad \text{and} \quad \det(M_{\mathcal{I}\mathcal{I}} - \lambda I) = \frac{t(\lambda)}{u(\lambda)},$$

for some $p/q, t/u \in \mathbb{W}$. Hence,

$$\det\big(R(M;\mathcal{B}) - \lambda I\big) = \frac{p(\lambda)u(\lambda)}{q(\lambda)t(\lambda)}.$$

Let $P = \{\lambda \in \mathbb{C} : p(\lambda) = 0\}$, $Q = \{\lambda \in \mathbb{C} : q(\lambda) = 0\}$, $T = \{\lambda \in \mathbb{C} : t(\lambda) = 0\}$, and $U = \{\lambda \in \mathbb{C} : u(\lambda) = 0\}$, with multiplicities. By canceling common linear factors, Definition 2.1 implies

$$\begin{aligned} \sigma\big(R(M;\mathcal{B})\big) &= \{\lambda \in \mathbb{C} : p(\lambda)u(\lambda) = 0\} - \{\lambda \in \mathbb{C} : q(\lambda)t(\lambda) = 0\} \\ &= (P \cup U) - (Q \cup T); \text{ and} \\ \sigma^{-1}\big(R(M;\mathcal{B})\big) &= \{\lambda \in \mathbb{C} : q(\lambda)t(\lambda) = 0\} - \{\lambda \in \mathbb{C} : p(\lambda)u(\lambda) = 0\} \\ &= (Q \cup T) - (P \cup U). \end{aligned}$$

Because $P = \sigma(M)$, $Q = \sigma^{-1}(M)$, $T = \sigma^{-1}(M_{\mathcal{I}\mathcal{I}})$, and $R = \sigma(M_{\mathcal{I}\mathcal{I}})$, the result follows. □

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APPENDIX B: PROPERTIES OF THE DEGREE OF A RATIONAL FUNCTION

Suppose $w_i = p_i(\lambda)/q_i(\lambda)$ where $p_i(\lambda), q_i(\lambda) \in \mathbb{C}[\lambda]$ and $q_i(\lambda)$ is non-zero for $1 \leqslant i \leqslant n$. Then, for $1 \leqslant i, j \leqslant n$, it is easy to show the following properties hold:

$$\pi\Big(\sum_{i=1}^{n} w_i\Big) = \max_{1\leqslant i\leqslant n} \big\{\pi(w_i) : w_i \neq 0\big\}; \tag{B1}$$

$$\pi\Big(\prod_{i=1}^{n} w_i\Big) = \begin{cases} \sum_{i=1}^{n} \pi(w_i) & \text{if } \forall i \in \{1, \cdots, n\}\ w_i \neq 0 \\ 0 & \text{otherwise;} \end{cases} \tag{B2}$$

$$\pi\big(w_i/w_j\big) = \begin{cases} \pi(w_i) - \pi(w_j) & \text{if } w_i \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } w_j \neq 0; \text{ and} \tag{B3}$$

$$\pi(w_i - \lambda) = 1 \ \text{ for } \ w_i \in \mathbb{W}_\pi. \tag{B4}$$

APPENDIX C: PROOF OF LEMMA 2.1 ON THE EXISTENCE OF ISOSPECTRAL REDUCTIONS

Proof
Let $M \in \mathbb{W}_\pi^{n\times n}$. The inverse of the matrix $M - \lambda I$ is given by

$$(M - \lambda I)^{-1} = \frac{1}{\det(M - \lambda I)} adj(M - \lambda I) \tag{C1}$$

where $adj(M - \lambda I)$ is the adjugate matrix of $M - \lambda I$, that is, the matrix with entries

$$adj(M - \lambda I)_{ij} = (-1)^{i+j} \det(\mathcal{M}_{ji}),\ 1 \leqslant i, j \leqslant n, \tag{C2}$$

where $\mathcal{M}_{ij} \in \mathbb{W}^{(n-1)\times(n-1)}$ is obtained by deleting the i-th row and j-th column of $M - \lambda I$.

Note that

$$\det(M - \lambda I) = \sum_{\rho\in\mathcal{P}_n} \left(sgn(\rho) \prod_{i=1}^{n} (M - \lambda I)_{i,\rho(i)}\right) \tag{C3}$$

where the sum is taken over the set $\mathcal{P}_n$ of permutations on N. The sign $sgn(\rho)$ of the permutation $\rho \in \mathcal{P}_n$ is 1 (resp. -1) if ρ is the composition of an even (resp. odd) number of permutations of two elements.

Using equations (B2) and (B4) in Appendix B, the term in (C3) corresponding to the identity permutation $\rho = id \in \mathcal{P}_n$ has degree n while for $\rho \neq id$ the other terms have degree strictly smaller than n. Equation (B1) then implies

$$\pi\big(\det(M - \lambda I)\big) = n. \tag{C4}$$

Therefore, $\det(M - \lambda I)$ is not identically zero, implying via Equation (C1) that the inverse $(M - \lambda I)^{-1}$ exists. Similarly, for $i \in N$, the matrix $\mathcal{M}_{ii}$ is equal to $\widetilde{\mathcal{M}}_{ii} - \lambda I$ for some $\widetilde{\mathcal{M}} \in \mathbb{W}_\pi^{(n-1)\times(n-1)}$. Hence,

$$\pi\big(\det(\mathcal{M}_{ii})\big) = n - 1,\ \text{for } i \in N. \tag{C5}$$

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For $i \neq j$, the matrices $\mathcal{M}_{ij} \in \mathbb{W}^{(n-1)\times(n-1)}$ contain $n-2$ entries of the form $M_{k\ell} - \lambda$ where all other entries of $\mathcal{M}_{ij}$ belong to the set $\mathbb{W}_\pi$. Hence, equations (B2) and (B4) imply that for $i \neq j$

$$\pi\left(\det(\mathcal{M}_{ij})\right) \leqslant n-2, \text{ for } i, j \in N \tag{C6}$$

because for $\rho \in \mathcal{P}_{n-1}$, at most $n-2$ terms in the product $\prod_{k=1}^{n-1} (\mathcal{M}_{ij})_{k,\rho(k)}$ have the form $M_{k\ell} - \lambda$.

Given that the degree of $\det(\mathcal{M}_{ij})$ in (C6) may be zero, Equations (C4)–(C6) together with (B3) imply that $\pi((M-\lambda I)^{-1}_{ij}) \leqslant 0$ for all $1 \leqslant i, j \leqslant n$. Hence, $(M-\lambda I)^{-1} \in \mathbb{W}_\pi^{n\times n}$. Therefore, if $\mathcal{B}$ and $\mathcal{I}$ form a non-empty partition of N, then

$$\left[(M-\lambda I)^{-1}\right]_{\mathcal{I}\mathcal{I}} \in \mathbb{W}_\pi^{|\mathcal{I}|\times|\mathcal{I}|}.$$

Definition 2.2 along with equations (B2) and (B4) then imply that $R(M;\mathcal{B})$ has entries in $\mathbb{W}_\pi$. □

APPENDIX D: EIGENVALUE INCLUSIONS AND EQUIVALENCE OF DEFINITIONS 3.2–6.1

Here, we first show that the three pseudoeigenvalue regions in Definition 3.2(a)–(c) are equivalent and include the eigenvalues of the matrix. The proof relies on the fact that the sets

(a) $\sigma_\epsilon(M) = \{\lambda \in \mathbb{C} : \|(M-\lambda I)\mathbf{v}\| < \epsilon \text{ for some } \mathbf{v} \in \mathbb{C}^n \text{ with } \|\mathbf{v}\| = 1\}$;
(b) $\sigma_\epsilon(M) = \{\lambda \in \mathbb{C} : \|(M-\lambda I)^{-1}\| > \epsilon^{-1}\} \cup \sigma(M)$; and
(c) $\sigma_\epsilon(M) = \{\lambda \in \mathbb{C} : \lambda \in \sigma(M+E) \text{ for some } E \in \mathbb{C}^{n\times n} \text{ with } \|E\| < \epsilon\}$.

given in Definition 3.2 are equivalent for any $M \in \mathbb{C}^{n\times n}$ and $\epsilon > 0$.

Theorem D.1
Let $M(\lambda) \in \mathbb{W}_\pi^{n\times n}$ and $\epsilon > 0$. Then, Definition 3.2 parts (a)–(c) are equivalent. Moreover, $\sigma(M) \subset \sigma_\epsilon(M)$.

Proof
For $M(\lambda) \in \mathbb{W}_\pi^{n\times n}$ and $\epsilon > 0$, let

(a) $\sigma_{\epsilon,a}(M) = \{\lambda \in \mathbb{C} : \|(M(\lambda)-\lambda I)\mathbf{v}\| < \epsilon \text{ for some } \mathbf{v} \in \mathbb{C}^n \text{ with } \|\mathbf{v}\| = 1\}$;
(b) $\sigma_{\epsilon,b}(M) = \{\lambda \in \mathbb{C} : \|(M(\lambda)-\lambda I)^{-1}\| > \epsilon^{-1}\} \cup \sigma(M)$; and
(c) $\sigma_{\epsilon,c}(M) = \{\lambda \in \mathbb{C} : \lambda \in \sigma(M+E) \text{ for some } E \in \mathbb{C}^{n\times n} \text{ with } \|E\| < \epsilon\}$.

Suppose $\lambda_0 \in \sigma(M)$. Then, there is a unit vector $\mathbf{v} \in \mathbb{C}^n$ such that

$$(M(\lambda) - \lambda I)\mathbf{v} = w(\lambda) \in \mathbb{W}_\pi^n$$

where $w(\lambda_0) = \mathbf{0}$. Because $\sigma(M)$ and $\mathbb{C} - \text{dom}(M)$ are finite, then there is a neighborhood $U \ni \lambda_0$ such that for $\tilde{U} = U - \{\lambda_0\}$:

(i) $\tilde{U} \subset \text{dom}\,(M)$;
(ii) $\|w(\lambda)\| < \epsilon$ for $\lambda \in \tilde{U}$; and
(iii) $(\sigma(M) - \{\lambda_0\}) \cap \tilde{U} = \emptyset$.

In particular, (ii) implies the set $\tilde{U} \subset \sigma_{\epsilon,a}(M)$.

For $\lambda \in \text{dom}\,(M)$ observe that the matrix $M(\lambda) \in \mathbb{C}^{n\times n}$. Because (a)–(c) are equivalent for any complex valued matrix, then

$$\tilde{U} \subset \sigma_{\epsilon,a}(M) - \{\lambda_0\}, \sigma_{\epsilon,b}(M) - \{\lambda_0\}, \sigma_{\epsilon,c}(M) - \{\lambda_0\}.$$

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This in turn implies

$$\sigma(M) \subset cl\left(\sigma_{\epsilon,a}(M)\right), cl\left(\sigma_{\epsilon,b}(M) - \sigma(M)\right), cl\left(\sigma_{\epsilon,c}(M)\right). \tag{D1}$$

In particular, if $\sigma_{\epsilon,b}(M) - \sigma(M)$ is open, then $\sigma_{\epsilon,b}(M)$ is open.

Note that the norm of a vector or matrix is continuous with respect to its entries. Also, the eigenvalues of a matrix depend continuously on the matrix entries. Thus, the sets $\sigma_{\epsilon,a}(M)$, $\sigma_{\epsilon,b}(M) - \sigma(M)$, and $\sigma_{\epsilon,c}(M)$ are open. Therefore, the set $\sigma_{\epsilon,b}(M)$ is also open.

Because the sets (a)–(c) are equivalent on dom (M) and $\mathbb{C} -$ dom (M) is finite, then

$$\sigma_{\epsilon,a}(M) \cap \text{dom}\,(M) = \sigma_{\epsilon,b}(M) \cap \text{dom}\,(M) = \sigma_{\epsilon,c}(M) \cap \text{dom}\,(M)$$

is an open set. Taking the closure, it follows that

$$cl\left(\sigma_{\epsilon,a}(M)\right) = cl\left(\sigma_{\epsilon,b}(M) - \sigma(M)\right) = cl\left(\sigma_{\epsilon,c}(M)\right)$$

implying that Definition 3.2 parts (a)–(c) are equivalent. Moreover, equation (D1) implies $\sigma(M) \subset \sigma_\epsilon(M)$. □

The proof that Definition 6.1 parts (a)–(c) are equivalent is very similar to the Proof of Theorem D.1 and is therefore omitted.

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