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ON THE STEADY-STATE TRANSVERSE VIBRATIONS OF A CRACKED PLATE

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Abstract—Using an integral formulation, the equation for a plate subjected to periodic transverse vibrations of frequency ω and containing a crack of length $2c$, is solved for the Kirchhoff bending stresses. The usual inverse square root singular behavior of the stresses is recovered and may be related to that of non-vibrating cracked plate by

$$\frac{\sigma_{y_{\text{dynamic}}}}{\sigma_{y_{\text{static}}}} \approx \frac{1}{1+f(\lambda^4)}\cdot\cos(\omega t+\phi)$$

where the function $f(\lambda^4)$ attains positive values for small λ.

NOTATION

$2c$	crack length
D	$Eh^3/[12(1-\nu^2)]$ = flexural rigidity of a plate
E	Young's modulus of elasticity
G	shear modulus of elasticity
h	thickness of a plate
k	foundation modulus of a plate
$M_y, My^{(P)}$	bending moments as defined in text
m_0	constant as defined in text
t	time variable
$U(s-\lambda)$	the unit step function
$V_y, V_y^{(P)}$	shear force as defined in text
w	transverse deflection of a plate in bending
W	as defined in text with $W^+ \equiv \lim_{y\to 0^+} W$ and $W^- \equiv \lim_{y\to 0^-} W$
X, Y, Z	rectangular coordinates in middle plane of plate
x	X/c, $y \equiv Y/c$, dimensionless rectangular coordinates
γ	0·578 Euler's constant
$\epsilon, \theta; \epsilon e^{i\theta}$	$x\pm 1+iy$
ζ	$x-\xi$
λ^4	$(2h\rho c/D)\omega^2$
ν	Poisson's ratio
ν_0	$1-\nu$
ρ	density of the material
$\sigma_x, \sigma_y, \tau_{xy}$	stress components
$\bar{\sigma}_b$	$(6D/h^2)(m_0/c^2)$
ω	vibrating frequency of plate

INTRODUCTION

In the field of Solid Mechanics considerable theoretical work has been done on the transverse vibrations of plates with different geometrics and different methods of support[1–7]. On the other hand no one, to the best of the author's knowledge, has attempted to investigate the effect of these transverse vibrations on cracked thin plates.

It is the intent, therefore, of this paper to study this effect for plates subjected to steady-state transverse vibrations.

FORMULATION OF THE PROBLEM

Consider a homogeneous and isotropic thin plate of uniform thickness h which contains a crack of finite length $2c$ and is subjected to transverse vibrations. Following Love[7], the classical differential equation governing the deflection $w(x, y, t)$ is

$$\nabla^4 w = -\frac{2h\rho c^4}{D}\frac{\partial^2 w}{\partial t^2}. \tag{1}$$

If furthermore, one assumes that the plate vibrates in a normal mode, w can be expressed as

$$w(x, y, t) = W(x, y)\cos(\omega t + \phi) \tag{2}$$

where the function $W(x, y)$ now satisfies the reduced equation

$$(\nabla^4 - \lambda^4) W(x, y) = 0. \tag{3}$$

As to boundary conditions, one must require the normal moment and equivalent vertical shear to vanish along the crack.

However, suppose that one has already found a particular solution satisfying (3) but there is a residual normal moment M_y and equivalent vertical shear V_y along the crack $|x| < 1$ of the form

$$M_y^{(P)} = -\frac{Dm_0}{c^2} \tag{4}$$

$$V_y^{(P)} = 0 \tag{5}$$

where, for simplicity, m_0 will be taken to be a constant. Specifically, one needs to find a function $W(x, y)$ such that it satisfies (3) and the following boundary conditions:

at $y = 0$ and $|x| < 1$

$$M_y(x, 0) = \frac{D}{c^2}\left[\frac{\partial^2 W}{\partial y^2} + \nu\frac{\partial^2 W}{\partial x^2}\right] = \frac{Dm_0}{c^2} \tag{6}$$

$$V_y(x, 0) = -\frac{D}{c^3}\left[\frac{\partial^3 W}{\partial y^3} + (2-\nu)\frac{\partial^3 W}{\partial x^2 \partial y}\right] = 0 \tag{7}$$

at $y = 0$ and $|x| > 1$ the continuity requirements

$$\lim_{|y|\to 0}\left[\frac{\partial^n}{\partial y^n}(W^+) - \frac{\partial^n}{\partial y^n}(W^-)\right] = 0 \quad (n = 0, 1, 2, 3). \tag{8}$$

Finally, to complete the formulation of the problem, we require that the function W together with its first derivatives to be finite far away from the crack.

METHOD OF SOLUTION

We construct the following integral representation for the function $W(x, y)$ with the proper symmetrical behavior in x

$$W(x, y^{\pm}) = \int_0^\infty \Big\{P_1(s)e^{-\sqrt{(s^2+\lambda^2)}|y|} + P_2(s)e^{-\sqrt{(s^2-\lambda^2)}|y|}U(s-\lambda) + iP_3(s)\sin\left[\sqrt{(\lambda^2 - s^2)}\,|y|\right]U(\lambda - s)\Big\}\cos xs\,ds, \tag{9}$$

where the P_i's are arbitrary functions of s to be determined from the boundary conditions, and the $\pm$ signs refer to $y > 0$ and $y < 0$ respectively.

Following the same method of solution as in[8], one may show (see Appendix) that the problem reduces to the solution of the following singular integral equation

$$\int_{-1}^{1} L[\lambda|x-\xi|]u(\xi)\,d\xi = -m_0\pi x;\ |x| < 1\dagger \tag{10}$$

where the kernel L is given by

$$L[\lambda|\xi|] \equiv \lim_{|y|\to 0} \int_0^\infty \left\{\frac{(\nu_0 s^2+\lambda^2)^2 e^{-\sqrt{(s^2+\lambda^2)}|y|}}{s\sqrt{(s^2+\lambda^2)}} - \frac{(\nu_0 s^2-\lambda^2)^2 e^{-\sqrt{(s^2-\lambda^2)}|y|}}{s\sqrt{(s^2-\lambda^2)}}\,U(s-\lambda)\right\} \sin\xi s\,ds \tag{11a}$$

and its asymptotic expansion for small $\lambda's$ is of the form

$$L[\lambda|\xi|] = \frac{(4-\nu_0)\nu_0\lambda^2}{\xi} + \frac{73\nu_0^2-180\nu_0+128}{2^6\cdot 3^2}\lambda^6\xi^3 - \frac{5\nu_0^2-12\nu_0+8}{2^4\cdot 3}\lambda^6\xi^3\left(\gamma+\ln\frac{\lambda|\xi|}{2}\right) + 0(\lambda^{10}\ln\lambda|\xi|). \tag{11b}$$

A method for constructing such a solution is given in[8]. Therefore, without going into the details (see Appendix), one may show that

$$W(x,y^\pm) = \int_0^\infty \left\{\frac{(\nu_0 s^2+\lambda^2)e^{-\sqrt{(s^2+\lambda^2)}|y|}}{\sqrt{(s^2+\lambda^2)}} - \frac{(\nu_0 s^2-\lambda^2)e^{-\sqrt{(s^2-\lambda^2)}|y|}}{\sqrt{(s^2-\lambda^2)}}U(s-\lambda) + \frac{(\nu_0 s^2-\lambda^2)}{\sqrt{(\lambda^2-s^2)}}\sin\sqrt{(\lambda^2-s^2)}\,|y|\,U(\lambda-s)\right\}\cdot \left\{A_1\frac{J_1(s)}{s}+0(\lambda^4)\right\}\cos xs\,ds \tag{12}$$

where

$$A_1 = -\frac{m_0}{(4-\nu_0)\nu_0\lambda^2}\cdot\frac{1}{1+\dfrac{3\nu_0^2-12\nu_0+6}{2\cdot(4-\nu_0)\nu_0}\lambda^4-\dfrac{3}{2^7}\dfrac{5\nu_0^2-12\nu_0+8}{(4-\nu_0)\nu_0}\lambda^4\left(\gamma+\ln\dfrac{\lambda}{4}\right)}. \tag{13}$$

In view of (2), (12), and (13) and the definition of the stresses, it is a simple exercise to show that the stresses at the surface $z = h/2c$ are of the form:

$$\sigma_x = \frac{P_b}{\sqrt{(2\epsilon)}}\left[-\frac{3-3\nu}{4}\cos\frac{\theta}{2}-\frac{1-\nu}{4}\cos\frac{5\theta}{2}\right]\cos(\omega t+\phi)+0(\epsilon^0) \tag{14}$$

$$\sigma_y = \frac{P_b}{\sqrt{(2\epsilon)}}\left[\frac{11+5\nu}{4}\cos\frac{\theta}{2}+\frac{1-\nu}{4}\cos\frac{5\theta}{2}\right]\cos(\omega t+\phi)+0(\epsilon^0) \tag{15}$$

$$\tau_{xy} = \frac{P_b}{\sqrt{(2\epsilon)}}\left[-\frac{7+\nu}{4}\sin\frac{\theta}{2}-\frac{1-\nu}{4}\sin\frac{5\theta}{2}\right]\cos(\omega t+\phi)+0(\epsilon^0) \tag{16}$$

†Where it is understood that the principal value of the integral is to be taken.

where the stress intensity factor P_b is

$$P_b = \frac{\bar{\sigma}_b}{(4-\nu_0)} \frac{1}{\left[1+\frac{3\nu_0^2-12\nu_0+6}{2^9(4-\nu_0)\nu_0}\lambda^4 - \frac{3}{2^7}\frac{5\nu_0^2-12\nu_0+8}{(4-\nu_0)\nu_0}\lambda^4\left(\gamma+\ln\frac{\lambda}{4}\right)\right]}, \quad (17a)$$

which for $\nu = \frac{1}{3}$ reduces to

$$P_b = \bar{\sigma}_b \frac{3}{10} \frac{1}{\left[1-\frac{3\lambda^4}{2^9 10} - \frac{3}{2^7}\lambda^4\left(\gamma+\ln\frac{\lambda}{4}\right)\right]}. \quad (17b)$$

Notice that the expression inside the brackets for small values of the parameter λ is positive. This indicates, at least for small frequencies, that a cracked plate subjected to transverse periodic vibrations is analogous to a cracked plate resting on an elastic foundation[9] with an equivalent spring constant

$$k = \frac{D}{c^4}\frac{\lambda^8}{100}(1-1.2\ln\lambda)^2 \quad (18)$$

Consequently, in order to find an estimate of the stress intensity factor for large values of the parameter λ, one might substitute the equivalent spring constant in the results of[10] to obtain

$$P_b \approx \frac{3\bar{\sigma}_b}{10} \cdot \frac{1.78}{\lambda\sqrt[4]{(|1-1.2\ln\lambda|)}}. \quad (17c)$$

A plot of the stress intensity factor, $P_b/\bar{\sigma}/_b$ for various values of the parameter λ is given in Fig. 1.

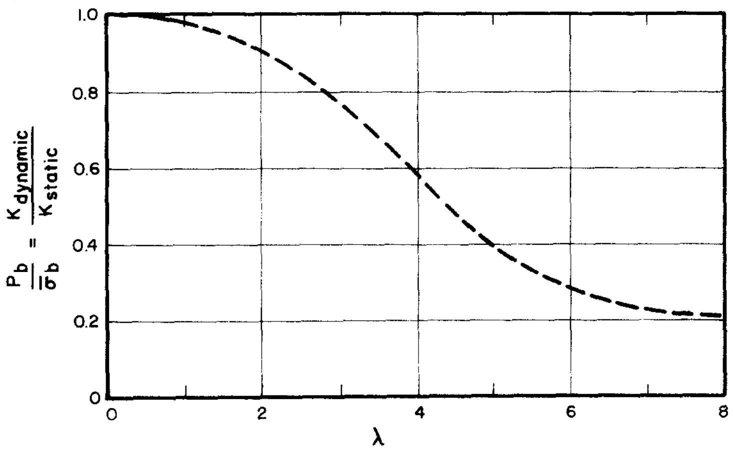

Fig. 1. Stress intensity factor vs. the parameter λ.

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APPENDIX

Assuming that one can differentiate under the integral sign, formally substituting (9) into (7), one obtains

$$\mp \int_0^\infty \Big\{ (\nu_0 s^2-\lambda^2)\sqrt{(s^2+\lambda^2)}\,P_1(s) + (\nu_0 s^2+\lambda^2)\sqrt{(s^2-\lambda^2)}\,P_2(s)\,U(s-\lambda) - i(\nu_0 s^2+\lambda^2)\sqrt{(\lambda^2-s^2)}\,P_3(s)\,U(\lambda-s)\Big\} \cos xs\,\mathrm{d}s = 0;\ |x|<1. \tag{19}$$

A sufficient condition, but not necessary, is to let the expression inside the braces to vanish, i.e. choose

$$\begin{aligned} P_1(s) &= (\nu_0 s^2+\lambda^2)\sqrt{(s^2-\lambda^2)}\,P(s)\\ P_2(s) &= -(\nu_0 s^2-\lambda^2)\sqrt{(s^2+\lambda^2)}\,P(s)\\ P_3(s) &= (\nu_0 s^2-\lambda^2)\sqrt{(s^2+\lambda^2)}\,P(s)\end{aligned} \tag{20}$$

where $P(s)$ is still largely arbitrary. Conditions (8) are satisfied if the following combination vanishes

$$\int_0^\infty \sqrt{(s^4-\lambda^4)}\,P(s)\cos xs\,\mathrm{d}s = 0;\ \ |x|>1. \tag{21}$$

Define next

$$\int_0^\infty \sqrt{(s^4-\lambda^4)}\,P(s)\cos xs\,\mathrm{d}s = u(x);\ \ |x|<1 \tag{22}$$

where $u(x)$ is an unknown function to be determined. By Fourier inversion of (22), one may express $P(s)$ as a function of $u(\xi)$, i.e.

$$P(s) = \frac{2}{\pi}\frac{1}{\sqrt{(s^4-\lambda^4)}}\int_0^1 u(\xi)\cos \xi s\,\mathrm{d}\xi. \tag{23}$$

Hence, by substituting (9) into (6), and utilizing (20) and (23), one can obtain by interchanging the order of integration the singular integral equation given by (10). Once $u(\xi)$ is determined, one may use (23), (20) and (9) to obtain (12).

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Résumé — Par l'intermédiaire d'une formule intégrale, l'équation définie pour une plaque soumise aux vibrations périodiques transversales de fréquence ω et contenant une rupture d'une longueur $2c$, est résolue pour les tensions de pliage de Kirchhoff. Le comportement usuel singulier de l'inverse de la racine carrée est retrouvé et peut être lié à celui d'une plaque fissurée non-vibrante par l'équation:

$$\frac{\sigma_{y_{\text{dynamique}}}}{\sigma_{y_{\text{statique}}}} = \frac{1}{1+f(\lambda^4)}\cdot\cos(\omega t+\phi)$$

dans laquelle la fonction $f(\lambda^4)$ atteind des valeurs positives pour petit λ.

Zusammenfassung — Mit Hilfe einer Integraldarstellung wird eine Gleichung für eine Platte, das periodischen transversalen Schwingungen von der Frequenz ω unterworfen wird, und die einen Riss von einer Länge von $2c$ aufweist, für die Kirchoffschen Biegespannungen gelöst. Es ergibt das übliche sich singuläre

verhalten der Spannungen (d.h. Zunahme mit $1/\sqrt{}$), welches durch die folgende Gleichung in Beziehung zur nichtschwingenden, rissbehafteten Platte gesetzt werden kann

$$\frac{\sigma_{y_{\text{dynamisch}}}}{\sigma_{y_{\text{statisch}}}} = \frac{1}{1+f(\lambda^4)} \cdot \cos(\omega t + \phi)$$

wobei für kleine Werte von λ die Funktion $f(\lambda^4)$ positive Werte annimmt.