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On the Steady-State Transverse Vibrations of a Cracked Spherical Shell

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ABSTRACT

Using an integral formulation, the problem of a spherical shell containing a through crack of length 2c and subjected to periodic transverse vibrations of frequency ω is solved for the in-plane and Kirchhoff bending stresses. The usual inverse square root singular behavior characteristic to crack problems is recovered. Furthermore, it is found that the transverse vibrations reduce the stresses in the vicinity of the crack tip, except when the forcing frequency ω reaches the natural frequency of the uncracked shell in which case they become infinite.

List of Symbols and Notations

c	= half crack length
D	= $Eh^3/[12(1-\nu^2)]$ = flexural rigidity
E	= Young's modulus of elasticity
$F(x, y, t)$, $\tilde{F}(x, y)$	= stress functions
$\tilde{F}^{(c)}(x, y)$	= complementary stress function
G	= shear modulus
h	= thickness
K_n	= modified Bessel function of the third kind of order n
L_i	= kernels as defined in text
m_0	= constant as defined in text
$M_x^{(c)}, M_y^{(c)}, M_{xy}^{(c)}$	= complementary bending forces
$M_x^{(p)}, M_y^{(p)}, M_{xy}^{(p)}$	= particular bending forces
n_0	= constant as defined in text
$N_x^{(c)}, N_y^{(c)}, N_{xy}^{(c)}$	= complementary membrane forces
$N_x^{(p)}, N_y^{(p)}, N_{xy}^{(p)}$	= particular membrane forces
N_n	= Newman function of order n
$q(x, y, t)$, $q(x, y)$	= internal pressure
r	= $\{(X-1)^2+Y^2\}^{\frac{1}{2}}$
R	= radius of curvature of the shell
$\bar{R}$	= $\{X^2+Y^2\}^{\frac{1}{2}}$
t	= time variables
$U(s-\lambda)$, $U(\lambda-s)$	= the unit step function
V_y	= equivalent shear
$W(x, y, t)$, $\tilde{W}(x, y)$	= displacement functions
$\tilde{W}^{(c)}(x, y)$	= complementary displacement function
$x = \dfrac{X}{c}$, $y = \dfrac{Y}{c}$, $z = \dfrac{Z}{c}$	= dimensionless rectangular coordinates
X, Y, Z	= rectangular cartesian coordinates

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γ	$= 0.5768 =$ Euler's constant
ω	$=$ frequency of the transverse vibration
ζ	$= x-\xi$
θ	$= \tan^{-1}(Y/X)$
λ^4	$= \frac{\rho h c^4}{D}\omega^2 - \frac{Ehc^4}{R^2 D}$
ν	$=$ Poisson's ratio
ν_0	$= 1-\nu$
ρ	$=$ density of the material
$\sigma_{x_b}, \sigma_{y_b}, \sigma_{xy_b}$	$=$ bending stress components
$\sigma_{x_e}, \sigma_{y_e}, \sigma_{xy_e}$	$=$ stretching stress components

Introduction

In the field of fracture mechanics, considerable theoretical work has been done [1 – 8] in order to assess analytically the effect of initial curvature upon the stress distribution in a thin sheet containing a finite line-crack. In practice, however, these curved sheets are, in addition to the external loadings, also subjected to longitudinal and/or transverse vibrations. Therefore, an exploratory study was undertaken to investigate the effect of these forcing frequencies on the stress distribution around the crack point.

The special case of a flat plate was recently investigated by Folias [9] and Sih and Loeber [10] independently. The authors, in this paper, will discuss the effect of the steady-state transverse vibrations on a spherical shell.

In the following, we consider bending and stretching of thin shells of revolution, as described by traditional two-dimensional linear theory, with the additional assumption of shallowness. In speaking of the formulation of two-dimensional differential equations, we mean the transition from the exact three-dimensional elasticity problem to that of two-dimensional approximate formulation, which is appropriate in view of the "thinness" of the shell. In this paper we limit ourselves to isotropic and homogeneous shallow segments* of elastic spherical shells of constant thickness. It is furthermore assumed that the shell is subjected to small deformations and strains so that the stress-strain relations may be established through Hooke's law.

Formulation of the Problem

Consider a spherical shell which contains a through crack of finite length $2c$ (see Fig. 1) and is subjected to transverse vibrations. Following Ref. [11], the system of equations governing the

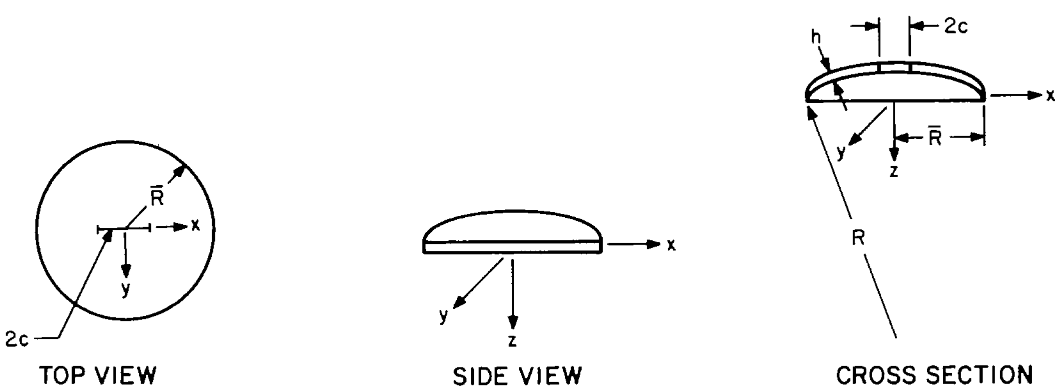

Figure 1. Geometry of the shell.

*A segment will be called shallow if the ratio of height to base diameter is less than, say, $\frac{1}{8}$.

deflection function $W(x, y, t)$ and the stress function $F(x, y, t)$, with x and y as dimensionalized rectangular coordinates of the base plane, can be written in the form

$$-\frac{Ehc^2}{R}\nabla^2 w+\nabla^4 F=0 \tag{1}$$

$$\nabla^4 w+\frac{c^2}{RD}\nabla^2 F+\frac{\rho hc^4}{D}\frac{\partial^2 w}{\partial t^2}=\frac{qc^4}{D}. \tag{2}$$

We shall consider harmonic vibrations. Then writing

$$q=\tilde{q}(x, y)\cos(\omega t+\phi), \tag{3}$$

and therefore

$$W=\tilde{W}(x, y)\cos(\omega t+\phi) \tag{4}$$

$$F=\tilde{F}(x, y)\cos(\omega t+\phi), \tag{5}$$

we reduce the system (1) and (2) to

$$-\frac{Ehc^2}{R}\nabla^2 w+\nabla^4 \tilde{F}=0 \tag{6}$$

$$\nabla^4 \tilde{w}-\frac{\rho hc^4}{D}\omega^2\tilde{w}+\frac{c^2}{RD}\nabla^2 \tilde{F}=\frac{\tilde{q}c^4}{D}. \tag{7}$$

As to the boundary conditions, one must require that the normal moment, equivalent vertical shear, and normal and tangential in-plane stresses vanish along the crack. However, suppose that one has already found a particular solution* satisfying Eqs. (6) and (7), but that there is a residual normal moment M_y, equivalent vertical shear V_y, normal in-plane stress N_y, and an in-plane tangential stress N_{xy}, along the real axis $|x|<1$, of the form

$$M_y^{(p)}=-\frac{D}{c^2}m_0, \tag{8}$$

$$V_y^{(p)}=0, \tag{9}$$

$$N_y^{(p)}=-\frac{n_0}{c^2}, \tag{10}$$

$$N_{xy}^{(p)}=0. \tag{11}$$

For simplicity, we assume m_0, n_0 to be constants.**

Mathematical Statement of the Complementary Problem

Assuming, therefore, that a particular solution has been found, we need to find now two functions of the dimensionless coordinates (x, y), $\tilde{W}(x, y)$, and $\tilde{F}(x, y)$ such that they satisfy the homogeneous partial differential equations (6) and (7) and the following boundary conditions. At $y=0$ and $|x|<1$:

$$M_y^{(c)}(x, 0)=\lim_{|y|\to 0}\ -\frac{D}{c^2}\left[\frac{\partial^2 \tilde{W}^{(c)}}{\partial y^2}+\nu\frac{\partial^2 \tilde{W}^{(c)}}{\partial x^2}\right]=Dm_0/c^2 \tag{12}$$

$$V_y^{(c)}(x, 0)=\lim_{|y|\to 0}\ -\frac{D}{c^3}\left[\frac{\partial^3 \tilde{W}^{(c)}}{\partial y^3}+(2-\nu)\frac{\partial^3 \tilde{W}^{(c)}}{\partial x^2\partial y}\right]=0 \tag{13}$$

*See reference 11.

**For m_0, n_0 non-constant, see reference 1.

$$N_y^{(c)}(x, 0) = \lim_{|y| \to 0} -\frac{1}{c^2} \frac{\partial^2 \tilde{F}^{(c)}}{\partial x^2} = n_0/c^2 \tag{14}$$

$$N_{xy}^{(c)}(x, 0) = \lim_{|y| \to 0} -\frac{1}{c^2} \frac{\partial^2 \tilde{F}^{(c)}}{\partial x \partial y} = 0 \,. \tag{15}$$

At $y=0$ and $|x|>1$ we must satisfy the continuity requirements, *i.e.*

$$\lim_{y \to 0} \left[\frac{\partial^n}{\partial y^n} \tilde{W}^{(c)}(x, 0+) - \frac{\partial^n}{\partial y^n} \tilde{W}^{(c)}(x, 0-) \right] \tag{16}$$

$$\lim_{y \to 0} \left[\frac{\partial^n}{\partial y^n} \tilde{F}^{(c)}(x, 0+) - \frac{\partial^n}{\partial y^n} \tilde{F}^{(c)}(x, 0-) \right] \tag{17}$$

for $n=0, 1, 2, 3$.

Furthermore, because we are limiting ourselves to a large radius of curvature for this shallow shell, *i.e.* small deviations from a flat sheet, we can apply certain boundary conditions at infinity even though we know physically that the stresses and displacements far away from the crack are finite. Therefore, to avoid infinite stresses and infinite displacements, we must require that the displacement function $\tilde{W}^{(c)}$ and the stress function $\tilde{F}^{(c)}$ with their first derivatives to vanish far away from the crack. These restrictions simplify the mathematical complexities of the problem considerably and correspond to the usual expectations of the St. Venant principle.

It should be pointed out that the boundary conditions at infinity are not geometrically feasible. However, if the crack is small compared to the dimensions of the shell, the approximation is good.

Integral Representations of the Solution

If one seeks the solution in the form

$$\tilde{W}^{(c)}(x, y) = \int_0^\infty P \, e^{ay} \cos xs \, ds \tag{18}$$

$$\tilde{F}^{(c)}(x, y) = \int_0^\infty Q \, e^{ay} \cos xs \, ds \,, \tag{19}$$

then direct substitution into the homogeneous parts of (6) and (7) leads to

$$\int_0^\infty \left\{ -\frac{Ehc^2}{R} (a^2 - s^2) P + (a^2 - s^2) Q \right\} e^{ay} \cos sx \, dx = 0 \tag{20}$$

$$\int_0^\infty \left\{ \left[(a^2 - s^2)^2 - \frac{\rho h c^4}{D} \omega^2 \right] P + \frac{c^2}{RD} (a^2 - s^2) Q \right\} e^{ay} \cos sx \, ds = 0 \,. \tag{21}$$

Sufficient conditions for the above integrals to vanish are

$$-\frac{Ehc^2}{R} (a^2 - s^2) P + (a^2 - s^2)^2 Q = 0 \tag{22}$$

$$\left[(a^2 - s^2)^2 - \frac{\rho h c^4}{D} \omega^2 \right] P + \frac{c^2}{RD} (a^2 - s^2) Q = 0 \,. \tag{23}$$

However, the necessary and sufficient conditions for the algebraic system (22) and (23) to have a solution are that its determinant vanishes, in particular

$$\begin{vmatrix} -\frac{Ehc^2}{R}(a^2-s^2) & (a^2-s^2)^2 \\ (a^2-s^2)^2 - \frac{\rho hc^4}{D}\omega^2 & \frac{c^2}{RD}(a^2-s^2) \end{vmatrix} = 0 \tag{24a}$$

$$(a^2-s^2)^2\left[(a^2-s^2)^2 + \frac{Ehc^4}{R^2 D} - \frac{hc^4}{D}\omega^2\right] = 0\,. \tag{24b}$$

The solution of the algebraic equation (24b) leads to the following eight roots

$$a = \pm s \text{ double root}; \quad a = \pm(s^2 \pm \lambda^2)^{\frac{1}{2}}$$

where $\lambda^4 \equiv \frac{\rho hc^4\omega^2}{D} - \frac{Ehc^4}{R^2 D}$.

Depending now as to the sign of λ^4, we will treat the following three cases (i) $\lambda^4 > 0$, (ii) $\lambda^4 = 0$, (iii) $\lambda^4 < 0$ separately.

Case I $\lambda^4 > 0$

In view of (18) and (19), we construct the following integral representations with the proper symmetrical behavior in x,

$$\tilde{W}^{(c)}(x, y^{\pm}) = \int_0^\infty \{P_1 e^{-(s^2+\lambda^2)^{\frac{1}{2}}|y|} + P_2 e^{-(s^2-\lambda^2)^{\frac{1}{2}}|y|}U(s-\lambda)$$
$$-iP_2 \sin\{\lambda^2 - s^2\}^{\frac{1}{2}}|y|\,U(\lambda-s) + P_3 e^{-s|y|}\}\cos xs\,ds\,, \tag{25}$$

$$\tilde{F}^{(c)}(x, y^{\pm}) = \frac{Ehc^2}{\lambda^2 R}\int_0^\infty \{P_1 e^{-(s^2+\lambda^2)^{\frac{1}{2}}|y|} - P_2 e^{-(s^2-\lambda^2)^{\frac{1}{2}}|y|}U(s-\lambda)$$
$$+iP_2 \sin\{(\lambda^2 - s^2)^{\frac{1}{2}}|y|\}\,U(\lambda-s) + P_4 e^{-s|y|}\}\cos xs\,ds\,, \tag{26}$$

where Pi's ($i = 1, 2, 3, 4$) are arbitrary functions of s to be determined from the boundary conditions, and the $\pm$ signs refer to $y > 0$ and $y < 0$, respectively.

Assuming that one can differentiate under integral sign, formally substituting Eq. (25) into Eqs. (12) and (13) one has, respectively

$$\lim_{|y|\to 0}\int_0^\infty \Big\{(v_0 s^2 + \lambda^2)P_1 e^{-(s^2+\lambda^2)^{\frac{1}{2}}|y|} + (v_0 s - \lambda^2)P_2 e^{-(s^2+\lambda^2)^{\frac{1}{2}}|y|}U(s-\lambda)$$
$$-iP_2(v_0 s^2 - \lambda^2)\sin\{(\lambda^2 - s^2)^{\frac{1}{2}}|y|\}\,U(\lambda-s) + P_3(v_0 s^2)e^{-s|y|}\Big\}\cos xs\,ds = -m_0\,;$$
$$|x| < 1 \tag{27}$$

and

$$\lim_{|y|\to 0} \mp \int_0^\infty \{(v_0 s^2 - \lambda^2)(s^2+\lambda^2)^{\frac{1}{2}}P_1 e^{-(s^2+\lambda^2)^{\frac{1}{2}}|y|} + (v_0 s^2 + \lambda^2)(s^2-\lambda^2)^{\frac{1}{2}} e^{-(s^2-\lambda^2)^{\frac{1}{2}}|y|}U(s-\lambda)$$
$$+iP_2(v_0 s^2 + \lambda^2)(\lambda^2 - s^2)^{\frac{1}{2}}\cos\{(\lambda^2 - s^2)^{\frac{1}{2}}|y|\}\,U(\lambda-s)$$
$$+v_0 s^3 P_3 e^{-s|y|}\}\cos xs\,ds = 0\,; \qquad |x| < 1\,, \tag{28}$$

where again the $\mp$ signs refer to $y > 0$ and $y < 0$, respectively. A sufficient condition for Eq. (28) is to set

$$v_0 s^3 p_3 = -\{(v_0 s^2 - \lambda^2)(s^2+\lambda^2)^{\frac{1}{2}}P_1 + (v_0 s^2 + \lambda^2)(s^2-\lambda^2)^{\frac{1}{2}}P_2\}\,. \tag{29}$$

Similarly, substituting Eq. (26) into Eqs. (14) and (15) one obtains respectively

$$\lim_{|y|\to 0} -\frac{Ehc^2}{\lambda^2 R}\int_0^\infty \{P_1 e^{-(s^2+\lambda^2)^{\frac{1}{2}}|y|} - P_2 e^{-(s^2-\lambda^2)^{\frac{1}{2}}|y|} U(s-\lambda) + iP_2 \sin\{(\lambda^2-s^2)^{\frac{1}{2}}|y|\} \cdot U(\lambda-s) + P_4 e^{-s|y|}\} s^2 \cos xs\, ds = n_0\ ; \qquad |x|<1 \tag{30}$$

and

$$\lim_{|y|\to 0} \mp\frac{Ehc^2}{\lambda^2 R}\int_0^\infty \{P_1 (s^2+\lambda^2)^{\frac{1}{2}} e^{-(s^2-\lambda^2)^{\frac{1}{2}}|y|} - P_2 (s^2-\lambda^2)^{\frac{1}{2}} e^{-(s^2-\lambda^2)^{\frac{1}{2}}|y|} U(s-\lambda) - iP_2(\lambda^2-s^2)^{\frac{1}{2}} \cos\{(\lambda^2-s^2)^{\frac{1}{2}}|y|\} U(\lambda-s) + sP_4 e^{-s|y|}\} s \sin xs\, ds = 0\ ; \qquad |x|<1\ . \tag{31}$$

Here again a sufficient condition for Eq. (31) is to set

$$P_4 = -\left\{\frac{(s^2+\lambda^2)^{\frac{1}{2}}}{s} P_1 - \frac{(s^2-\lambda^2)^{\frac{1}{2}}}{s} P_2\right\}. \tag{32}$$

Furthermore, it can be shown that the continuity conditions on $\tilde{W}^{(c)}$ and $\tilde{F}^{(c)}$ are satisfied if one considers the following combinations to vanish

$$\int_0^\infty \frac{P_1}{s^2}(s^2+\lambda^2)^{\frac{1}{2}} \cos xs\, ds = 0\ ; \qquad |x|>1 \tag{33}$$

$$\int_0^\infty \frac{P_2}{s^2}(s^2-\lambda^2)^{\frac{1}{2}} \cos xs\, ds = 0\ ; \qquad |x|>1\ . \tag{34}$$

Therefore the problem has been reduced to solving the dual integral equations (27), (30), (33) and (34) for the unknown functions $P_1(s)$ and $P_2(s)$.

Because we are unable, however, to solve directly dual integral equations of this type, we will cast the problem to singular integral equations. Let

$$u_1(x) = \int_0^\infty P_1(s^2+\lambda^2)^{\frac{1}{2}} \frac{\cos sx}{s^2} ds\ ; \qquad |x|<1 \tag{35}$$

$$u_2(x) = \int_0^\infty P_2(s^2-\lambda^2)^{\frac{1}{2}} \frac{\cos xs}{s^2} ds\ ; \qquad |x|<1 \tag{36}$$

which by Fourier inversion gives

$$P_1(s^2+\lambda^2)^{\frac{1}{2}} = \frac{2s^2}{\pi}\int_0^1 u_1(\xi)\cos\xi s\, ds \tag{37}$$

$$P_2(s^2-\lambda^2)^{\frac{1}{2}} = \frac{2s^2}{\pi}\int_0^1 u_2(\xi)\cos\xi s\, ds\ , \tag{38}$$

where the functions $u_1(\xi)$ and $u_2(\xi)$, due to the symmetry of the problem, are even.

Formally substituting Eqs. (37) and (38) into (27) and (30), we find after changing the order of integration and rearranging

$$\int_{-1}^1 \{u_1(\xi)L_1 - u_2(\xi)L_2\} d\xi = \frac{-\pi\lambda^2 R n_0}{Ehc^2} x\ ; \qquad |x|<1 \tag{39}$$

$$\int_{-1}^1 \{u_1(\xi)L_3 + u_2(\xi)L_4\} d\xi = -\pi m_0 x\ ; \qquad |x|<1 \tag{40}$$

where the kernels L_1, L_2, L_3, L_4 are given by

$$L_1 = \lim_{|y|\to 0} \int_0^\infty \left\{ \frac{s^3 e^{-(s^2+\lambda^2)^{\frac{1}{2}}|y|}}{(s^2+\lambda^2)^{\frac{1}{2}}} - s^2 e^{-s|y|} \right\} \sin \zeta s \, ds \tag{41}$$

$$L_2 = \lim_{|y|\to 0} \int_0^\infty \left\{ \frac{s^3 e^{-(s^2-\lambda^2)^{\frac{1}{2}}|y|}}{(s^2-\lambda^2)^{\frac{1}{2}}} U(s-\lambda) - s^2 e^{-s|y|} \right\} \sin \zeta s \, ds \tag{42}$$

$$L_3 = \lim_{|y|\to 0} \int_0^\infty \left\{ \frac{s(\nu_0 s^2+\lambda^2)}{(s^2+\lambda^2)^{\frac{1}{2}}} e^{-(s^2+\lambda^2)^{\frac{1}{2}}|y|} - (\nu_0 s^2-\lambda^2) e^{-s|y|} \right\} \sin \zeta s \, ds \tag{43}$$

$$L_4 = \lim_{|y|\to 0} \int_0^\infty \left\{ \frac{s(\nu_0 s^2-\lambda^2)}{(s^2-\lambda^2)^{\frac{1}{2}}} e^{-(s^2-\lambda^2)^{\frac{1}{2}}|y|} U(s-\lambda) - (\nu_0 s^2+\lambda^2) e^{-s|y|} \right\} \sin \zeta s \, ds . \tag{44}$$

The integration in Eqs. (41)–(44) may be carried out explicitly in terms of modified Bessel and Newman functions by making use of the Fourier cosine transform table of Appendix I. Asymptotic expansions of K_n and N_n also are listed in the appendix.

Without going into any details, the expressions (41)–(44) then become

$$L_1 = \frac{-\lambda^2}{\zeta} K_0(\lambda|\zeta|) - \frac{\lambda^2 \zeta}{|\zeta|} K_1(\lambda|\zeta|) - \frac{2\lambda}{\zeta|\zeta|} K_1(\lambda|\zeta|) + \frac{2}{\zeta^3}$$

$$= \frac{-\lambda^2}{2\zeta} + \lambda^4 \zeta \left\{ \frac{5}{32} - \frac{3\gamma}{8} - \frac{3}{8} \ln \frac{|\lambda\zeta|}{2} \right\} + 0(\lambda^6 \ln |\lambda\zeta|) \tag{41a}$$

$$L_2 = \frac{-\lambda^2 \pi}{2\zeta} N_0(\lambda|\zeta|) - \frac{\lambda^3 \pi}{2} \frac{\zeta}{|\zeta|} N_1(\lambda|\zeta|) + \frac{\lambda\pi}{\zeta|\zeta|} N_1(\lambda|\zeta|) + \frac{2}{\zeta^3}$$

$$= \frac{\lambda^2}{2} + \lambda^4 \zeta \left\{ \frac{5}{32} - \frac{3\gamma}{8} - \frac{3}{8} \ln \frac{|\lambda\zeta|}{2} \right\} + 0(\lambda^2 \ln |\lambda\zeta|) \tag{42a}$$

$$L_3 = -\frac{\nu_0 \lambda^2}{\zeta} K_0(\lambda|\zeta|) - \nu_0 \lambda^3 \frac{\zeta}{|\zeta|} K_1(\lambda|\zeta|) - \frac{2\nu_0}{\zeta|\zeta|} K_1(\lambda|\zeta|) + \frac{\lambda^3 \zeta}{|\zeta|} K_1(\lambda|\zeta|) + \frac{2\nu_0}{\zeta^3} + \frac{\lambda^2}{\zeta}$$

$$= -\frac{\lambda^2 (4-\nu_0)}{2\zeta} + \lambda^4 \left\{ \frac{5\nu_0 - 8}{32} + \frac{4-3\nu_0}{8} \left(\gamma + \ln \frac{|\lambda\zeta|}{2} \right) \right\} + 0(\lambda^6 \ln |\lambda\zeta|) \tag{43a}$$

$$L_4 = -\frac{\nu_0 \lambda^2}{2\zeta} N_0(\lambda|\zeta|) - \frac{\nu_0 \lambda^3 \pi \zeta}{|\zeta|} N_1(\lambda|\zeta|) + \frac{\nu_0 \lambda \pi}{\zeta|\zeta|} N_1(\lambda|\zeta|) + \frac{\lambda^3 \pi \zeta}{2|\zeta|} N_1(\lambda|\zeta|) + \frac{2\nu_0}{\zeta^3} - \frac{\lambda^2}{\zeta}$$

$$= -\frac{\lambda^2 (4-\nu_0)}{2} + \lambda^4 \zeta \left\{ \frac{5\nu_0 - 8}{32} + \frac{4-3\nu_0}{8} \left(\gamma + \ln \frac{|\lambda\zeta|}{2} \right) \right\} + 0(\lambda^6 \ln |\lambda\zeta|) . \tag{44a}$$

The kernels L_1, L_2, L_3, L_4 have singularities of the order $1/\zeta \equiv 1/(x-\xi)$. We require that the solutions $u_1(x)$, $u_2(x)$ be Holder continuous for some positive Holder indices μ_1 and μ_2 for all x in the closed interval $[-1, 1]$. Thus, in particular $u_1(x)$, $u_2(x)$ are to be bounded near the ends of the crack. The problem of obtaining a solution to the coupled integral equations (39) and (40) can be reduced to the problem of solving two coupled Fredholm integral equations with a bounded kernel. See Ref. [1b].

Following the same method of solution as that described in Ref. [1], one may let

$$u_1(\xi) = (1-\xi^2)^{\frac{1}{2}} [A_1 + \lambda^2 A_2 (1-\xi^2) + \ldots] \; ; \qquad |\xi| < 1 \tag{45}$$

$$u_2(\xi) = (1-\xi^2)^{\frac{1}{2}} [B_1 + \lambda^2 B_2 (1-\xi^2) + \ldots] \; ; \qquad |\xi| < 1 , \tag{46}$$

where the coefficients A_1, A_2, ... B_1, B_2, ... are functions of λ but not of ξ.

Substituting Eqs. (45) and (46) into (39) and (40) and making use of the integrals given in Part III of the appendix, we obtain respectively

$$A_1\left\{-\frac{\lambda^2\pi}{2}x+\lambda^4\left(\frac{5}{32}-\frac{3\gamma}{8}\right)\frac{\pi}{2}x-\frac{3\lambda^4}{8}\left[\frac{\pi}{4}\left(1+\ln\frac{\lambda^2}{16}\right)x+\frac{\pi}{6}x^3\right]\right\}$$

$$+B_1\left\{-\frac{\lambda^2\pi}{2}x-\lambda^4\left(\frac{5}{32}-\frac{3\gamma}{8}\right)\frac{\pi}{2}x+\frac{3\lambda^4}{8}\left[\frac{\pi}{4}\left(1+\ln\frac{\lambda^2}{16}\right)x+\frac{\pi}{6}x^3\right]\right\}$$

$$+A_2\left\{-\frac{\lambda^4}{2}\frac{3\pi}{2}x+\frac{\lambda^4\pi}{2}x^3\right\}$$

$$+B_2\left\{-\frac{\lambda^4}{2}\frac{3\pi}{2}x+\frac{\lambda^4\pi}{2}x^3\right\}+0(\lambda^6\ln\lambda) = -\frac{\pi\lambda^2 Rn_0}{Ehc^2} \tag{47}$$

and

$$A_1\left\{(4-\nu_0)\frac{\lambda^2\pi}{2}x+\lambda^4\left(\frac{5\nu_0-8}{32}\gamma\right)\frac{\pi}{2}x+\lambda^4\left(\frac{4-3\nu_0}{8}\right)\left[\frac{\pi}{4}\left(1+\ln\frac{\lambda^2}{16}\right)x+\frac{\pi}{16}x^3\right]\right\}$$

$$+B_1\left\{-(4-\nu_0)\frac{\lambda^2\pi}{2}x+\lambda^4\left(\frac{5\nu_0-8}{32}+\frac{4-3\nu_0}{8}\gamma\right)\frac{\pi}{2}x+\lambda^4\left(\frac{4-3\nu_0}{8}\right)\right.$$

$$\left.\cdot\left[\frac{\pi}{4}\left(1+\ln\frac{\lambda^2}{16}\right)x+\frac{\pi}{16}x^3\right]\right\}$$

$$+A_2\left\{(4-\nu_0)\frac{\lambda^4}{2}\left(\frac{3\pi}{2}x-\pi x^3\right)\right\}$$

$$+B_2\left\{-(4-\nu_0)\frac{\lambda^4}{2}\left(\frac{3\pi}{2}x-\pi x^3\right)\right\}+0(\lambda^6\ln\lambda) = -m_0\pi x\,. \tag{48}$$

Next we equate coefficients. In particular, we first require the coefficients of the x^3 terms to vanish which gives

$$A_2+B_2 = \tfrac{1}{8}(A_1-B_1) \tag{49}$$

$$A_2+B_2 = \frac{4-3\nu_0}{24(4-\nu_0)}(A_1+B_1)\,.$$

Then substituting Eqs. (49) and (50) into (47) and (48) and solving for A_1 and B_1 one has

$$A_1 = \frac{Rn_0}{Ehc^2}\left\{1-\frac{\lambda^2(8-7\nu_0)}{32(4-\nu_0)}-\frac{\lambda^2(4-3\nu_0)}{8(4-\nu_0)}\gamma-\frac{\lambda^2(4-3\nu_0)}{16(4-\nu_0)}\ln\frac{\lambda^2}{16}\right\}$$

$$+\frac{m_0}{\lambda^2(4-\nu_0)}\left\{-1+\frac{7}{32}\lambda^2+\frac{3\lambda^2}{8}\gamma+\frac{3\lambda^2}{16}\ln\frac{\lambda^2}{16}\right\}+0(\lambda^2\ln\lambda) \tag{51}$$

$$B_1 = \frac{Rn_0}{Ehc^2}\left\{1+\frac{\lambda^2(8-7\nu_0)}{32(4-\nu_0)}+\frac{\lambda^2(4-3\nu_0)}{8(4-\nu_0)}\gamma+\frac{\lambda^2(4-3\nu_0)}{16(4-\nu_0)}\ln\frac{\lambda^2}{16}\right\}$$

$$+\frac{m_0}{\lambda^2(4-\nu_0)}\left\{1+\frac{7}{32}\lambda^2+\frac{3\lambda^2}{8}\gamma+\frac{3\lambda^2}{16}\ln\frac{\lambda^2}{16}\right\}+0(\lambda^2\ln\lambda)\,. \tag{52}$$

We should point out that if coefficients of A_1 and B_1 of higher accuracy are desired, say up to order λ^{2n}, then it is necessary to solve an $n\times n$ algebraic system. In effect, this is a method of successive approximations for which the question of convergence is investigated in Ref. [1b].

In the view of Eqs. (29), (32), (37–39), (45–46), and the relation

$$\int_0^\infty (u^2-x^2)^{\nu-\frac{1}{2}}\cos(as)\mathrm{d}x = \frac{\sqrt{\pi}}{2}\left(\frac{2u}{a}\right)^\nu \Gamma(\nu+\tfrac{1}{2})J_\nu(au) \qquad [a>0, u>0\ \mathrm{Re}\,\nu>-\tfrac{1}{2}]\,, \tag{53}$$

which can be found on page 427 in Ref. [12], one has

$$P_1(s) = \frac{s}{(s^2+\lambda^2)^{\frac{1}{2}}}\left\{A_1 J_1(s) + 3\lambda^2 A_2 \frac{J_2(s)}{s} + 0(\lambda^4)\right\} \tag{54}$$

$$P_2(s) = \frac{s}{(s^2-\lambda^2)^{\frac{1}{2}}}\left\{B_1 J_1(s) + 3\lambda^2 B_2 \frac{J_2(s)}{s} + 0(\lambda^4)\right\} \tag{55}$$

$$P_3(s) = -(A_1+B_1)J_1(s) - 3\lambda^2(A_2+B_2)\frac{J_2(s)}{s} + \frac{\lambda^2}{v_0 s^2}(A_1-B_1)J_1(s) + 0(\lambda^4) \tag{56}$$

$$P_4(s) = -(A_1+B_1)J_1(s) - 3\lambda^2(A_2+B_2)\frac{J_2(s)}{s} + 0(\lambda^4)\,, \tag{57}$$

where the coefficients A_1 and B_1 are given by Eqs. (51) and (52) respectively.

Therefore, a substitution of the above relations into (25) and (26) will determine the bending deflection $\tilde{W}^{(c)}$ and membrane stress function $\tilde{F}^{(c)}$ as follows:*

$$\begin{aligned} \tilde{W}^{(c)}(x, y^{\pm}) = \int_0^{\infty}\Big\{ & \frac{s}{(s^2+\lambda^2)^{\frac{1}{2}}}\left[A_1 J_1(s) + 3\lambda^2 A_2 \frac{J_2(s)}{s} + \ldots\right] e^{-(s^2-\lambda^2)^{\frac{1}{2}}|y|} \\ & + \frac{s}{(s^2+\lambda^2)^{\frac{1}{2}}}\left[B_1 J_1(s) + 3\lambda^2 B_2 \frac{J_2(s)}{s} + \ldots\right] e^{-(s^2-\lambda^2)^{\frac{1}{2}}|y|}\, U(s-\lambda) \\ & - \frac{s}{(\lambda^2-s^2)^{\frac{1}{2}}}\left[B_1 J_1(s) + 3\lambda^2 B_2 \frac{J_2(s)}{s} + \ldots\right] \sin\{(\lambda^2-s^2)^{\frac{1}{2}}|y|\}\, U(\lambda-s) \\ & + \Big[-(A_1+B_1)J_1(s) - 3\lambda^2(A_2+B_2)\frac{J_2(s)}{s} \\ & + \frac{\lambda^2}{v_0 s^2}(A_1-B_1)J_1(s) + \ldots\Big] e^{-s|y|}\Big\} \cos xs\, ds \end{aligned} \tag{58}$$

$$\begin{aligned} \tilde{F}^{(c)}(x, y^{\pm}) = \frac{Ehc^2}{\lambda^2 R}\int_0^{\infty}\Big\{ & \frac{s}{(s^2+\lambda^2)^{\frac{1}{2}}}\left[A_1 J_1(s) + 3\lambda^2 A_2 \frac{J_2(s)}{s} + \ldots\right] e^{-(s^2+\lambda^2)^{\frac{1}{2}}|y|} \\ & - \frac{s}{(s^2-\lambda^2)^{\frac{1}{2}}}\left[B_1 J_1(s) + 3\lambda^2 B_2 \frac{J_2(s)}{s} + \ldots\right] e^{-(s^2-\lambda^2)^{\frac{1}{2}}|y|}\, U(s-\lambda) \\ & + \frac{s}{(\lambda^2-s^2)^{\frac{1}{2}}}\left[B_1 J_1(s) + 3\lambda^2 B_2 \frac{J_2(s)}{s} + \ldots\right] \sin\{(\lambda^2-s^2)^{\frac{1}{2}}|y|\}\, U(\lambda-s) \\ & + \left[-(A_1-B_1)J_1(s) - 3\lambda^2(A_2-B_2)\frac{J_2(s)}{s} + \ldots\right] e^{-s|y|}\Big\} \cos xs\, ds\,. \end{aligned} \tag{59}$$

Without going into any details, the bending and extensional stresses may be computed from Eqs. (58) and (59). The results are:

Bending stresses: on the surface $Z=(h/2)$

$$\sigma_{x_b} = P_b(c/2r)^{\frac{1}{2}}\left(-\frac{3-3v}{4}\cos\frac{\theta}{2} - \frac{1-v}{4}\cos\frac{5\theta}{2}\right)\cos(\omega t+\phi) + 0(r^0) \tag{60}$$

$$\sigma_{y_b} = P_b(c/2r)^{\frac{1}{2}}\left(\frac{11+5v}{4}\cos\frac{\theta}{2} + \frac{1-v}{4}\cos\frac{5\theta}{2}\right)\cos(\omega t+\phi) + 0(r^0) \tag{61}$$

*The terms leading to non-singular stresses have been omitted.

$$\tau_{xy_b} = P_b(c/2r)^{\frac{1}{2}} \left(-\frac{7+\nu}{4} \sin\frac{\theta}{2} - \frac{1-\nu}{4} \sin\frac{5\theta}{2}\right) \cos(\omega t+\phi) + 0(r^0) \tag{62}$$

where

$$P_b = \frac{\lambda^2 Eh(A_1 - B_1)}{4(1-\nu^2)c^2} = -\frac{\lambda^4 Rn_0}{2(1-\nu^2)c^4(4-\nu_0)}\left\{\frac{8-7\nu_0}{32} + \frac{4-3\nu_0}{8}\gamma + \frac{4-3\nu_0}{16}\ln\frac{\lambda^2}{16}\right\} + \frac{m_0 Eh}{2(1-\nu^2)c^2(4-\nu_0)} + 0(\lambda^4 \ln \lambda). \tag{63}$$

Extensional stresses: through the thickness

$$\sigma_{x_e} = P_e(c/2r)^{\frac{1}{2}} \left(\frac{3}{4}\cos\frac{\theta}{2} + \frac{1}{4}\cos\frac{5\theta}{2}\right) \cos(\omega t+\phi) + 0(r^0) \tag{64}$$

$$\sigma_{y_e} = P_e(c/2r)^{\frac{1}{2}} \left(\frac{5}{4}\cos\frac{\theta}{2} - \frac{1}{4}\cos\frac{5\theta}{2}\right) \cos(\omega t+\phi) + 0(r^0) \tag{65}$$

$$\tau_{xy_e} = P_e(c/2r)^{\frac{1}{2}} \left(\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\cos\frac{5\theta}{2}\right) \cos(\omega t+\phi) + 0(r^0), \tag{66}$$

where

$$P_e = \frac{E}{2R}(A_1+B_1) = \frac{n_0}{hc^2} + \frac{m_0 E}{R(4-\nu_0)}\left\{\frac{7}{32} + \frac{3\gamma}{8} + \frac{3}{16}\ln\frac{\lambda^2}{16}\right\} + 0(\lambda^4 \ln \lambda). \tag{67}$$

Case II $\lambda^4 = 0$

One can simply investigate this case from the previous results by letting $\lambda \to 0$. It is easily seen from Eq. (67) that the stress coefficient P_e becomes infinite. This, however, is to be expected for

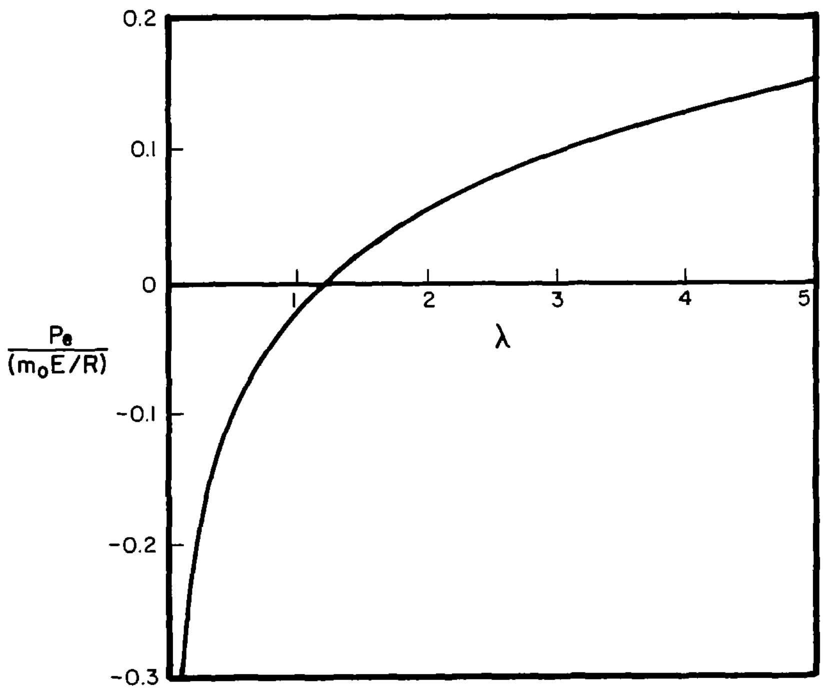

Figure 2. Stress intensity factor *vs.* λ for $n_0 = 0$.

when $\lambda=0$ (*i.e.* $\omega=\{(E/\rho)(1/R^2)\}^{\frac{1}{2}}$) corresponds to the natural frequency of the uncracked shell [11].

A plot of the stress coefficient P_e, given by Eq. (67), for $n_0=0$ and various values of λ is given in Fig. 2.

Case III $\lambda^4<0$

For $\lambda^4<0$ the integral representations for $\tilde{W}^{(c)}$ and $\tilde{F}^{(c)}$ are exactly the same as those of the non-vibrating cracked spherical shell. The stresses, therefore, from Ref. [1] are given by:

Bending stresses: on the surface $Z=(h/2)$

$$\sigma_{x_b}=P_b(c/2r)^{\frac{1}{2}}\left(-\frac{3-3\nu}{4}\cos\frac{\theta}{2}-\frac{1-\nu}{4}\cos\frac{5\theta}{2}\right)\cos(\omega t+\phi)+0(r^0) \tag{68}$$

$$\sigma_{y_b}=P_b(c/2r)^{\frac{1}{2}}\left(\frac{11+5\nu}{4}\cos\frac{\theta}{2}+\frac{1-\nu}{4}\cos\frac{5\theta}{2}\right)\cos(\omega t+\phi)+0(r^0) \tag{69}$$

$$\tau_{xy_b}=P_b(c/2r)^{\frac{1}{2}}\left(-\frac{7+\nu}{4}\sin\frac{\theta}{2}-\frac{1-\nu}{4}\sin\frac{5\theta}{2}\right)\cos(\omega t+\phi)+0(r^0)\,, \tag{70}$$

where

$$P_b=\frac{\alpha^2\lambda^2Eh(A_1-B_1)}{4(1-\nu^2)c^2}=-\frac{\lambda^4Rn_0}{2(1-\nu^2)c^4(4-\nu_0)}\left\{\frac{8-7\nu_0}{32}+\frac{4-3\nu_0}{8}+\frac{4-3\nu_0}{16}\ln\frac{\lambda^2}{16}\right\}$$
$$+\frac{m_0Eh}{2(1-\nu^2)c^2(4-\nu_0)}\left\{1+\frac{\pi\lambda^2}{32}\frac{4-3\nu_0}{4-\nu_0}\right\}. \tag{71}$$

Extensional stresses: through the thickness

$$\sigma_{x_e}=P_e(c/2r)^{\frac{1}{2}}\left(\frac{3}{4}\cos\frac{\theta}{2}+\frac{1}{4}\cos\frac{5\theta}{2}\right)\cos(\omega t+\phi)+0(r^0) \tag{72}$$

$$\sigma_{y_e}=P_e(c/2r)^{\frac{1}{2}}\left(\frac{5}{4}\cos\frac{\theta}{2}-\frac{1}{4}\cos\frac{5\theta}{2}\right)\cos(\omega t+\phi)+0(r^0) \tag{73}$$

$$\tau_{xy_e}=P_e(c/2r)^{\frac{1}{2}}\left(-\frac{1}{4}\sin\frac{\theta}{2}+\frac{1}{4}\cos\frac{5\theta}{2}\right)\cos(\omega t+\phi)+0(r^0)\,, \tag{74}$$

where

$$P_e=\frac{E(A_1+B_1)}{2R}=\frac{n_0}{hc}\left\{1+\frac{3\pi}{32}\lambda^2\right\}+\frac{m_0E}{R(4-\nu_0)}\left\{\frac{7}{32}+\frac{3\gamma}{8}+\frac{3}{16}\ln\frac{\lambda^2}{16}\right\}+0(\lambda^4\ln\lambda)\,. \tag{75}$$

As a practical matter, it is of some value to compare the dynamic with the static stress along the line of crack prolongation. For $c=1$ in., $h=0.1$ in., $R=32.6$ in., $\nu=1/3$, $E=16\times10^6$ psi, $\rho=0.315$ lbf/in^3

(i) for $n_0\neq0$, $m_0=0$:

$$\frac{\sigma_{y_{\text{dynamic}}}}{\sigma_{y_{\text{static}}}}=\begin{cases}\left[0.67(1+0.29\lambda^2)-1.24\lambda^4\left(0.25+0.13\ln\frac{\lambda^2}{16}\right)\right]\cos(\omega t+\phi)\,; & \lambda^4<0\\ \left[0.67-1.24\lambda^2\left(0.25+0.13\ln\frac{\lambda^2}{16}\right)\right]\cos(\omega t+\phi)\,; & \lambda^4>0\,;\end{cases} \tag{76}$$

(ii) for $n_0=0$, $m_0\neq0$:

$$\frac{\sigma_{y_{\text{dynamic}}}}{\sigma_{y_{\text{static}}}} = \begin{cases} \left[0.87(1+0.18\lambda^2)+0.14\left(0.43+0.19\ \ln\frac{\lambda^2}{16}\right)\right]\cos(\omega t+\phi)\ ; & \lambda^4<0 \\ \left[0.87+0.14\left(0.43+0.19\ \ln\frac{\lambda^2}{16}\right)\right]\cos(\omega t+\phi)\ ; & \lambda^4>0\,, \end{cases} \tag{77}$$

where $\lambda^4 = 2.1\times10^{-5}\,\omega^2-1$. The plots of the ratio

$$I = \frac{\sigma_{y_{\text{dynamic}}}}{\sigma_{y_{\text{static}}}\cdot\cos(\omega t+\phi)}$$

for various values of ω are given in Figs. 3 and 4.

It is evident from the figures that the general effect of the transverse vibrations on the shell is to reduce the stresses in the neighborhood of the crack point. However, when $\omega\to[(E/\rho)^{\frac{1}{2}}/R$, which is the natural frequency of the uncracked shell, the stress intensity factor becomes infinite.

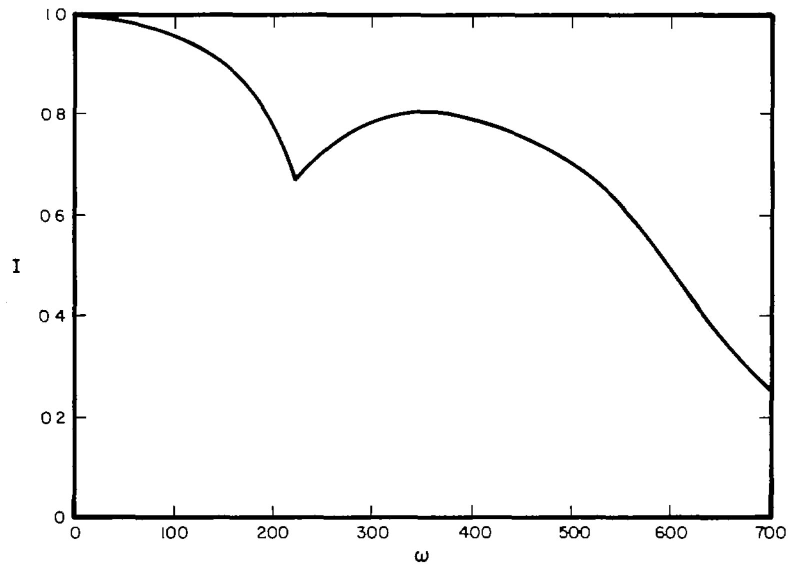

Figure 3. Ratio of dynamic and static stresses *vs.* ω for $m_0=0$.

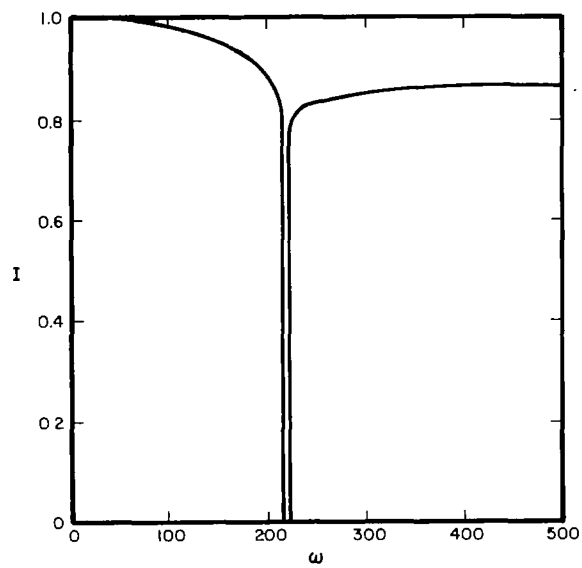

Figure 4. Ratio of dynamic and static stresses *vs.* ω for $n_0=0$.

Conclusions

The local stresses near the crack point are found to be proportional to the $\sqrt{c/r}$ which is characteristic for crack problems. Furthermore, the angular distribution around the crack tip is exactly the same as that of a flat sheet, and the stress intensity factors are functions of the shell geometry and the forcing frequency ω. From the solution the following special limiting cases of practical interest can be examined:

1. If $\omega \to 0$ and $R \neq \infty$, the stresses of a nonvibrating cracked spherical shell are recovered and coincide with those obtained in Ref. [1].
2. If $\omega \neq 0$ and $R \to \infty$, we recover the vibrating cracked plate expressions in Reference [9].
3. If $\omega = 0$ and $R \to \infty$, the stresses of a flat sheet are recovered and coincide with those obtained previously for bending [13] and extension [14].
4. If $\lambda \to 0$, i.e. when the forcing frequency ω reaches the matural frequency $(E/\rho)^{\frac{1}{2}}(1/R)$ of the uncracked shell, the extensional stress intensity factor becomes infinite.

The analysis has shown that transverse vibrations in general reduce the stresses in the vicinity of the crack tip, except when the forcing frequency reaches the natural frequency of the uncracked shell. This fact, therefore, coupled with the inverse square root singular behavior, causes the structure to fail catastrophically.

Acknowledgements

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Appendix

I. Tables of F. C. Transforms

$$\int_0^\infty e^{-|y|s} \cos \zeta s \, ds = \frac{|y|}{|y|^2+\zeta^2} \tag{1}$$

$$\int_0^\infty s e^{-s|y|} \cos \zeta s \, ds = -\frac{1}{|y|^2+\zeta^2} + \frac{2|y|^2}{(|y|^2+\zeta^2)^2} \tag{2}$$

$$\int_\lambda^\infty \frac{s \sin \zeta s}{(s^2-\lambda^2)^{\frac{1}{2}}} ds = -\frac{\lambda\pi}{2} N_1(\lambda\zeta) \tag{3}$$

$$\int_\lambda^\infty \frac{s^3 \sin \zeta s}{(s^2-\lambda^2)^{\frac{1}{2}}} ds = -\frac{\lambda^2\pi}{2\zeta} N_0(\lambda\zeta) - \frac{\lambda^3\pi}{2} N_1(\lambda\zeta) + \frac{\lambda\pi}{\zeta^2} N_1(\lambda\zeta) \tag{4}$$

$$\int_\lambda^\infty \frac{\cos \zeta s}{(s^2-\lambda^2)^{\frac{1}{2}}} ds = -\frac{\pi}{2} N_0(\lambda\zeta) \tag{5}$$

$$\int_\lambda^\infty \frac{s^2 \cos \zeta s}{(s^2-\lambda^2)^{\frac{1}{2}}} ds = -\frac{\lambda^2\pi}{2}\left\{N_0(\lambda\zeta) - \frac{N_1(\lambda\zeta)}{\lambda\zeta}\right\} \tag{6}$$

$$\int_{-\infty}^\infty \frac{e^{i\zeta s}}{(s^2+\lambda^2)^{\frac{1}{2}}} = 2K_0(\lambda\zeta) \tag{7}$$

$$\int_{-\infty}^\infty \frac{s e^{i\zeta s}}{(s^2+\lambda^2)^{\frac{1}{2}}} ds = 2i\lambda K_1(\lambda\zeta) \tag{8}$$

$$\int_{\infty}^\infty \frac{s^3 e^{i\zeta s}}{(s^2+\lambda^2)^{\frac{1}{2}}} ds = \frac{2i\lambda^2}{\zeta} K_0(\lambda\zeta) - 2i\lambda^3 K_1(\lambda\zeta) - \frac{4i\lambda}{\zeta^2} K_1(\lambda\zeta) \tag{9}$$

II Asmptotic expansions of K_n and N_n; for small arguments Z the expansions are:

$$K_0(Z) = -\left(\gamma+\ln\frac{Z}{2}\right)\left[1+\left(\frac{Z}{2}\right)+\frac{1}{2^2}\left(\frac{Z}{2}\right)^4+\left(\frac{Z}{2}\right)^2\frac{1}{(1!)^2}+\right.$$
$$+\frac{3}{2\cdot(2!)^2}\left(\frac{Z}{2}\right)^4+0(Z^6\ln Z) \tag{10}$$

$$K_1(Z) = \frac{1}{Z}+\left(\gamma+\ln\frac{Z}{2}\right)\left[\frac{Z}{2}+\left(\frac{Z}{2}\right)^3\frac{1}{1^2\cdot 2}+\left(\frac{Z}{2}\right)^5\frac{1}{1^2\cdot 2^2\cdot 3}\right]-\tfrac{1}{2}\left(\frac{Z}{2}\right)$$
$$-\frac{5}{4\cdot 2!}\left(\frac{Z}{2}\right)^3-\frac{10}{6}\frac{1}{2!3!}\left(\frac{Z}{2}\right)^5+0(Z^7\ln Z) \tag{11}$$

$$\pi N_0(Z) = \left(\ln\frac{Z}{2}+\gamma\right)\left[2-\frac{Z^2}{2}+\frac{Z^4}{2^5}+\ldots\right]+\frac{Z^2}{2}-\frac{3Z^4}{2^6}+0(Z^6\ln Z) \tag{12}$$

$$\pi N_1(Z) = \left(\ln\frac{Z}{2}+\gamma\right)\left[Z-\frac{Z^3}{2^3}+\frac{Z^5}{2^6\cdot 3}+\ldots\right]-\frac{2}{Z}-\frac{Z}{2}+\frac{5Z^3}{2^5}-\frac{5Z^5}{2^6 3^2}+0(Z^7\ln Z) \tag{13}$$

III. Tables of proper and improper integrals

$$\text{C.P.V.}\int_{-1}^{1}\frac{(1-\xi^2)^{\frac{1}{2}}}{x-\xi}\,d\xi = \pi x \tag{14}$$

$$\text{C.P.V.}\int_{-1}^{1}\frac{(1-\xi^2)^{\frac{3}{2}}}{x-\xi}\,d\xi = \pi(\tfrac{3}{2}x-x^3) \tag{15}$$

$$\text{C.P.V.}\int_{-1}^{1}\frac{(1-\xi^2)^{\frac{3}{2}}}{x-\xi}\,d\xi = \pi(\tfrac{3}{2}x-x^3) \tag{15}$$

$$\text{C.P.V.}\int_{-1}^{1}\frac{(1-\xi^2)^{\frac{5}{2}}}{x-\xi}\,d\xi = \pi(\tfrac{15}{8}x-\tfrac{5}{2}x^3+x^5) \tag{16}$$

$$\text{C.P.V.}\int_{-1}^{1}(1-\xi^2)^{\frac{1}{2}}(x-\xi)\ln\frac{\lambda|x-\xi|}{2}\,d\xi = \frac{\pi}{4}\left(1+\ln\frac{\lambda^2}{16}\right)x+\frac{\pi}{6}x^3 \tag{17}$$

$$\text{C.P.V.}\int_{-1}^{1}(1-\xi^2)^{\frac{1}{2}}(x-\xi)^3\ln\frac{\lambda|x-\xi|}{2}\,d\xi = \pi\left(\frac{3}{32}-\frac{1}{4}\ln\frac{\lambda}{4\sqrt{2}}\right)x+\frac{\pi}{4}x^3+\frac{\pi}{20}x^5 \tag{18}$$

IV. Some integrals of the bessel functions $J_1(s)$

$$\int_0^\infty J_1(s)e^{-s|y|}\cos xs\,ds = R_e\int_0^\infty J_1(s)e^{-s(|y|-ix)}ds = (2\varepsilon)^{-\frac{1}{2}}\cos\frac{\theta}{2}+\ldots \tag{19}$$

$$\int_0^\infty J_1(s)e^{-s|y|}\sin xs\,ds = -(2\varepsilon)^{-\frac{1}{2}}\sin\frac{\theta}{2}+\ldots \tag{20}$$

$$|y|\int_0^\infty sJ_1(s)e^{-s|y|}\cos xs\,ds = \tfrac{1}{4}(2\varepsilon)^{-\frac{1}{2}}\left[\cos\frac{\theta}{2}-\cos\frac{5\theta}{2}\right]+\ldots \tag{21}$$

$$|y|\int_0^\infty sJ_1(s)e^{-s|y|}\sin xs\,ds = -\tfrac{1}{4}(2\varepsilon)^{-\frac{1}{2}}\left[\sin\frac{\theta}{2}-\sin\frac{5\theta}{2}\right]+\ldots \tag{22}$$

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RÉSUMÉ

On analyse le problème de l'enveloppe sphérique ayant une fissure de longeur $2c$ de part en part de son épaisseur, et sujette à des vibrations transversales de pulsation ω; on résoud ce problème à l'aide de fonctions intégrales, pour les contraintes coplanaires et les contraintes de flexion de Kirchhoff.

On retrouve le comportement singulier habituel d'ordre 1/2, caractéristique des problèmes de fissuration. En outre, on trouve que des vibrations transversales ont tendance à réduire les contraintes au voisinage de l'extrémité des fissures, sous réserve que leur fréquence ω atteigne la fréquence naturelle de l'enveloppe non fissurée; dans ces conditions les contraintes deviennent en effet infinies.

ZUSAMMENFASSUNG

Das Problem einer sphärischen Hülle, mit einem sich über die gesamte Dicke der Hülle hinziehenden Riß der Länge $2c$, welche Querschwingungen mit einer Pulsierung unterworfen ist, wurde für die Fälle von koplanaren und von Kirchhoff-Biegebeanspruchungen mit Hilfe einer Integralformulierung gelöst. Hierbei ergab sich wiederum das für Rißprobleme charakteristische Gesetz der umgekehrten Quadratwurzel.

Außerdem zeigte sich, das Querschwingungen die Spannungen in der Umgebung der Rißspitze vermindern, ausgenommen der Fall, wo die Frequenz ω der aufgezwungenen Schwingung mit der Eigenfrequenz der unbeschädigten Hülle übereinstimmt, wo sie dann ins Unendliche ansteigen.