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2 Cooley, M. D. A., and O'Neill, M. E., "On the Slow Motion of Two Spheres in Contact Along Their Line of Centers Through a Viscous Fluid," *Proceedings of the Cambridge Philosophical Society*, Vol. 66, 1969, pp. 407-415.

3 Goren, S. L., "The Normal Force Exerted by Creeping Flow on a Small Sphere Touching a Plane," *Journal of Fluid Mechanics*, Vol. 41, 1970, pp. 619-625.

4 Reed, L. D., and Morrison, F. A., Jr., "The Slow Motion of Two Touching Fluid Spheres Along Their Line of Centers," *International Journal of Multiphase Flow*, Vol. 1, 1974, pp. 573-584.

5 Majumdar, S. R., "The Motion of Two Spheres in Contact Parallel to the Common Line of Centers in an Incompressible, Homogeneous Nonviscous Fluid," *Buletinul Institutului Politehnic din Iasi*, Vol. 7, 1961, pp. 51-56.

Authors' Closure

We thank Professor Morrison for his interest in our work and are happy to hear that he plans to continue this study.

As is well known both series and integral representations may be used to express a solution to Laplace's equation in both finite and infinite regions (see Sneddon [6][4] or any text on partial differential equations for further details). Indeed, some of the most elementary solutions in classical fluid mechanics, e.g., potential flow over a sphere in an infinite fluid [6, Chapter 4] are obtained by means of series. We, therefore, look forward to seeing Professor Morrison's analysis which should serve as a useful check on our results.

We are also grateful for his calling to our attention Majumdar's paper [7], which we were not familiar with, and to this date have not been able to locate.

Additional References

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7 Majumdar, S. R., "The Motion of Two Spheres in Contact Parallel to the Common Line of Centers in an Incompressible, Homogeneous Nonviscous Fluid," *Buletinul Institutului Politechnic din Iasi*, Vol. 7, 1961, pp. 51-56.

[4] Numbers in brackets designate References at end of Closure.

On the Three-Dimensional Theory of Cracked Plates[1]

J. P. Benthem[2] and W. T. Koiter.[3] The problem of the present paper is both extremely important and enormously difficult. The author has undertaken a gigantic task in his attempt to achieve an analytic solution. Unfortunately, the results obtained are highly questionable. This is undoubtedly due to a lack of justification for his method of analysis in many places. Moreover, the results obtained, qualified as "unorthodox" by the author, are contradicted by physical intuition and reasoning, as well as by experimental evidence. These general criticisms are substantiated by the following detailed comments:

1 The author applies Lure's symbolic method of solution for an elastic layer $-\infty < x,y < \infty$, $|z| \leq h$ (the author's reference [1]), a *generalisation of Heaviside's symbolic analysis.* Whereas Heaviside's method may be justified by Laplace transforms, Lure's method may be justified by double Fourier transforms. This justification, however, applies to the *infinite* layer (without internal holes or cracks), and the author has *not* investigated whether the method may also be justified in the presence of internal cracks.

2 The author's basic results are the displacements and stresses (52)-(61) in the form of infinite integrals of doubly or triply infinite series. It has not been verified that these results are meaningful.

3 The method of solution of the doubly infinite system of equations (72) for the coefficients $A_\nu^{(k)}$, $\nu = 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$ by truncating the system at $k = 0$ has not been justified.

4 The results (90)–(95) show that the stress-intensity factor over the thickness of the plate behaves like $1/(1 - z/h)^{2\nu} + 1/(1 + z/h)^{2\nu}$ (except near the end points $z/h = 1$), where ν is Poisson's ratio. Thus the *distribution* of the stress-intensity factor over the plate thickness is *independent* of the ratio c/h of crack width $2c$ and plate thickness $2h$. This simple result is too good to be true.

5 Near the corner points $x = \pm c$, $y = 0$, $z = \pm h$ the stresses and displacements have common factors $\rho^{-1/2-2\nu}$ and $\rho^{1/2-2\nu}$, respectively, where ρ is the distance to the corner point in question and ν is Poisson's ratio. The fact that the displacements are even unbounded for $\rho \to 0$ in the cases where $\nu > 1/4$ seems to be incompatible with sound physical intuition. It is true that linear elasticity is no longer valid in this case, but this theory indicates that the displacements should be very large.

6 The strain energy (127) is unbounded for $\nu \to 1/2$ (i.e., compression modulus $\to \infty$, Young's modulus remains finite). This result cannot be accepted. Besides, from the physical *and* mathematical point of view there is hardly any reason why $\nu = 1/2$ should be such an exceptional value in the present problem where only stresses are prescribed and no displacements.

7 The three-dimensional analysis gives at the plate surface a high increase of stresses compared with the usual two-dimensional analysis of a state of plane strain or generalized plane stress. This is in contradistinction to the results for a plate with a circular cylindrical hole [1][4] and author's reference [5]. In that case an important stress *relief* at the plate surfaces is achieved, in accordance with physical intuition.

8 The author's results along the crack front through the plate thickness are contradicted by available theoretical *and* experimental evidence [a.o. 2, 3]. The results near the corner points are contradicted by such evidences in [4].

References

1 Youngdahl, C. K., and Sternberg, E., "Three-Dimensional Stress Concentration Around a Cylindrical Hole in a Semi-Infinite Elastic Body," JOURNAL OF APPLIED MECHANICS, Vol. 33, TRANS. ASME, Vol. 88, Series E, Dec. 1966, p. 855.

2 Hartranft, R. J., and Sih, G. C., "An Approximate Three-Dimensional Theory of Plates With Application to Crack Problems," *International Journal of Engineering Sciences*, Vol. 8, 1970, pp. 711-729.

3 Villareal, G., Sih, G. C., and Hartranft, R. J., "Photoelastic Investigation of a Thick Plate With a Transverse Crack," JOURNAL OF APPLIED MECHANICS, Vol. 42, TRANS. ASME, Vol. 97, Series E, Mar. 1975, pp. 9-14.

4 Benthem, J. P., "Three-Dimensional State of Stress at the Vertex of a Quarter-Infinite Crack in a Half Space," Report WTHD No. 74, Department of Mechanical Engineering, Delft University of Technology, The Netherlands, Sept. 1975.

Author's Closure

I am very pleased that Professors Benthem and Koiter share my views as to the importance of this problem in the field of fracture mechanics and as to its magnitude of difficulties. However, I respectfully disagree with their comments for the following reasons.

1 It is true that Lure's method may be justified by the use of

[1] By E. S. Folias, and published in the September, 1975, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 42, No. 3, TRANS. ASME, Vol. 97, Series E, pp. 663-674.

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[3] Chair for the Theory of Stiffness, Strength and Stability of Structures, Department of Mechanical Engineering, Delft University of Technology, The Netherlands.

[4] Numbers in brackets designate References at end of Discussion.

double Fourier transforms. Although the author hasn't proven a theorem for the justification of the method (particularly as to the question of completeness), he has given it considerable thought and he believes that the method should work. The nonexistence, of course, of such a theorem is certainly no evidence that the method is not applicable. Be that as it may, the question is an important one which eventually will have to be answered.

But then Professor Benthem's work (see discussion reference [4]) may also be subject to the same criticism inasmuch as the author has not proven completeness and furthermore whether the problem admits a solution in a separable form, particularly in θ and ϕ.

2 Due to the space requirements that the Journal imposes, the author was obligated to restrict himself to a short and formal discussion of the method thus sacrificing mathematical rigor. One, however, if he so desires, may investigate the question of whether the displacement and stress functions (52)–(61) are meaningful.

3 From the statement of the paragraph following equation (77), it is clear that the exact value of the coefficients $A_\nu^{(0)}$ must be determined from equations (72) and (75). However, due to insufficient research funds, the author had no other alternative but to truncate the system (72) for $k = j = 0$. While we have no (mathematical) error analysis for the justification of the truncation per se, we have (numerically) studied the system carefully and have found plenty of evidence that equation (78) does represent the main contribution. However, it should be emphasized that equation (79) which leads to the author's three-dimensional singularities involves no truncation whatsoever. Be that as it may, the author well recognizes the fact that the coefficients $A_\nu^{(0)}$ should inevitably be computed exactly. However, we have not as yet been successful in securing the necessary research funds to complete this and other important questions of the foregoing problem.

4 All along the inner layers, the stresses are proportional to Λ which is a function of Poisson's and (c/h) ratios. Consequently the stress-intensity factor is dependent on the ratio (c/h).

5 The author does not understand comment 5. The displacements for $\nu = 1/4$ are definitely finite! Perhaps the comment refers to the stresses rather than the displacements.

The expressions for the stresses are derivable from the corresponding displacements by the process of differentiation. Thus, since for $\nu = 1/4$, the displacements behave like

$$u_i^{(c)} \sim \rho^{1/2 - 2(1/4)} + \ldots = \rho^0 + \ldots,$$

therefore upon differentiation one has

$$\sigma_{ij}^{(c)} \sim 0 + \ldots$$

Hence the stresses in the form

$$\sigma_{ij}^{(c)} \sim \rho^{-1/2 - 2\nu}$$

are valid for all $\nu \neq 1/4$.

6 I quite agree that an unbounded energy is unacceptable. In fact, the uniqueness theorem (reference [13]) specifically requires the solution to satisfy the edge condition of a local finite energy.

But this is certainly the case. I believe that the discussers have overlooked the fact that

$$\sum_{\nu=1}^{\infty} \sum_{k=0}^{\infty} A_\nu^{(k)} \ldots \sim \left(\frac{1-2\nu}{\nu}\right) \ldots,$$

as can easily be verified by equation (72). Thus the total strain energy is finite as $\nu \to 1/2$. Incidentally, it should be pointed out that for $\nu = 1/2$ Navier's equations are no longer valid (see Sokolnikoff, *Mathematical Theory of Elasticity*, P. 79).

7 It is the author's opinion that such a comparison between a circular hole and a sharp-edge-plane crack may not be valid.

8 The results of reference [2] (see discussion) are based on an approximate theory. Consequently they may or may not depict the correct trend.

As for the experimental evidence, I believe that the experimentalists are more qualified to decide as to whether it constitutes a valid contradiction to my theoretical (linear elastic, sharp-edge) results.

Professor Benthem's results, however, do constitute a contradiction. My only questions here are: (*i*) Is the solution really separable, particularly in θ and ϕ? (*ii*) Should the numerical truncation of the system for the determination of the singularity be trusted?

It appears, therefore, that a third independent analytical solution is most desirable.

I would very much like to thank Professors Benthem and Koiter for taking the time to read my work and thus comment on it.

Finally, the author would like to take this opportunity and bring to the attention of the readers the following error in his work.

The homogeneous solution of equation (82) should also include a linear term in ξ. Therefore, equation (84) becomes

$$f(\xi) = C_0 \left\{(2-\xi)^{2-2\nu} + \frac{m-1}{m} 2^{2-2\nu}\xi - \left(\frac{3m-2}{m}\right) 2^{1-2\nu}\right\} + \frac{3B}{m-1}\left(\frac{\xi^3}{3} - \frac{\xi^2}{2}\right) + \frac{(m-1)}{m^2} C_0\, 2^{-2\nu}\{(5m-2)\xi^2 - 2(2m-1)\xi^3\}$$

where

$$B = \frac{(m-1)^2}{m^3}\frac{2^{-2\nu}}{3} C_0\{12m^2 - 7m + 2\}$$

In view of this correction, the stresses are given by equations (90)–(92)

$$\frac{2}{2-2^{-2\nu}}\{(1-\zeta)^{-2\nu} + (1+\zeta)^{-2\nu} - 2^{-2\nu}\}$$

Effect of Sliding Friction on Contact Stresses in a Rectangle Compressed by Rigid Planes[1]

G. D. Gupta.[2] The authors have presented an interesting solution to the problem of compression of an elastic rectangle by rigid, rough planes, with finite coefficient of friction. This discussion primarily pertains to the nature of stress singularity at the corners of the rectangle. The authors derive the extent of adhesion (c) which depends on the values of frictional coefficient (μ), Poisson's ratio (ν), and the aspect ratio (a). In the section on Numerical Results and Discussions, they remark, "Thus, when $c \to 1$, μ tends to infinity, because the shear stress must be unbounded near $z = 1$." Contrary to what this sentence implies, it must be noted that the shear stresses near $z = 1$ are unbounded in all cases, except for $\mu = 0$ (where $c = 0$) and for $\nu = 0$. Furthermore, μ does not have to be infinite when $c \to 1$. In fact, there exists a minimum value of μ (depends on Poisson's ratio) above which c is always equal to one.

In the Appendix, the authors present a relationship (equation (57)) between the power of stress singularity (α), the coefficient of friction, and Poisson's ratio. This is also plotted in Fig. 6 of the papers. The authors make a remark, "It is interesting to note that when $\nu \to 0$, the power of stress singularity does not tend to zero. The shear stress, however, approaches zero everywhere with $\nu \to 0$." This unresolved paradox pointed out by the authors can be easily explained by recognizing the fact that equation (57) of the paper is not valid for all values of μ. In order to clarify this, let us consider the semi-infinite strip [1][3] and the finite strip problems [2] in which no sliding is allowed. In both of these references, an expression for the minimum values of the coefficient of friction has been derived above which sliding cannot occur. Let us call this

[1] By S. N. Prasad, and S. Dasgupta, and published in the September, 1975, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 42, TRANS. ASME, Vol. 97, Series E, pp. 656–662.

[2] Foster Wheeler Energy Corporation, Livingston, N. J. Assoc. Mem. ASME.

[3] Numbers in brackets designate References at end of Discussion.