On the Prediction of Fatigue Cracks at Holes

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INTRODUCTION

It is well known that the majority of fractures that occur in engineering members are due to repeated loads or fatigue. It is important, therefore, that designers have a complete understanding of this phenomenon and how to deal with it. When we speak of repeated loads we are usually referring to moving parts or members which are found in engines, turbines, pumps, motors, etc. Fatigue failure is the phenomenon of progressive cracking and, unless detected early it can lead to catastrophic failures.

The exact mechanism of the initiation of a fatigue crack is extremely complex and thus not very well understood. Nevertheless, discussions of some fatigue theories can be found in the existing literature, Broek, (1974), Hertzberg, (1983), Cherepanov, (1977), and will not be addressed here. It is more important for us to examine where, when, and under what conditions a fatigue crack is most likely to initiate. If a repeated load is large enough to cause a fatigue crack, the crack will initiate at a point of maximum stress. This maximum stress is usually due to a stress concentration often referred to as a stress riser.

Stress concentrations can occur in the interior of a member as a result of the inclusion of a foreign matter or voids in the material. They can also occur on the exterior surface of the member in the form of scratches, rust pits, machining marks or even sharp corners. There exists an overwhelming experimental evidence which points to the undesirability of the presence of such flaws and emphasizes how important it is for the engineer to take great measures to eliminate all adverse conditions which may lead to the initiation of a fatigue crack. For example, high quality control may reduce the chances of interior imperfections. Polishing of the surface of certain critical areas may also be necessary. Finally, a careful design may also reduce stress concentrations. The ultimate strategy, therefore, is to take appropriate measures so that no crack, however small, will manifest itself.

Despite careful design, practically every structure contains stress concentrations due to holes. Bolt holes and rivet holes are necessary components for structural joints. It is not surprising, therefore, that the majority of service cracks nucleate in the vicinity of a hole. The subject of eventual concern is to derive reliable design criteria which can be used to insure the structural integrity of the joints for the entire service life of the structure. In deriving such criteria, the knowledge of the three-dimensional stress concentration factor is a prerequisite.

Sternberg and Sadowsky, (1948), used a modified version of the Ritz method to find an approximate solution to the stress field of a plate weakened by a cylindrical hole. Far away from the hole, the plate was subjected to a uniform tensile load. Subsequently, Alblas, (1957), investigated the same problem whereby, assuming a certain form of the three-dimensional solution to Navier's equations, was able to express the stress field in terms of a set of complex eigenfunctions. Finally, Reiss, (1963), employed a formal asymptotic expansion method to obtain "three-dimensional" corrections to those of plane stress theory. In all of the above papers the focus of the investigation has been on diameter to thickness ratios greater than or equal to 0.5. The author suspects that this was due to the enormous mathematical difficulties that this problem presents. Be that as it may, their results showed the stress concentration factor to attain its maximum in the middle of the plate and to decrease parabolically as one approaches the, free of stress, plate surfaces.

This trend, however, does not seem to explain the experimental observations that for thin plates fatigue cracks either develop at the corner, ie., where the hole meets the free surface, or at the center of the plate. On the other hand, for relatively thick plates the crack almost always appears at the corner. Moreover, there exists no precise definition as to what constitutes a thin or a thick plate.

For this reason, Youngdahl and Sternberg, (1966), investigated the stress field in an elastic half-space with a semi-infinite transverse circular cylindrical hole. Their results showed that all of the three-dimensional effects were highly sensitive to changes in Poisson's ratio and became more pronounced at larger values of this parameter.

The purpose of this paper is two-fold. First to show that the solution is derivable from the general three-dimensional solution which the author constructed in a previous paper, Folias, (1980), and second to obtain the stress concentration factor for all diameter to thickness ratios thus bridging the gap between thin and thick plates and furthermore offering a definition of what constitutes a thick plate.

FORMULATION OF THE PROBLEM

Consider the equilibrium of a homogeneous, isotropic, linear elastic plate which occupies the space $|x| < \infty$, $|y| < \infty$ and $|z| \leq h$ and contains a cylindrical hole of radius a, whose generators are perpendicular to the boundary planes. Let the body be subjected to a uniform shear load (see Fig. 1) which is parallel to the boundingplanes.

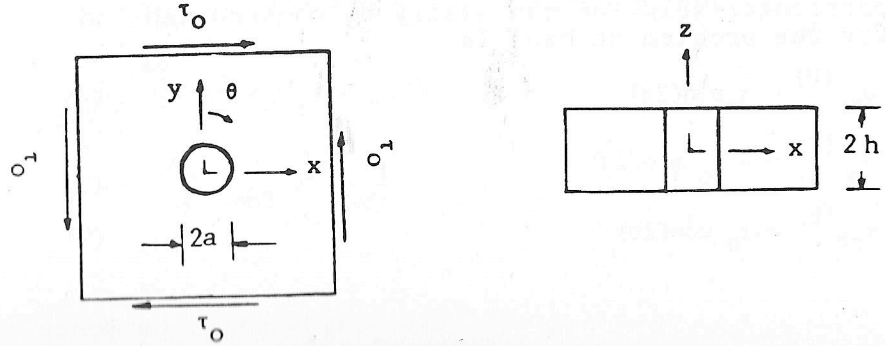

Fig. 1. Geometrical configuration and loading.

In the absence of body forces, the coupled differential equations governing the displacement functions u, v, and w are

$$\frac{m}{m-2}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)e + \nabla^2(u,v,w) = 0, \tag{1}$$

where ∇^2 is the Laplacian operator,

$m \equiv \frac{1}{\nu}$, ν is Poisson's ratio,

$$e \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \tag{2}$$

and the stress-displacement relations are given by Hooke's law as

$$\sigma_{xx} = 2G\left\{\frac{\partial u}{\partial x} + \frac{e}{m-2}\right\}, \ldots, \tau_{xy} = G\left\{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\}, \ldots \tag{3}$$

with G being the shear modulus.

As to the boundary conditions, one must require that

$$\text{as } |x| \to \infty: \quad \tau_{xy} = \tau_{xz} = \sigma_{xx} = 0 \tag{4}$$

$$\text{as } |y| \to \infty: \quad \tau_{xy} = \tau_o, \tau_{yz} = 0, \sigma_{yy} = 0 \tag{5}$$

$$\text{at } |z| = h: \quad \tau_{xz} = \tau_{xz} = \sigma_{zz} = 0. \tag{6a}$$

$$\text{at } r = a: \quad \sigma_{rr} = \tau_{r\theta} = \tau_{rz} = 0 \tag{6b}$$

In treating this type of problem it is found convenient to seek the solution into two parts, the "undisturbed" or "particular" solution which satisfies Equation (1) and the loading and support conditions but leaves residual forces along the crack, and the "complementary" solution which precisely nullifies these residuals and offers no contributions far away from the crack. Such a particular solution can easily be constructed and for the problem at hand is:

$$\sigma_{rr}^{(P)} = \tau_o \sin(2\theta) \tag{7}$$

$$\sigma_{\theta\theta}^{(P)} = -\tau_o \sin(2\theta) \tag{8}$$

$$\tau_{r\theta}^{(P)} = \tau_o \cos(2\theta) \tag{9}$$

MATHEMATICAL STATEMENT OF THE COMPLEMENTARY PROBLEM

In view of the particular solution, we need to construct three functions $u^{(c)}(x,y,z)$, $v^{(c)}(x,y,z)$, and $w^{(c)}(x,y,z)$ such that they satisfy simultaneously the partial differential Equation (1) and the following boundary condition:

$$\text{at } r = a: \quad \sigma_{rr}^{(c)} = -\sigma_{rr}^{(P)} \tag{10}$$

$$\tau_{r\theta}^{(c)} = -\tau_{r\theta}^{(P)} \tag{11}$$

$$\tau_{rz}^{(c)} = -\tau_{rz}^{(P)} \tag{12}$$

$$\text{at } |z| = h: \quad \sigma_{zz}^{(c)} = \tau_{xz}^{(c)} = \tau_{yz}^{(c)} = 0 \tag{13}$$

Furthermore, in order to complete the formulation of the problem we require that the complementary displacements and the complementary stresses do vanish as $r \to \infty$.

METHOD OF SOLUTION

A general solution to Navier's equations for plates of uniform thickness, $2h$, and with plate faces free of stress has already been constructed by the author and the results can be found in Folias, (1975 and 1980). In particular,

<u>The displacement field:</u>

$$\begin{aligned} u^{(c)} = \frac{1}{m-2}\sum_{\nu=1}^{\infty} A_\nu \frac{\partial H_\nu}{\partial x} \{&2(m-1)\cos(\beta_\nu h)\cos(\beta_\nu z) \\ &+ m\beta_\nu h \sin(\beta_\nu h)\cos(\beta_\nu z) - m\beta_\nu z \cos(\beta_\nu h)\sin(\beta_\nu z)\} \\ + \sum_{n=1}^{\infty} B_n &\frac{\partial H_n^*}{\partial y}\cos(\alpha_n h)\cos(\alpha_n z) \\ + I_1 - y &\frac{\partial I_3}{\partial x} + \frac{1}{m+1} z^2 \frac{\partial^2 I_3}{\partial x \partial y} \end{aligned} \tag{14}$$

$$v^{(c)} = \frac{1}{m-2} \sum_{\nu=1}^{\infty} A_\nu \frac{\partial H_\nu}{\partial y} \{2(m-1) \cos(\beta_\nu h) \cos(\beta_\nu z)$$

$$+ m\beta_\nu h \sin(\beta_\nu h) \cos(\beta_\nu z) - m\beta_\nu z \cos(\beta_\nu h) \sin(\beta_\nu z)\}$$

$$- \sum_{n=1}^{\infty} B_n \frac{\partial H_n^*}{\partial x} \cos(\alpha_n h) \cos(\alpha_n z) + \frac{3m-1}{m+1} I_3 + I_2$$

$$- y \frac{\partial I_3}{\partial y} - \frac{1}{m+1} z^2 \frac{\partial^2 I_3}{\partial x^2} \qquad (15)$$

$$w^{(c)} = \frac{1}{m-2} \sum_{\nu=1}^{\infty} A_\nu H_\nu \beta_\nu \{(m-2) \cos(\beta_\nu h) \sin(\beta_\nu z)$$

$$- m\beta_\nu h \sin(\beta_\nu z) - m\beta_\nu z \cos(\beta_\nu h) \cos(\beta_\nu z)\}$$

$$- \frac{2}{m+1} z \frac{\partial I_3}{\partial y} \qquad (16)$$

where A_ν and B_n are functions of β_ν and α_n, respectively,

$$\alpha_n \equiv \frac{n\pi}{h} \quad n = 1,2,3 ,\ldots, \qquad (17)$$

β_ν are the roots of the transidental equation

$$\sin(2\beta_\nu h) = (-2\beta_\nu h) \qquad (18)$$

and the functions H_ν and H_n^* satisfy the reduced wave equation, ie.

$$\frac{\partial^2 H_\nu}{\partial x^2} + \frac{\partial^2 H_\nu}{\partial y^2} - \beta_\nu^2 H_\nu = 0$$

and

$$\frac{\partial^2 H_n^*}{\partial x^2} + \frac{\partial^2 H_n^*}{\partial y^2} - \alpha_n^2 H_n^* = 0,$$

and I_1, I_2 and I_3 are, two-dimensional, harmonic functions. By direct substitution into the governing equations and the appropriate boundary conditions, one can easily show that the above displacement field does indeed satisfy Navier's equations and that the stresses $\sigma_{zz}^{(c)}$, $\tau_{xz}^{(c)}$, and $\tau_{yz}^{(c)}$ do vanish at the plate faces. Moreover, the displacement field is complete and general enough so that the remaining boundary conditions can indeed be satisfied.

A careful examination of the stress field, the governing equations, and the appropriate boundary conditions suggest the following functional representations for

$$H_\nu = \frac{K_2(\beta_\nu r)}{\beta_\nu^2} \sin(2\theta) \qquad (19)$$

$$H_n^* = \frac{K_2(\alpha_n r)}{\alpha_n^2} \cos(2\theta) \qquad (20)$$

$$I_1 = \frac{2B}{r^3} \cos(3\theta) - \frac{D}{r} \cos\theta \qquad (21)$$

$$I_2 = - \frac{2B}{r^3} \sin(3\theta) + \frac{D}{r} \sin\theta \qquad (22)$$

$$I_3 = \frac{C}{r} \sin\theta \qquad (23)$$

where $K_2(\ )$ represents the modified Bessel function of the second kind and of order two.

Finally, it remains to satisfy the boundary conditions along the surface of the hole. By taking advantage of the local coordinate system, one can show after some manipulations that the remaining boundary conditions become:

$$\frac{2}{m-2} \sum_{\nu=1}^{\infty} A_\nu K_2(\beta_\nu a) \cos(\beta_\nu h) \cos(\beta_\nu z) + \frac{1}{m-2} A_\nu \frac{\partial^2 K_2(\beta_\nu a)}{\partial(\beta_\nu a)^2}$$

$$\bullet \{2(m-1) \cos(\beta_\nu h) \cos(\beta_n z) + m\beta_\nu h \sin(\beta_\nu h) \cos(\beta_\nu z) -$$

$$-m\beta_\nu z\cos(\beta_\nu h)\sin(\beta_\nu z)\} +$$

$$+\sum_{n=1}^{\infty} B_n\left\{\frac{2}{\alpha_n a}\frac{\partial K_2(\alpha_n a)}{\partial(\alpha_n a)} - \frac{2}{(\alpha_n a)^2}K_2(\alpha_n a)\right\}$$

$$\bullet\ \cos(\alpha_n h)\cos(\alpha_n z) = -\frac{\tau_o}{2G} +$$

$$\frac{1}{m+1}\frac{6C}{a^4}z^2 - \frac{6B}{a^4} + \frac{2C}{a^2};\ |z| < h \qquad (24)$$

$$\frac{1}{m-2}\sum_{\nu=1}^{\infty} A_\nu\left\{\frac{4}{\beta_\nu a}\frac{\partial K_2(\beta_\nu a)}{\partial(\beta_\nu a)} - \frac{4}{(\beta_\nu a)}K_2(\beta_\nu a)\right\}$$

$$\bullet\ \{2(m-1)\cos(\beta_\nu h)\cos(\beta_\nu z) + m\beta_\nu h\sin(\beta_\nu h)\cos(\beta_\nu z)$$

$$-m\beta_\nu z\cos(\beta_\nu h)\sin(\beta_\nu z)\} +$$

$$+\sum_{n=1}^{\infty} B_n\left\{\frac{\partial^2 K_2(\alpha_n a)}{\partial(\alpha_n a)^2} - \frac{1}{(\alpha_n a)}\frac{\partial K_2(\alpha_n a)}{\partial(\alpha_n a)} + \frac{4}{(\alpha_n a)^2}K_2(\alpha_n a)\right\}$$

$$\bullet\ \cos(\alpha_n h)\cos(\alpha_n z) = \frac{\tau_o}{G} - \frac{12}{m+1}\frac{Cz^2}{a^4} + \frac{12B}{a^4} - \frac{2C}{a^2},\ |z| < h \qquad (25)$$

$$-\frac{2m}{m-2}\sum_{\nu=1}^{\infty} A_\nu\frac{\partial K_2(\beta_\nu a)}{\partial(\beta_\nu a)}\{\cos(\beta_\nu h)\sin(\beta_\nu z)$$

$$+\beta_\nu h\sin(\beta_\nu h)\sin(\beta_\nu z)$$

$$+\beta_\nu z\cos(\beta_\nu h)\cos(\beta_\nu z)\} - \sum_{n=1}^{\infty} B_n\frac{2}{(\alpha_n a)}\cdot$$

$$\bullet\ K_2(\alpha_n a)\cos(\alpha_n h)\sin(\alpha_n z) = 0;\ |z| < h. \qquad (26)$$

It is interesting to note that the θ-dependence has now been eliminated and that it remains to solve for the unknown complex coefficients A_ν, and for the real coefficients B_n. Moreover, the right hand side constants B and C can be shown to be functions of A_ν and B_n, which in turn are proportional to:

$$\sim \frac{\nu}{1+\nu}, \qquad (27)$$

Thus:

$$\lim_{\nu\to o}\sigma_{zz} = o \qquad (28)$$

$$\lim_{\nu\to o}\sigma_{\theta\theta}\Big|_{\substack{r=\alpha\\ \theta=-\pi/4}} = 4\,\tau_o. \qquad (29)$$

These results are compatible with our expectations for they represent the results of an exact solution.

DISCUSSION OF THE RESULTS

Without going into the mathematical details the system of equations (24) through (26) was solved numerically, and the stress concentration factor at $\theta = -\pi/4$ and $r = a$ was computed for various radius to thickness ratios, (a/h), and for Poisson's ratio of $\nu = 0.3$. All numerical work was carried out in double precision and a sophisticated algorithm was used for the evaluation of the Bessel modified function of the second kind. The results are shown in figures 2 through 6.

It is observed that at least for the ratios of $(a/h) > 0.5$ the stress concentration factor attains its maximum in the middle of the plate and it decreases as one approaches the free surfaces. On the other hand, for ratios of $(a/h) \leqslant 0.2$ the stress concentration factor attains its maximum close to the plate faces. For example, for $(a/h) = 0.05$ the stress concentration factor attains its maximum at $(z/h) = 0.95$ where its magnitude is by 11% higher than that of the center of the plate.

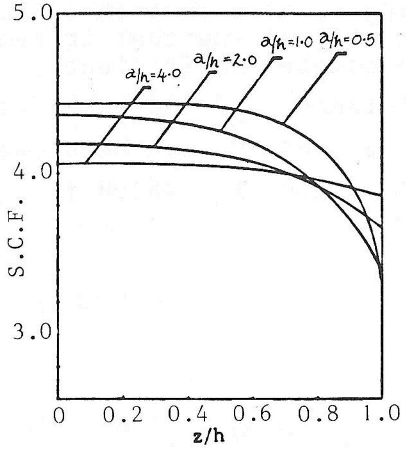

Fig. 2. Stress concentration factor through the thickness for Poisson's ratio of $\nu = 0.3$. .

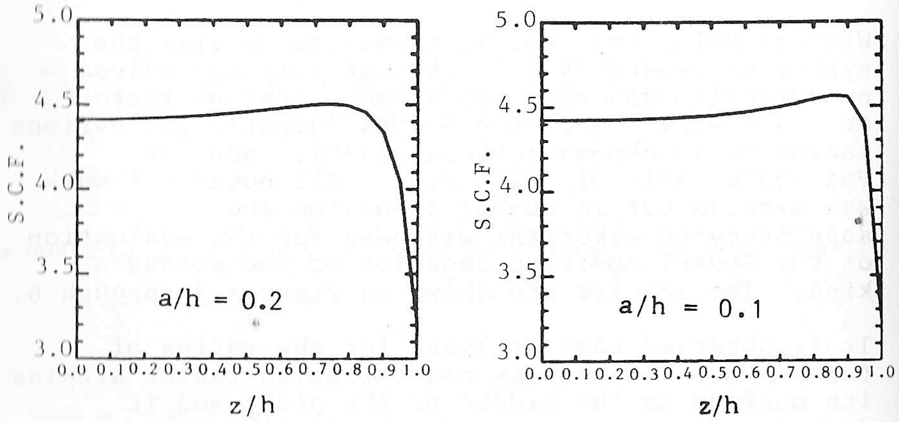

Fig. 3 Stress concentration factor through the thickness of Poisson's ratio of $\nu = 0.3$

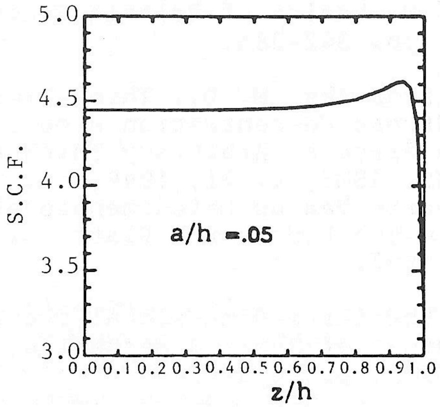

Fig. 4 Stress concentration factor through the thickness for Poisson's ratio of $\nu = 0.3$

As a practical matter, one may draw the following conclusions:

1. For ratios $a/h > 0.3$ and all other conditions being equal, fatigue cracks are most likely to appear at the center of the plate.
2. For ratios $a/h \leq 0.2$ fatigue cracks are most likely to appear close to the free surfaces of the plate, e.g., see Broek, (1974).
3. The fatigue life of the member may be substantially shorter than that predicted by the two-dimensional elasticity theory.

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