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# RECENT ADVANCES ON 3D ELASTICITY PROBLEMS RELATED TO FRACTURE

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ABSTRACT

This paper discusses and establishes a general three-dimensional analytical solution for the equilibrium of a linear elastic plate of uniform thickness, 2h, and with plate faces free of stress. This general solution can now be used to solve a whole class of three-dimensional elasticity problems, e.g., a plate weakened by a circular hole, an elliptical hole, a crack, an inclusion etc.

Results for the special case of a plate with a circular hole are also presented and shown to be derivable directly from the general solution. Special attention is given to the neighborhood of the intersection of the hole and the free surface where the solution is examined for possible stress singularities. Finally, answers to some fundamental questions pertaining to a 3D Griffith crack are suggested.

INTRODUCTION

There exist in the literature a considerable number of papers which deal with the stress distribution across the thickness of a plate, both from an analytical as well as finite elements point of view. However, most of these studies have focused on two dimensional considerations. This is because analytical solutions to the full three dimensional linear equations of elasticity are difficult to obtain, and three dimensional numerical studies generally require a large amount of computer resources. A somewhat complete historical discussion to the three-dimensional Griffith crack problem is given by Burton et al. (1984).

Be that as it may, the effect that a specimen thickness has on the mechanism of failure, so far, is not very well understood. In fact, previous analyses have raised some very important questions. For example, do plane strain conditions prevail at the crack front in the interior core of a plate? Do plane stress conditions prevail at a layer adjacent to the free surfaces? What is the actual shape of the crack front immediately after deformation and prior to fracture? What is the strength of the stress singularity at the corner where the crack front meets the free of stress surface of the plate? Do stress fields associated with long cracks have the same characteristics as those associated with short cracks?

Complete answers to the above questions have so far defied researchers, yet the answers are of great importance for the complete understanding of the phenomenon of fracture. For example, it is well recognized that the inadequacy of linear elastic fracture mechanics (LEFM) to predict the behavior of short cracks to the same degree of accuracy obtained for long cracks is usually attributed to one of the following two reasons (or possibly both). These are, that either LEFM is not the appropriate analysis technique, or that other effects, not normally accounted for, are important and should be included.

Factors which are often neglected in LEFM analysis but which are likely to be important in short crack conditions are: (i) surface conditions (e.g. residual stresses due to machining, cold working or chemical finishing, and applied stresses due to fretting), (ii) three dimensional consideration, (iii) plasticity considerations (e.g. yielded zone at the crack tip as well as crack closure due to the wake of the yielded material). While undoubtedly all three categories are equally important for the proper understanding of short crack growth behavior, the consideration of realistic, 3D, specimen geometries becomes ever more essential. Thus, linear elasticity is a logical fountainhead for detailed theoretical study for it represents a relatively simple mathematical model.

In this paper, the author will present some past and recent results related to the 3D Griffith crack problem, and will try to answer some fundamental questions pertaining to 3D elasticity problems. Moreover, he will bring to the attention of the reader the existence of a general, 3D, analytical solution for the equilibrium of linear elastic plates.

## REVIEW OF AUTHOR'S PAST WORK

A general solution for the equilibrium of linear elastic plates was constructed by Folias (1975) and was then specialized to the case of a plate containing a, 3D, Griffith crack (Fig. 1). The solution was expressed in terms of integral representations which were subsequently expanded, asymptotically, in the inner core of the plate in order to obtain the displacement and stress fields (Folias, 1975):

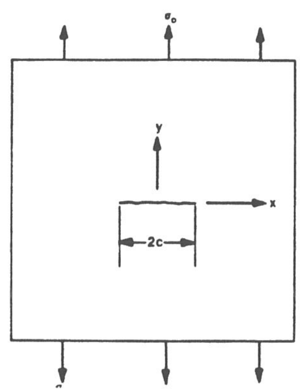


Fig. 1 Cracked plate of thickness 2h.

(i) <u>the displacement field</u>:

$$u^{(c)} = \sigma_0 \frac{\Lambda}{2G} f(z/h) \sqrt{\frac{c\varepsilon}{2}} \left\{\frac{m-2}{m} \cos\left(\frac{\phi}{2}\right) + \frac{1}{2} \sin\phi \sin\left(\frac{\phi}{2}\right)\right\} + O(\varepsilon') \tag{1}$$

$$v^{(c)} = \sigma_0 \frac{\Lambda}{2G} f(z/h) \sqrt{\frac{c\varepsilon}{2}} \left\{2\left(\frac{m-1}{m}\right) \sin\left(\frac{\phi}{2}\right) - \frac{1}{2} \sin\phi \cos\left(\frac{\phi}{2}\right)\right\} + O(\varepsilon') \tag{2}$$

$$w^{(c)} = 0 + O(\varepsilon') \tag{3}$$

(ii) the stress field:

$$\sigma_{xx}^{(c)} = \sigma_0 \Lambda\, f(z/h) \sqrt{\frac{c}{2\varepsilon}} \left\{\frac{1}{2} \cos\left(\frac{\phi}{2}\right) - \frac{1}{4} \sin\phi \sin\left(\frac{3\phi}{2}\right)\right\} + O(\varepsilon^0) \tag{4}$$

$$\sigma_{yy}^{(c)} = \sigma_0 \Lambda\, f(z/h) \sqrt{\frac{c}{2\varepsilon}} \left\{\frac{1}{2} \cos\left(\frac{\phi}{2}\right) + \frac{1}{4} \sin\phi \sin\left(\frac{3\phi}{2}\right)\right\} + O(\varepsilon^0) \tag{5}$$

$$\sigma_{zz}^{(c)} = \nu\sigma_0 \Lambda\, f(z/h) \sqrt{\frac{c}{2\varepsilon}} \cos\left(\frac{\phi}{2}\right) + O(\varepsilon^0) \tag{6}$$

$$\tau_{xy}^{(c)} = \sigma_0 \Lambda\, f(z/h) \sqrt{\frac{c}{2\varepsilon}} \left\{\frac{1}{4} \sin\phi \cos\left(\frac{3\phi}{2}\right)\right\} + O(\varepsilon^0) \tag{7}$$

$$\tau_{yz}^{(c)} = -\, \nu\sigma_0 \frac{\Lambda}{h}\, g(z/h) \sqrt{\frac{c\varepsilon}{2}} \left\{\frac{1}{2} \sin\phi \cos\left(\frac{\phi}{2}\right)\right\} + O(\varepsilon') \tag{8}$$

$$\tau_{xz}^{(c)} = \nu\sigma \frac{\Lambda}{h}\, g(z/h) \sqrt{\frac{c\varepsilon}{2}} \left\{(1-2\nu) \cos\left(\frac{\phi}{2}\right) + \frac{1}{2} \sin\phi \sin\left(\frac{\phi}{2}\right)\right\} + O(\varepsilon'), \tag{9}$$

where

$$f(z/h) = \left\{\frac{1}{(1 - \frac{z}{h})^{2\nu}} + \frac{1}{(1 + \frac{z}{h})^{2\nu}}\right\} \tag{10}$$

$$g(z/h) = \left\{\frac{1}{(1 - \frac{z}{h})^{2\nu+1}} - \frac{1}{(1 + \frac{z}{h})^{2\nu+1}}\right\}, \tag{11}$$

$\in$ and $\phi$ are the usual cylindrical coordinates, and $\Lambda$ is a function of Poisson's ratio $\nu$ and of the crack to thickness ratio $(2c/2h)$.

It should be emphasized that the above asymptotic solution is valid only in the interior core of the plate[1], i.e., for all $|z| < h$. Certain other features are also worthy of note:

(1) the stresses possess the usual $(1/\sqrt{\in})$ singular behavior
(2) the stresses possess the usual angular distribution
(3) the stress intensity factor is a function of $z$
(4) a state of pseudo[2] plane strain exists where

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[1]This is because the function $f(z/h)$ represents the dominant part of the solution of a difference-differential equation (Folias, 1980).

[2]According to the definition of plane strain, the displacements must be independent of $z$, in this case they are functions of $z$ hence the term 'pseudo' plane strain.

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \qquad (12)$$

(5) as the Poisson's ratio $\nu \to 0$, the plane stress solution is recovered
(6) as $h \to \infty$, the z-dependency is eliminated
(7) the crack opening displacement is

$$v^{(c)}_{y=0} \approx \pm \left(\frac{\sigma^{o}}{2G}\right) (1-\nu)\Lambda\, f(z/h) \sqrt{c^2 - x^2} \qquad (13)$$

Inasmuch as the solution represents an asymptotic expansion which is valid only in the inner core of the plate, naturally the stress field cannot be expected to satisfy the boundary conditions on the plate faces, i.e., at $z = \pm h$. However, Folias (1975) shows that in the neighborhood of the corner points additional terms also contribute to the same order of singularity and therefore must be accounted for[3]. In fact, the integral representations for the stresses do indeed satisfy the boundary conditions at the free surfaces.

The author, subsequently, ventured to examine (Folias, 1975) the character of the stress singularity that prevails at such corner points, i.e., the points where the crack front meets the free surfaces of the plate. After considerable effort it was concluded at the time that the displacements are proportional to $\rho^{1/2-2\nu}$ and that the stresses are proportional to $\rho^{-1/2-2\nu}$. Moreover, it was concluded that all stresses there, inclusive $\tau^{(c)}_{xz}$ and $\tau^{(c)}_{yz}$, possess the same order of singularity.

Perhaps it is appropriate at this point to note that Folias at that time considered his primary contribution to be the presentation of a systematic method for the construction of a general solution to a certain class of, 3D, elasticity problems and used the Griffith crack problem as a vehicle to show how the solution may be specialized. Instead, researchers in the field concentrated on the strength of the singularity which, because of the unbounded nature of the displacements for certain Poisson's ratios, has caused some discussion[4] (Bentham et al., 1976; Folias, 1976).

As a result, the author compiled the following list of fundamental questions that needed to be addressed:

(1) Is the solution of this notoriously difficult problem unique and if so under what conditions?
(2) Is an infinite displacement field in such regions admissible?
(3) Does the symbolic method adopted (Folias, 1975) generates a 'complete' set of eigenfunctions for the solution to Navier's equations?
(4) What is the actual order of the stress singularity at such neighborhoods?

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[3]This matter was discussed by the author in some detail at a National workshop on 3D Fracture at Battelle Memorial Institute in 1978. However, the matter has recently resurfaced (Burton et al., 1984). Referring to equation (110) (Folias, 1975) "... the terms of the same order vanish" because of the factor (h-z) which is present. The author there does not imply that the contributions from the integrals vanish.

[4]It may be noted that Folias did not exclude the possibility that the displacement field be of the form

$$u_i \sim \rho^{1/2 - 2\nu}\{f_i(\theta\phi) + g_i(\theta,\phi)\ \ln\rho\}$$

which explains the special case when $\nu = 1/4$. Incidentally, comment five of the Discussion was revised prior to printing.

(5) Is the solution at such neighborhoods separable particularly in cylindrical or spherical coordinates?
(6) Can a plate be characterized by a plane strain core sandwiched between two thin plane stress layers at the surfaces?
(7) When is a plate classified as being thick or thin?
(8) Is it appropriate to compare results between the problems of a hole and a crack?

The answers to the first two questions were given by Wilcox (1979). He was successful in proving that a displacement field which satisfies the condition of local finite energy is unique. This of course is quite a departure from our traditional 2D fracture mechanics thinking, for the displacements may now be allowed to be singular.[5] Consequently, one may not a priori assume them to be finite as it is customarily done. In general, such an assumption makes the class of solutions too restrictive and, as a result, one may not find the complete solution to the problem. On the other hand, the solution may very well lead to finite displacements everywhere! Be that as it may, physical intuition should be used with extreme caution.

The answer to the third question was given by Folias (1977) and independently by Wilcox (1978), who using a double Fourier integral transform in $x$ and $y$ and a contour integration recovered precisely the same integral representations as those reported by Folias (1975). The analysis establishes, therefore, the validity of the symbolic method and the completeness of the eigenfunctions. Thus, it remains for us to determine the order of the stress singularity which prevails at such neighborhoods.

## GENERAL SOLUTION TO NAVIER'S EQUATIONS

As it was previously noted, a general three-dimensional solution to Navier's equations for plates of uniform thickness, 2h, and with plate faces free of stress has been constructed by the author (Folias, 1975). The results, were subsequently put in a more convenient form (Folias, 1985) which makes the solution much more suitable for direct applications. Without going into the mathematical details, the expressions for the general displacement and stress fields can be found in (Folias et al., 1986).

Next, by appropriately choosing some five remaining arbitrary functions, one may now solve a whole class of 3D linear elastic problems, e.g., the problem of a cylindrical hole, an elliptical hole, a crack, a cylindrical inclusion, etc. For example, Folias et al. (1986) considered the problem of a plate with a cylindrical hole of radius a [see Fig. 2], to show how the general solution can be used to construct such a 3D stress field. The analysis showed the stress concentration factor to be sensitive to the ratio (a/h) as well as to the Poisson's ratio $\nu$. Typical results are shown in Figures 3 through 5. Certain important features are worth mentioning. For $\nu = 0.33$ and for ratios $(a/h) > 0.5$ it is found that the stress concentration factor attains its maximum in the middle core of the plate and decreases parabolically as one approaches the free surfaces. On the other hand, for $(a/h) < 0.5$, the stress concentration factor attains its maximum in the vicinity of the free surfaces. Moreover, as the ratio of a/h decreases further, the following numerical trends are observed for the stress concentration factor (i) the magnitude of the rise slowly increases (ii) the maximum occurs approximately

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[5]Consider a hemisphere with center the corner point $z = h$. The strain energy now becomes

$$w_{local} \sim \int_V \sigma_{ij}^2 \, dv \sim (1-2\nu)^2 \int_0^\rho \{\rho^{-1-4\nu} + \dots\} \, \rho^2 d\rho = \frac{(1-2\nu)}{2} \rho^{2-4\nu} + \dots$$

Thus $w_{local} \to 0$ as $\rho \to 0$.

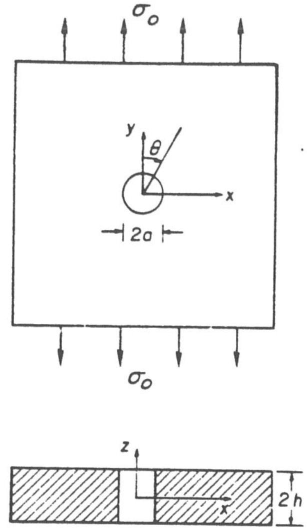


Fig. 2 Geometrical configuration of a plate weakened by a circular hole of radius a.

one hole radius away from the surface (iii) at the surface of the plate it drops rather abruptly and its magnitude slowly decreases.

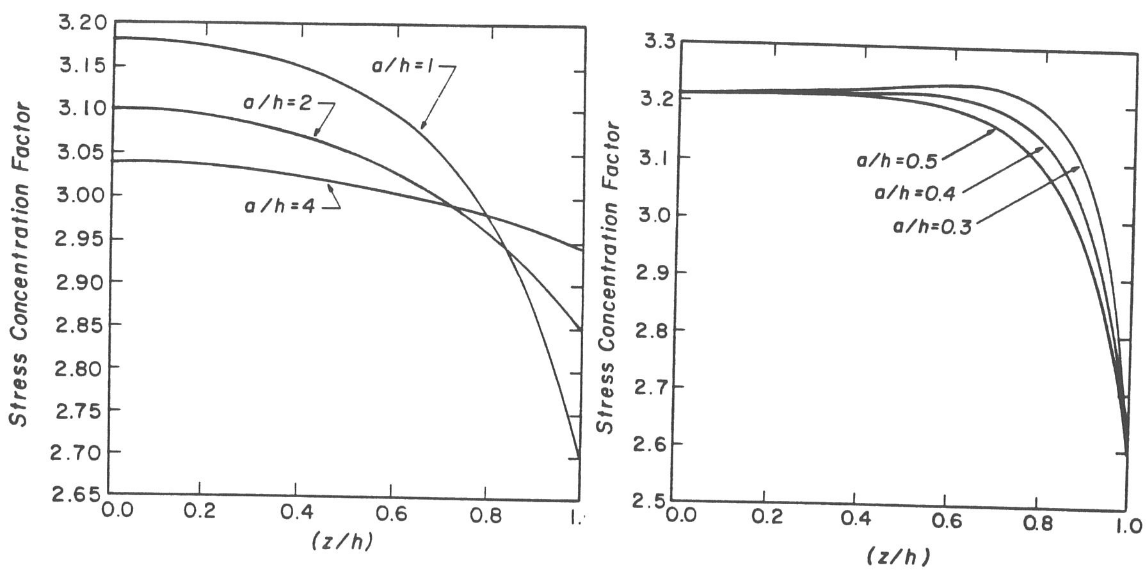


Fig. 3,4 Stress concentration factor across the thickness for Poisson's ratio $\nu = 0.33$ and various a/h ratios.

The results of Folias et al. (1986) clearly substantiate the existence of a boundary layer in the vicinity of the intersection of the hole and the free surface of the plate. The presence of this boundary layer was first reported by Youngdahl et al. (1966). In fact, if we take into account the shear loading and compare the out of the plane displacement w (see Fig. 6), we see that the agreement is very good. Similarly, for $(a/h) > 1$ the results are in agreement with those obtained by Alblas (1957) as well as by Reiss (1963). Thus the solution of Folias et al. (1986) recovers in the limit the existing in the literature results for thin as well as for thick plates.

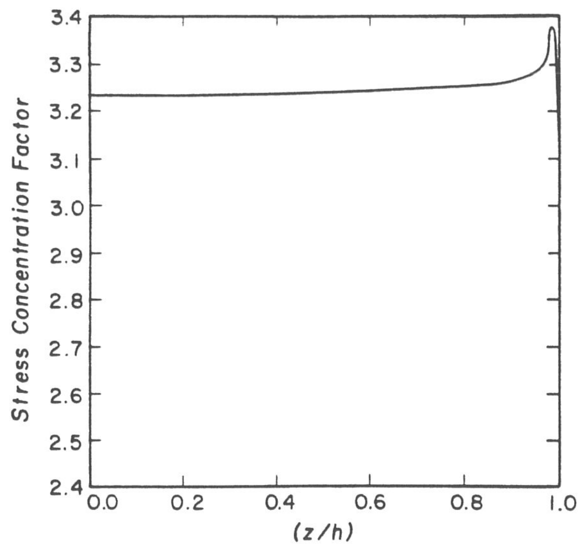


Fig. 5 Stress concentration factor across the thickness for $\nu = 0.33$ and $a/h = 0.01$.

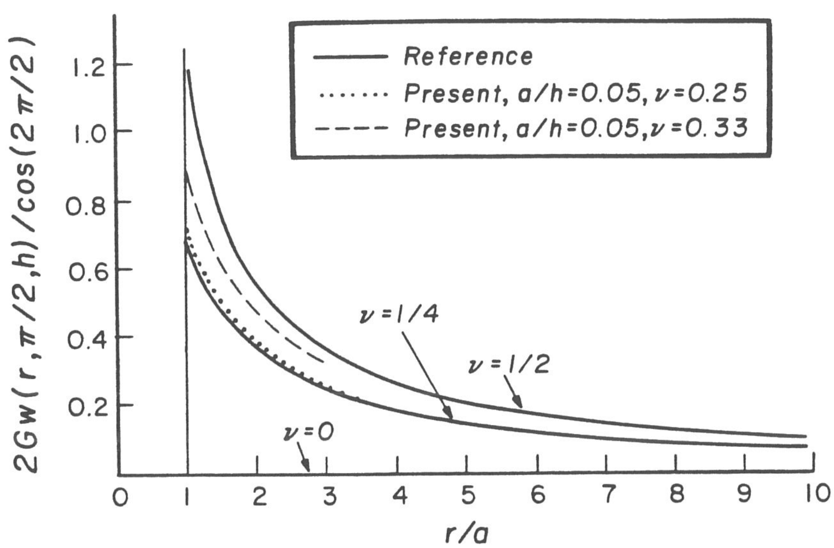


Fig. 6 Comparison of the displacement $w$ at $z = h$ with the results of Youngdahl et. al.

THE STRESS FIELD AT THE CORNER OF A HOLE

It is well recognized that at the vertex of a sector plate, in stretching or in bending, unbounded stresses may occur for certain vertex angles. In the case of 2D problems, this has been investigated analytically by Williams (1952, 1951) subject to various edge conditions. Such information is not only of academic interest but also of practical importance. However, because of the difficult mathematical nature of these type of problems, the author believes that such information should only be extracted by analytical means. By utilizing the inherent form of the general solution (Folias, 1975), it is now possible to construct explicitly the displacement and stress fields in such neighborhoods, at least for certain type of geometries.

For example, in the case of a cylindrical hole, we assume the complementary displacement field to be of the form:

$$u^{(c)} = \frac{1}{m-2} \frac{\partial}{\partial x} \{2(m-1)f_2 + mh \frac{\partial f_1}{\partial z} + mz \frac{\partial f_2}{\partial z}\} + \frac{\partial g}{\partial y} \tag{14}$$

$$v^{(c)} = \frac{1}{m-2} \frac{\partial}{\partial y} \{2(m-1)f_2 + mh \frac{\partial f_1}{\partial z} + mz \frac{\partial f_2}{\partial z}\} - \frac{\partial g}{\partial x} \tag{15}$$

$$w^{(c)} = \frac{1}{m-2} \frac{\partial}{\partial z} \{-2(m-1)f_2 + mh \frac{\partial f_1}{\partial z} + mz \frac{\partial f_2}{\partial z}\}, \tag{16}$$

where the functions $f_1$, $f_2$ and $g$ represent 3D harmonic functions and $m \equiv (1/\nu)$. Furthermore, we assume the functions to be of the form

$$f_i = r^{-1/2} H_i(r-a,\ h-z) \cos(2\theta);\ i = 1,2 \tag{17}$$

and

$$g = r^{-1/2} H_3(r-a,\ h-z) \sin(2\theta), \tag{18}$$

where

$$H_k = \sum_{n=0}^{\infty} \rho^{\alpha+n} \{A_n^k \cos(\alpha+n)\phi + B_n^k \sin(\alpha+n)\phi\};\ k = 1,2,3. \tag{19}$$

with $A_n^k$, $B_n^k$ and $\alpha$ as constants to be determined from the boundary conditions. Without going into the mathematical details (Folias, 1987a) one finds the following complimentary stress field (see Fig. 7):

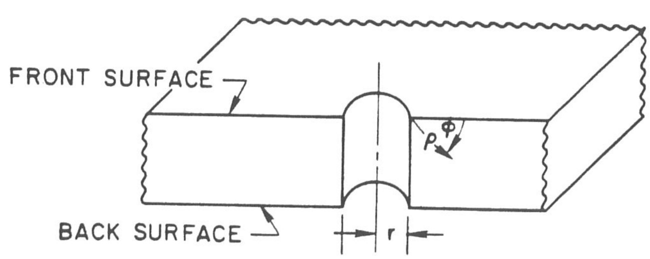


Fig. 7 Definition of local coordinates at the corner.

$$\sigma_{rr}^{(c)} = \frac{2mG}{m-2}\alpha(\alpha-1)\rho^{\alpha-2}B_0^{(2)}\{2(\frac{1-\alpha}{\alpha})\tan(\frac{a\Pi}{2})\cos(\alpha-2)\phi$$

$$+\sin(\alpha-2)\phi-(\alpha-2)\sin\phi[(\frac{1-\alpha}{\alpha})\tan(\frac{\alpha\Pi}{2})\sin(\alpha-3)\phi \tag{20}$$

$$-\cos(\alpha-3)\phi]\}\cos(2\theta)+0(\rho^{\alpha-1})$$

$$\sigma_{\theta\theta}^{(c)} = -\frac{2mG}{m-2}\alpha(\alpha-1)\rho^{\alpha-2}B_0^{(2)}\{2(\frac{m+1}{m})(\frac{1-\alpha}{m})\tan(\frac{\alpha\Pi}{2})\cos(\alpha-2)\phi$$

$$+(\frac{m+2}{m})\sin(\alpha-2)\phi-(\alpha-2)\sin\phi[(\frac{1-\alpha}{\alpha}\tan(\frac{\alpha\Pi}{2})\sin(\alpha-3)\phi \tag{21}$$

$$-\cos(\alpha-3)\phi]\}\cos(2\theta)+0(\rho^{\alpha-1})$$

$$\sigma_{zz}^{(c)} = \frac{2mG}{m-2}\alpha(\alpha-1)\rho^{\alpha-2}B_0^{(2)}\{\sin(\alpha-2)\phi+(\alpha-2)\sin\phi$$
$$[(\frac{1-\alpha}{\alpha})\tan(\frac{\alpha\Pi}{2})\sin(\alpha-3)\phi-\cos(\alpha-3)\phi]\}\cos(2\theta)+0(\rho^{\alpha-1}) \tag{22}$$

$$\tau_{rz}^{(c)} = \frac{2mG}{m-2}\alpha(\alpha-1)\rho^{\alpha-2}B_0^{(2)}\{(\frac{1-\alpha}{\alpha})\tan(\frac{\alpha\Pi}{2})\sin(\alpha-2)\phi$$

$$+(\alpha-2)\sin\phi[(\frac{1-\alpha}{\alpha})\tan(\frac{\alpha\Pi}{2})\cos(\alpha-3)\phi+\sin(\alpha-3)\phi]\}\cos(2\theta) \tag{23}$$

$$+0(\rho^{\alpha-1})$$

$$\tau_{xz}^{(c)} = \tau_{rz}\sin\theta+0(\rho^{\alpha-1}) \tag{24}$$

$$\tau_{yz}^{(c)} = \tau_{rz}\cos\theta+0(\rho^{\alpha-1}) \tag{25}$$

$$\tau_{r\theta}^{(c)} = 0(\rho^{\alpha-1}), \tag{26}$$

where the min Re $\alpha = 3.73959 \pm i1.11902$ and the coefficient $[B_0^2/m-2]$ has been shown to be (Folias et al., 1986) to be proportional to Poisson's ratio $\nu$ as well as the applied load.

It is interesting to note that the characteristic value of $\alpha$ is precisely that obtained by Williams (1952) for a 90° corner with free-free of stress boundaries. Moreover, an extension of this analysis to other angles of intersection with the free surface reveals the same results as those predicted by Williams. The results have also been extended to the case of an inclusion (Folias, 1987b) and were found to be identical to the corresponding 2D results (Boggy, 1968).

DISCUSSION

In view of our previous discussion, one may draw the following conclusions concerning the effect that specimen thickness has on the stress concentration factor:

- for ratios of $(a/h) > 0.5$ the maximum stress occurs at the middle plane
- for ratios of $(a/h) < 0.5$ the maximum stress occurs close to the free surface, approximately one radius distance away from the surface
- depending on the value of the ratio $a/h$, the fatigue life of the structure may be substantially shorter than that predicted by 2D elasticity theory
- a state of pseudo plane strain exists in the interior core of the plate where

$$\sigma_{zz} = \nu[\sigma_{rr} + \sigma_{\theta\theta}] \tag{27}$$

- at $z = 0$ the stress concentration factor attains a value which depends on the ratios of $(a/h)$ and $\nu$ and which is slightly higher than the value of 3.
- the notion that a thick plate may be characterized as a plane strain core sandwiched between two thin plane stress layers is in serious error
- there exists a highly complicated three-dimensional stress field in the neighborhood of the free surfaces of the plate
- assuming that the radius of the hole $a$ is sufficiently large, the stress field at the corner has been shown to be non-singular.

Perhaps it is noteworthy to comment on the conditions of plane stress and plane strain. For $(a/h) \to \infty$, our numerical results give precisely the value of plane stress, ie the value of 3. In the case of plane strain, the boundary planes $z = \pm h$ are thrown to infinity and simultaneously the boundary conditions must be relaxed by requiring that all stresses and displacements there be finite.

So, where do we stand in reference to the crack problem?

First of all, inasmuch as the solution of both the hole and the crack problem can be derived from the same general 3D solution, similar trends are expected to prevail. Therefore, by analogy, all of the above remarks are also applicable to the 3D Griffith crack problem, except perhaps the last remark concerning the stress singularity. This is because, in the vicinity of the free surfaces, a highly complicated 3D stress field appears to exist where the solution is not separable either in cylindrical or spherical coordinates. Thus, the strength of the actual stress singularity there remains to be established. On the other hand, in the interior core of the plate the stresses possess the usual $1/\sqrt{\epsilon}$ singular behavior.

Second, it has been observed experimentally that on the free surfaces of the plate and around the neighborhood of the crack tip, a small dip appears during the processes of deformation. As the load is applied, the material immediately tries to smooth out this ninety degree corner before any relaxation due to fracture takes place. Moreover, the larger the load is the more noticeable the dip is. As in the 2D case, large forces of atomic or molecular attraction of the order of the "theoretical strength" may prevail at such corners. These "cohesive forces," attempting to smooth out the right angle, pull the material towards the center of the plate thus forcing the crack front to 'buckle,' if you will, and tunnel its way into the center core of the plate. Naturally, this is a supposition which can only be substantiated when the explicit stress field at such neighborhoods has been established.

The time to develop 3D fracture mechanics is now upon us. We believe that the complete 3D solution to the Griffith crack problem will not only enable us to understand the mechanism of fracture propagation better, but will also expand our horizons for future research and, hopefully, will contribute to the advancement of the field to higher levels of safe design against fracture.

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