Int. J. Solids Structures Vol. 29, No. 2, pp. 145–170, 1992
Printed in Great Britain.

0020-7683/92 $3.00+ .00

EFFECT OF STRESS WAVES ON LAMINATED COMPOSITE PLATES

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(*Received* 31 *July* 1989; *in revised form* 2 *April* 1991)

Abstract—Several types of waves in a layered composite material were examined. In order to obtain explicit approximate solutions, a perturbation method was used. This approach views the nonhomogeneous layering as a perturbation on a homogeneous but anisotropic solid. For propagation normal to the direction of layering, it was necessary to include the second-order perturbation terms to get qualitative agreement. Also considered were waves which travel parallel to the laminae. The material in these waves follows an elliptical path, the orientation of which depends on the position in the material. All these waves are also examined numerically, and there is excellent agreement between the perturbation expansions and the numerical results.

INTRODUCTION

The purpose of this paper is to present a perturbation approach in dealing with layered materials. The approach is simple to use and, for a particular carbon fiber composite examined, yields results which agree very well with various numerical approaches. The basic idea is to write the layering effect on the material constants in terms of a Fourier series, solve the averaged (homogenized) problem, and then take a truncated series solution of the rest of the resulting first-order perturbation problem. In this introduction, we mention some of the previous work, and then outline the results of this paper.

Brillouin (1946) did much early work in waves in layered media. Though he did not look at waves in an elastic solid, he did consider waves in two- and three-dimensional lattices and continuous media. He obtained exact solutions for some simple problems, and used some perturbation techniques for more general cases. His approaches have had a lasting influence on the topic of dispersion in a layered material.

Rytov (1956) examined wave propagation in a medium composed of alternating layers of two isotropic materials. He calculated the wave speed for shear and longitudinal waves travelling both normal and parallel to the lamination. We will compare these results with our approach for this case in the sequel. He then went on to calculate effective elastic moduli for the material, now viewed as anisotropic. He was particularly interested in the thin layers (as compared to wavelength of the elastic wave) case.

Considerable work has been done subsequently on wave propagation in layered solids. Some was done with a view towards understanding earthquakes, as the earth can be viewed as a layered solid and the earthquakes as waves passing through it. Ivakin (1960) studied wave propagation in a periodically layered material by an analogy with electric circuits. The approach used involved matching impedances at each boundary and is rather tedious. In fact, the author concludes "... the determination of the velocity of propagation and of the amplitudes of sinusoidal waves in fine-scale nonhomogeneous media constitutes a laborious problem ..." (p. 108).

More recently, some work has been done particularly with reference to artificial materials such as carbon composites. Sun *et al.* (1968) present a continuum theory and display dispersion curves for various waves in a layered material. They linearly expand the displacements about the midplanes of the layers, and then require that some continuity conditions be satisfied at the layer boundaries. An expression for the energy is derived, and Hamilton's principal is used to obtain (approximate) equations of motion.

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Nayfeh and Nemat-Nasser (1972) examined one-dimensional waves in harmonic media, and were able to reduce the problem to the study of a Mathieu equation and a Hill equation. They examined in detail their Hill-type equation. Lee and Yang (1973) also examined this problem, and looked at the behavior of solutions of the one-dimensional problem near the discontinuities in the dispersion curve.

In two papers, Hegemier and Nayfeh (1973) and Hegemier and Bache (1974) developed a continuum theory for layered media with a similar approach. The latter paper, in fact, attacked the problem of waves travelling at an arbitrary angle in a layered solid and included phase velocity versus angle of propagation curves.

In a series of two papers, Ben-Amoz (1975a,b) examined wave propagation in a direction parallel to the laminae and normal to the laminae. This was done by assuming relative orders of magnitude for the solution, and then discarding the pieces of the solid equation which would then be negligible. He concludes that the behavior of a laminated composite material is "predominantly that of a macroscopically homogeneous medium" (page 43).

Delph *et al.* (1979) consider plane strain harmonic waves in a periodic medium, and show that the dispersion spectrum is governed by the eigenvalues of an 8×8 matrix with complex entries. The approach used was to write the displacements in terms of complex exponentials.

This leads to the purpose of this paper. Herein will be described a perturbation method, and some results from it, to obtain dispersion curves and displacements for harmonic waves in an anisotropic layered medium. First, we will describe the notation we will use, and write down the equations of elasticity which we will assume to hold. We will view the material as a continuous medium with noncontinuous elastic constants, and thus avoid needing to deal with boundary conditions. For comparison purposes with some of the above work, we will in particular consider the case of a material made of alternating layers of two anisotropic materials. The methods presented apply to much more general N layers situations, but this case is presented as an example and to show the validity of the technique.

In the one dimensional simplification, the method will give rise to the solution of a nonlinear ordinary differential equation. This equation is first derived, and then an approximate perturbation solution is presented. It is found that to obtain qualitative agreement with the dispersion curve it is necessary to include the leading terms of the second-order perturbation.

Next, the equations are derived for propagation parallel to the laminae, and perturbation solutions are obtained for the two cases of on average longitudinal and on average shear waves. It will be found that to the first-order perturbation these are nondispersive.

The one-dimensional case will then be returned to, and examined numerically. Agreement will be shown with the perturbation approach, the numerical solution, and Rytov's solution for the specific example case. The interesting behavior of the solution of the nonlinear ordinary differential equation will lead to a more general discussion of it, in particular the question of existence of solutions. This relates back to the question of lack of continuity of the dispersion curve. The change of behavior of solutions in the vicinity of the break in the dispersion curve is demonstrated.

Finally, the parallel waves are examined numerically. This necessitates the solution of a generalized eigenvalue problem, which is done by factoring the problem, and rewriting it in standard eigenvalue form. For a range of frequencies the numerical solution is shown to be nondispersive.

It is hoped that the methods used in this paper present a straightforward approach to the propagation of waves in a layered solid, and that it will throw more light on the behavior of such waves.

THE APPROACH

Consider a material made of N layers of an anisotropic, elastic material. A periodic layering of height h is assumed, with the layering occurring in the y direction. This means the thickness of these N layers would add up to h. Each of these layers will be assumed to be linearly elastic, with the appropriate elastic constants. After having put our material together, we will view the "constants" in Hooke's law as being dependent upon y. If $c_{ij}(y)$ are the various constants in Hooke's law, then they may be written in the form

$$c_{ij}(y) = c_{ij} + d_{ij} p_{ij}(y) \tag{1}$$

where there is no implied summation. The c_{ij} (without the (y)) terms are the average values

$$c_{ij} = \frac{1}{h} \int_0^h c_{ij}(y)\,\mathrm{d}y, \tag{2}$$

the constants d_{ij} are the local deviations, and the p_{ij} function takes into account the actual structure of the laminae. The reason we are choosing to write $d_{ij}p_{ij}(y)$ rather than $d_{ij}(y)$ is that often there are symmetries in a lay-up which produce the same p_{ij} for different ij.

In general, layers will have different densities, and so we will also write

$$\rho(y) = \rho + d_\rho p_\rho(y). \tag{3}$$

If laminae are made of the same material at different angles with respect to the anisotropy, the density will be constant throughout, and d_ρ will be zero.

To actually solve the equations coming from the perturbation, smooth p_{ij} are needed. Since they are in a sense the periodic deviation from homogeneity for the material, it is natural to expand them in a Fourier series:

$$p_{ij}(y) = \sum_{n=1}^{\infty} \{p_{ij,n} \sin(n\alpha y) + q_{ij,n} \cos(n\alpha y)\}, \quad \alpha = \frac{2\pi}{h}. \tag{4}$$

The α term will be appearing often, and one should view it as 2π times the "frequency" of the layering.

This general approach of using a Fourier series to describe the inhomogeneity of the material makes it possible to model many situations. By picking appropriate p_{ij}, one can describe layers of different thickness, layers with glue between the layers, and layers of completely different material.

A PARTICULAR PROBLEM

As an example of a way to use the above ideas, and some results from them, wave propagation in an anisotropic layered material will be considered. One way of viewing this problem is as separate layers where the above sets of constants hold, and trying to match the solutions in each layer at the layer boundaries. This is very difficult, but it is the way the problem is often attacked. We have chosen to write the material parameters as functions which are position-dependent. For the infinite periodic medium there are no boundaries, but the partial differential equations no longer have constant coefficients. This naturally leads to a perturbation approach.

To examine waves travelling in a plate, a plane strain model was used, as this corresponds to taking a slice through the plate. The equations are (see Malvern, 1969)

$$\partial_1(c_{11}(y)\varepsilon_{11} + c_{12}(y)\varepsilon_{22}) + 2\partial_2(c_{66}(y)\varepsilon_{12}) = \rho u_{tt} \tag{5}$$

$$2\partial_1(c_{66}(y)\varepsilon_{12}) + \partial_2(c_{12}(y)\varepsilon_{11} + c_{22}(y)\varepsilon_{22}) = \rho v_{tt}, \tag{6}$$

where the ε_{ij} are the strains, the $c_{ij}(y)$ are the coefficients from Hooke's law, ρ is the density, and subscripts 1 and 2 refer to x and y directions, respectively.

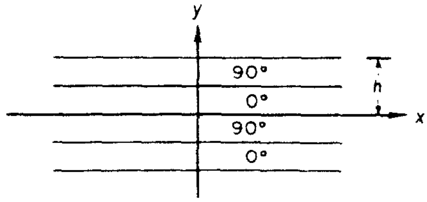

Fig. 1. The material set-up and local coordinate system.

A PARTICULAR MATERIAL

As a practical matter, we consider the material Hercules A.S. 4/3501-6. This material is made of thin carbon fibers (very thin—they are about 10 μm, or 0.001 cm in diameter) which are placed in an epoxy prepreg and pressed into thin sheets, where the fibers all run in one direction. These sheets are then stacked so that in each layer the fibers run perpendicularly to the fibers in the previous layer. At this point the sheets are heated and pressed together, so that the epoxy is a continuous matrix of supporting fibers which run in two different directions. The resulting plate is of the form ...0°/90°/0°/90°... (see Fig. 1). Each layer is quite thin, and a plate of this material one-quarter inch thick has about 25 layers. With this configuration, $h = 2(1/100)$ inch $= 0.02$ inch.

Let us consider, therefore, such a composite plate made of alternating layers, with layer 1 having fibers running in the x direction, and layer 2 having fibers running in the z direction (out of the plane). The resulting c_{ij} and d_{ij} values are†

$$
\begin{aligned}
c_{11} &= 11.1900 \qquad & d_{11} &= 9.6350 \\
c_{22} &= 1.7180 \qquad & d_{22} &= 0.0000 \\
c_{12} &= 0.6276 \qquad & d_{12} &= 0.0426 \\
c_{66} &= 0.6060 \qquad & d_{66} &= 0.1040
\end{aligned}
$$

in million pounds (force) per square inch. For this case, we have

$$
p_{ij}(y) = \begin{cases} 1, & 0 \leqslant y < \dfrac{h}{2} \\[2ex] -1, & \dfrac{h}{2} \leqslant y < h \end{cases}. \tag{7}
$$

Since $p_{ij}(y)$ is the same for all ij, we let $p_{ij}(y) = p(y)$, and expand $p(y)$ in a Fourier series to get

$$
p_n = \begin{cases} 4/\pi n, & n \text{ odd} \\ 0, & n \text{ even}, \end{cases} \tag{8}
$$

with all the q_n being zero. Also, since the material is the same throughout, the density is constant and we have $d_\rho = 0$.

NORMAL PROPAGATION, AND A LOCAL WAVESPEED

In this section, the problem of normal propagation in the layering is considered. Both the longitudinal and transverse waves give rise to the same type of equation, with the material constants being different. In order to relate to the specific material discussed above,

† Data from Hercules, S.L.C., Utah.

the case of the transverse wave will be considered, since, for the longitudinal wave $d_{22} = 0$, and one has a general travelling wave (see below).

The form of the wave solution will generally be

$$f = \sin\,(m(\varphi(y)-t)). \tag{9}$$

In this case the wave is propagating in the $+y$ direction (for the case of waves propagating in the x direction replace y by x). To compare with the more usual approach, one takes

$$f = \sin\,(ky-mt) \tag{10}$$

and then examines the relationship between the phase speed and the wave number k, where k is independent of y. The approach here is quite different. The wave speed is allowed to vary locally, and by following a constant angle or phase in the sine term one arrives at

$$m(\varphi(y)-t) = \text{constant}, \tag{11}$$

so that for a constant frequency

$$\frac{\mathrm{d}\varphi}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} = 1, \tag{12}$$

or

$$\text{wave (phase) speed} = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{\varphi'} \tag{13}$$

where the prime denotes differentiation with respect to y. If the wave is dispersive, φ' will have an m dependence. The frequency ν of the wave, or number of oscillations per unit time for a fixed point in space, is then $2\pi\nu = m\cdot 1$ or

$$\text{frequency} = \nu = \frac{m}{2\pi}. \tag{14}$$

Notice that

$$\frac{\text{frequency of wave}}{\text{frequency of material}} = \frac{\nu}{\frac{1}{h}} = \frac{m}{\alpha}. \tag{15}$$

We will find that in the solutions to be obtained, m and α will always appear in the ratio m/α.

For the transverse case, we assume a solution of the form

$$u = \eta(y)f(\varphi(y)-t) \tag{16}$$

$$v = 0, \tag{17}$$

where $\eta(y)$ now represents an amplitude modulation of the overlying travelling wave. This amplitude modulation is fixed in space.

Equations (5) and (6) yield

$$\partial_2(c_{66}(y)\partial_2 u) = \rho u_{tt}. \tag{18}$$

The reader may also note that for the longitudinal case the equation is of the same form, i.e.

$$\partial_2(c_{22}(y)\partial_2 v) = \rho v_{tt}. \tag{19a}$$

and that if $c_{22}(y)$ is a constant ($d_{22} = 0$) then

$$c_{22}\partial_2 v = \rho v_{tt}. \tag{19b}$$

which is the equation for a general travelling wave. This justifies the previous remark.

Carrying out the substitution $u = \eta f$ one finds

$$\partial_2[c_{66}(y)\{\eta' f + \eta f'\varphi'\}] = \rho\eta f'' \tag{20}$$

where the prime denotes differentiation. Letting $c_{66}(y) = c(y)$, as there is only one c_{ij} which appears, and expanding one finds

$$c'\{\eta' f + \eta f'\varphi'\} + c[\eta'' f + 2\eta' f'\varphi' + \eta\{f''(\varphi')^2 + f'\varphi''\}] = \rho\eta f''. \tag{21}$$

Our next step is to remove the f' term by requiring its coefficient to vanish, i.e.

$$c'\eta\varphi' + c\{2\eta'\varphi' + \eta\varphi''\} = 0. \tag{22}$$

If this is divided by $c\eta\varphi'$, one obtains

$$\frac{c'}{c} + 2\frac{\eta'}{\eta} + \frac{\varphi''}{\varphi'} = 0, \tag{23}$$

which can be integrated to give

$$\varphi' = \frac{A}{c\eta^2} \tag{24}$$

where A is a constant of integration (A is nonzero as it is the exponential of the actual integration constant). Placing this rather nice result back in eqn (21) gives

$$c'\eta' f + c\left\{\eta'' f + \eta f''\left(\frac{A}{c\eta^2}\right)^2\right\} = \rho\eta f'', \tag{25}$$

which can finally be reduced to

$$(c\eta')' f + \frac{A^2}{c\eta^3} f'' = \rho\eta f''. \tag{26}$$

This equation has periodic boundary conditions with period h. To proceed further, some assumptions on f must be made. Suppose that $f'' = -m^2 f$, as it would if it were $\sin(m(\bullet))$ or $\cos(m(\bullet))$. This yields

$$(c\eta')' - \frac{A^2m^2}{c\eta^3} + m^2\rho\eta = 0 \tag{27}$$

as an equation for $\eta(y)$. Now fix m, where m can be any real number.

In this case, the wave (phase) speed is from eqn (24):

$$\text{phase speed} = \frac{1}{\varphi'} = \frac{c\eta^2}{A}. \tag{28}$$

This leads one to wonder about the effect of the constant A in the η equation. If we let η be a solution with A_η as the constant, and if we let μ be a solution with A_μ as the constant, then $A_\mu = kA_\eta$ for some k (A is nonzero), and making the substitution in the μ equation gives

$$(c\mu')' - \frac{k^2 A_\eta^2 m^2}{c\mu^3} + m^2\rho\mu = 0. \tag{29}$$

If this is divided by $\sqrt{k}$, it is concluded that $\mu/\sqrt{k} = \eta$ and that

$$\frac{A_\mu}{\mu^2} = \frac{kA_\eta}{(\sqrt{k}\eta)^2} = \frac{A_\eta}{\eta^2}. \tag{30}$$

One concludes, therefore, that A is quite arbitrary as far as physical meaning goes. (This can also be seen by dividing eqn (27) by $\sqrt{A}$.) Thus, in the following, we let $A = 1$.

As nonlinear equations are difficult to solve, a perturbation approach is used to linearize the equation in order to glean some information on η. The quantity ε will be used as an expansion parameter to obtain equations in various powers of ε, after which ε will then be set equal to 1. Suppose

$$\eta \approx \eta^0 + \varepsilon\eta^1 + \varepsilon^2\eta^2. \tag{31}$$

This leads to

$$\frac{1}{\eta^3} \approx \left(\frac{1}{\eta^0}\right)^3 \left(1 - \varepsilon 3\frac{\eta^1}{\eta^0} + \varepsilon^2\left(-3\frac{\eta^2}{\eta^0} + 6\left(\frac{\eta^1}{\eta^0}\right)^2\right)\right). \tag{32}$$

Using a similar expansion for the coefficient involving p (let p_{66} be written εp), one has

$$\frac{1}{c} = \frac{1}{c_{66} + \varepsilon d_{66} p(y)} = \frac{1}{c_{66}}\left\{1 - \varepsilon\frac{d_{66}}{c_{66}}p(y) + \varepsilon^2\left(\frac{d_{66}}{c_{66}}p(y)\right)^2\right\}. \tag{33}$$

Upon substitution into eqn (27), and letting ρ be written as $\rho + \varepsilon d_\rho p_\rho$, one obtains an equation with various powers of ε.

Separating and equating powers of ε in this equation yields three equations:

$$\varepsilon^0:\ c_{66}(\eta^0)'' - \frac{m^2}{c_{66}(\eta^0)^3} + m^2\rho\eta^0 = 0, \tag{34}$$

$$\varepsilon^1:\ c_{66}(\eta^1)'' + (d_{66}p(\eta^0)')' + \frac{m^2}{c_{66}(\eta^0)^3}\left(\frac{d_{66}}{c_{66}}p + \frac{3}{\eta^0}\eta^1\right) + m^2\rho\eta^1 + m^2\eta^0 d_\rho p_\rho = 0, \tag{35}$$

$$\varepsilon^2:\ (\eta^1)'' d_{66}p + (\eta^1)' d_{66}p' + c_{66}(\eta^2)'' - m^2\frac{1}{\eta_0^3 c_{66}}\left[6\left(\frac{\eta^1}{\eta^0}\right)^2 - 3\left(\frac{\eta^2}{\eta^0}\right) + 3\frac{\eta^1}{\eta^0}\frac{d_{66}}{c_{66}} + \left(\frac{d_{66}}{c_{66}}\right)^2\right] + m^2(\eta_1 d_\rho + \rho\eta_2) = 0. \tag{36}$$

There is a simple solution to the ε^0 equation, namely

$$\eta^0 = (\rho c_{66})^{-1/4}, \tag{37}$$

a constant. This simplifies the ε^1 equation giving

$$c_{66}(\eta^1)'' + 4m^2\rho\eta^1 = -\eta^0 m^2 \left\{ \rho \frac{d_{66}}{c_{66}} p + d_\rho p_\rho \right\}. \tag{38}$$

This equation has periodic boundary conditions of period h.

Using the Fourier series expansion for the p and p_ρ, a solution to this ε^1 equation can be found using the orthogonality of the basis. The form of the solution η^1 is

$$\eta^1(y) = \sum_{n=1}^{\infty} \left\{ a_n \sin(n\alpha y) + b_n \cos(n\alpha y) \right\}. \tag{39}$$

A substitution, use of orthogonality and some algebra gives

$$\left[4\rho - c_{66}\left(\frac{n\alpha}{m}\right)^2 \right] a_n = -\eta^0 \left\{ \rho \frac{d_{66}}{c_{66}} p_n + d_\rho p_{\rho,n} \right\}, \tag{40}$$

$$\left[4\rho - c_{66}\left(\frac{n\alpha}{m}\right)^2 \right] b_n = -\eta^0 \left\{ \rho \frac{d_{66}}{c_{66}} q_n + d_\rho q_{\rho,n} \right\}. \tag{41}$$

As will also occur for the waves travelling parallel to the layering, $n(\alpha/m)$ always occur together. α/m is a measure of the frequency of the wave to the "frequency" of the material. Moreover, α is large for thin layers and m is large for high frequencies. For this case, the perturbation is expected to be good for both large and small $n(\alpha/m)$. Equation (40) implies that a_n is approximately a constant multiplied by p_n for small $n(\alpha/m)$, and that a_n is roughly inversely proportional to $(n(\alpha/m))^2$ for large $n(\alpha/m)$.

By examining eqns (40) and (41) it is seen that if

$$4\rho - c_{66} n^2 \frac{\alpha^2}{m^2} = 0, \tag{42}$$

a solution to the ε^1 problem does not always exist. This nonexistence occurs for

$$v = \frac{n}{2h}\sqrt{\frac{c_{66}}{\rho}}. \tag{43}$$

This will be explored more fully in a later section.

It was found in the course of the investigation that the dispersion curve of the first-order perturbation solution had some qualitative disagreement with exact solution in the neighborhood of the nonexistent first-order perturbation. This led us to consider the second-order perturbation. Since a full second-order term would be difficult to obtain, an approximate one will be found. As we are interested in regions where eqn (42) nearly holds we fix n where this holds, and see that a_n and b_n are large. This implies the $(\eta^1)^2$ term dominates the non-η^2 terms of (36), leaving

$$\varepsilon^2: \; c_{66}(\eta^2)'' + 4m^2\rho\eta^2 \approx 6m^2 \frac{1}{\eta^0} \rho (a_n \sin(n\alpha y) + b_n \cos(n\alpha y))^2 \tag{44}$$

(n is fixed here). A solution to this equation is

$$\eta^2 = g_{1n} \sin^2 (n\alpha y) + g_{2n} \sin (n\alpha y) \cos (n\alpha y) + g_{3n} \cos^2 (n\alpha y) \tag{45}$$

where

$$\begin{bmatrix} 4\rho - 2c_{66}\left(\dfrac{n\alpha}{m}\right)^2 & 2c_{66}\left(\dfrac{n\alpha}{m}\right)^2 \\ 2c_{66}\left(\dfrac{n\alpha}{m}\right)^2 & 4\rho - 2c_{66}\left(\dfrac{n\alpha}{m}\right)^2 \end{bmatrix} \begin{pmatrix} g_{1n} \\ g_{3n} \end{pmatrix} = 6\frac{\rho}{\eta^0}\begin{pmatrix} a_n^2 \\ b_n^2 \end{pmatrix} \tag{46}$$

and

$$g_{2n} = \frac{12 a_n b_n \rho/\eta^0}{4\rho - 4c_{66}\left(\dfrac{n\alpha}{m}\right)^2}. \tag{47}$$

Using the fact that $4\rho \approx c_{66}(n\alpha/m)^2$ and doing some simplification, we obtain

$$\left.\begin{aligned} g_{1n} &\approx \frac{1}{2\eta^0}(a_n^2 + 2b_n^2) \\ g_{2n} &\approx -\frac{1}{\eta^0} a_n b_n \\ g_{3n} &\approx \frac{1}{2\eta^0}(2a_n^2 + b_n^2) \end{aligned}\right\}. \tag{48}$$

When the dispersion curve results are later presented, this second-order result will be included.

This at least gives a better feel for the solution. It is interesting to note that for high frequencies (large m), eqn (27) gives

$$\eta \approx (\rho(c_{66} + d_{66}p(y)))^{-1/4} \tag{49}$$

while for low frequencies $\eta \approx (\rho c_{66})^{-1/4}$. To see an interesting sidelight of eqn (40), we let for simplicity $\{q_n\}$ and d_ρ be zero, then for large m and small n,

$$a_n \approx -(\rho c_{66})^{-1/4} \frac{d_{66}}{4c_{66}} p_n, \tag{50}$$

which gives

$$\eta \approx \eta^0 + \eta^1 \approx (\rho c_{66})^{-1/4}\left\{1 - \sum_{\text{truncated}} \frac{d_{66}}{4c_{66}} p_n \sin (n\alpha y)\right\} \approx (\rho c_{66})^{-1/4}\left\{1 - \frac{d_{66}}{4c_{66}} p(y)\right\}. \tag{51}$$

This is the first term of the Taylor's expansion of eqn (49). Thus the perturbation solution agrees with both the large and small m limit solutions.

In summary, it is of some practical interest to note that at high frequencies there is a surprising inverse quarter-power amplitude modulation of the wave, while at low frequencies the wave hardly notices the inhomogeneity of the material at all. There is a discrete spectrum of frequencies where ε^1 perturbation problem does not have a solution.

WAVES PROPAGATING PARALLEL TO THE LAMINAE

Here there do not exist purely longitudinal or purely transverse waves, but the same idea applies. A perturbation method is used to obtain some approximate solutions.

First, the following form of the solution is assumed:

$$u = \eta(y)\cos(m(\lambda x - t)) \tag{52}$$

$$v = \mu(y)\sin(m(\lambda x - t)), \tag{53}$$

where λ is a constant. This form is chosen since a travelling wave is being looked for, and the solution is expected to have a y dependence in the amplitude. In this case a variable phase speed is not being considered because if the phase speed did depend on y, the wave would separate in the layers, which is not desirable. In addition, an x-dependent wave speed is not expected since the material properties do not change in the x direction.

It seems wise to elucidate some assumptions in the above equations. It has been assumed that the actual solid "particles" follow an elliptical path. To see this, fix y and note that

$$\frac{u^2}{\eta(y)^2} + \frac{v^2}{\mu(y)^2} = 1. \tag{54}$$

As will become apparent, the "longitudinal" wave will have a major axis in the direction the wave travels, while the "transverse" wave will have a major axis normal to the direction of wave propagation. On a physical note, it has been assumed that the frequency is not so high that the layers are acting as waveguides. Waveguiding will probably occur at high frequencies, meaning the wave will separate in each of the layers, the wave in one layer travelling faster than the wave in the other layer. Though the assumed form of solution leads to solutions of the solid equations for almost all frequencies, we suspect that experimentally these type waves would be very difficult to produce for high frequencies, as waveguiding would more naturally occur.

The assumed u and v are placed in the original solid equations, eqns (5) and (6), to give

$$\begin{aligned}(c_{11}+d_{11}p_{11})(-m^2\lambda^2)\eta\cos(\cdot)+\partial_2(c_{66}+d_{66}p_{66})\eta'\cos(\cdot)+(c_{12}+d_{12}p_{12})m\lambda\mu'\cos(\cdot)\\+\partial_2(c_{66}+d_{66}p_{66})m\lambda\mu\cos(\cdot) = -m^2\rho\eta\cos(\cdot),\end{aligned} \tag{55}$$

$$\begin{aligned}(c_{66}+d_{66}p)(-m\lambda)\eta'\sin(\cdot)+\partial_2(c_{12}+d_{12}p)(-m\lambda)\eta\sin(\cdot)+(c_{66}+d_{66}p_{66})(-m^2\lambda^2)\mu\sin(\cdot)\\+c_{22}\mu''\sin(\cdot) = -m^2\rho\mu\cos(\cdot),\end{aligned} \tag{56}$$

where $\cdot$ corresponds to $m(\lambda x - t)$ and the prime denotes differentiation with respect to y. The cos $(\cdot)$ can be factored out of the first equation and the sin $(\cdot)$ can be factored out of the second equation, since they do not depend on y.

This is what is left:

$$\begin{aligned}(c_{11}+d_{11}p_{11})(-m^2\lambda^2)\eta+((c_{66}+d_{66}p_{66})\eta')'+(c_{12}+d_{12}p_{12})m\lambda\mu'\\+m\lambda((c_{66}+d_{66}p_{66})\mu)' = -m^2\rho\eta,\end{aligned} \tag{57}$$

$$\begin{aligned}(c_{66}+d_{66}p_{66})(-m\lambda)\eta'-m\lambda((c_{12}+d_{12}p_{12})\eta)'+(c_{66}+d_{66}p_{66})(-m^2\lambda^2)\mu\\+c_{22}\mu'' = -m^2\rho\mu.\end{aligned} \tag{58}$$

These are coupled ordinary differential equations, with periodic boundary conditions.

Once again the system only seems amenable to a perturbation approach: thus one may proceed as follows. Think of inserting an ε in front of the p term, and use an expansion of the form

$$\eta \approx \eta^0 + \varepsilon\eta^1, \tag{59}$$

$$\eta \approx \eta^0 + \varepsilon\mu^1. \tag{60}$$

The zero order (ε^0) equation is

$$-c_{11}m^2\lambda^2\eta^0 + c_{66}(\eta^0)'' + (c_{12}+c_{66})m\lambda(\mu^0)' = -m^2\rho\eta^0, \tag{61}$$

$$-(c_{66}+c_{12})m\lambda(\eta^0)' - c_{66}m^2\lambda^2\mu^0 + c_{22}(\mu^0)'' = -m^2\rho\eta^0. \tag{62}$$

Let $a = c_{12}+c_{66}$ as this term will occur again. The first order (ε^1) equation is

$$-c_{11}m^2\lambda^2\eta^1 + c_{66}(\eta^1)'' + am\lambda(\mu^1)' + m^2\rho\eta^1 = m^2\lambda^2 d_{11}p_{11}\eta^0 - (d_{66}p_{66}(\eta^0)')' \\ - m\lambda d_{12}p_{12}(\mu^0)' - m\lambda(d_{66}p_{66}\mu^0)' - m^2 d_\rho p_\rho \eta^0, \tag{63}$$

$$-am\lambda(\eta^1)' - c_{66}m^2\lambda^2\mu^1 + c_{22}(\mu^1)'' + m^2\rho\mu^1 = m\lambda d_{66}p_{66}(\eta^0) + m\lambda(d_{12}p_{12}\eta^0)' \\ + m^2\lambda^2 d_{66}p_{66}\mu^0 - m^2 d_\rho p_\rho \mu^0. \tag{64}$$

There are two simple solutions to the equations, first

$$\eta_1^0 = 1, \quad \mu_1^0 = 0, \quad \lambda_1 = \sqrt{\frac{\rho}{c_{11}}}, \tag{65}$$

where the $\eta_1^0 = 1$ is an arbitrary selection as the equations are linear, and

$$\eta_2^0 = 0, \quad \mu_2^0 = 1, \quad \lambda_2 = \sqrt{\frac{\rho}{c_{66}}}. \tag{66}$$

The subscripts will distinguish these two cases. Similar to the normal wave, λ is inversely proportional to the wave speed, and it is immediately seen that the above two solutions are two different types of waves, as they travel at different speeds. This is, of course, not unexpected: transverse (shear) and longitudinal waves in a homogeneous, isotropic solid also travel at different speeds.

PARALLEL LONGITUDINAL WAVES

This section examines the longitudinal waves, or those arising from eqn (65). The first-order perturbation equation (eqns (63) and (64)) becomes

$$c_{66}(\eta_1^1)'' + am\sqrt{\frac{\rho}{c_{11}}}(\mu_1^1)' = m^2\rho\frac{d_{11}}{c_{11}}p_{11}(\alpha y) - m^2 d_\rho p_\rho, \tag{67}$$

$$-am\sqrt{\frac{\rho}{c_{11}}}(\eta_1^1)' + c_{22}(\mu_1^1)'' + \left(1 - \frac{c_{66}}{c_{11}}\right)m^2\rho\mu_1^1 = m\sqrt{\frac{\rho}{c_{11}}}d_{12}p_{12}'(\alpha y). \tag{68}$$

One has a solution by assuming the following forms:

$$\eta_1^1(y) = \sum_{n=1}^{\infty} \{a_n \sin(n\alpha y) + \alpha_n \cos(n\alpha y)\}, \tag{69}$$

$$\mu_1^1(y) = \sum_{n=1}^{\infty} \{b_n \cos(n\alpha y) + \beta_n \sin(n\alpha y)\}. \tag{70}$$

Placement of these in the first-order eqns (67) and (68) gives a system for the a_n and b_n, namely

$$-c_{66}n^2\alpha^2 a_n - am\sqrt{\frac{\rho}{c_{11}}}\, n\alpha b_n = m^2\rho\frac{d_{11}}{c_{11}}p_{11,n} - m^2 d_\rho p_{\rho,n}, \tag{71}$$

$$-am\sqrt{\frac{\rho}{c_{11}}}\, n\alpha a_n + \left\{-c_{22}n^2\alpha^2 + \left(1 - \frac{c_{66}}{c_{11}}\right)m^2\rho\right\} b_n = m\sqrt{\frac{\rho}{c_{11}}}\, d_{12} n\alpha p_{12,n}. \tag{72}$$

and a similar system for the α_n and β_n.

Solution of these gives the first-order perturbation for the longitudinal case. Dividing both equations by m^2 leaves the equations in a form where m and α do not exist independently, but only the ratio α/m occurs:

$$\begin{pmatrix} -c_{66}n^2\dfrac{\alpha^2}{m^2} & -a\sqrt{\dfrac{\rho}{c_{11}}}\, n\dfrac{\alpha}{m} \\ -a\sqrt{\dfrac{\rho}{c_{11}}}\, n\dfrac{\alpha}{m} & -c_{22}n^2\dfrac{\alpha^2}{m^2} + \left(1 - \dfrac{c_{66}}{c_{11}}\right)\rho \end{pmatrix}\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} \rho\dfrac{d_{11}}{c_{11}}p_{11,n} - d_\rho p_{\rho,n} \\ \sqrt{\dfrac{\rho}{c_{11}}}\, d_{12} n\dfrac{\alpha}{m} p_{12,n} \end{pmatrix}. \tag{73}$$

Solution of this system gives the a_n and the b_n. The α_n and β_n are given by

$$\begin{pmatrix} -c_{66}n^2\dfrac{\alpha 2}{m^2} & a\sqrt{\dfrac{\rho}{c_{11}}}\, n\dfrac{\alpha}{m} \\ a\sqrt{\dfrac{\rho}{c_{11}}}\, n\dfrac{\alpha}{m} & -c_{22}n^2\dfrac{\alpha^2}{m^2} + \left(1 - \dfrac{c_{66}}{c_{11}}\right)\rho \end{pmatrix}\begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} = \begin{pmatrix} \rho\dfrac{d_{11}}{c_{11}}q_{11,n} - d_\rho q_{\rho,n} \\ -\sqrt{\dfrac{\rho}{c_{11}}}\, d_{12} n\dfrac{\alpha}{m} p_{12,n} \end{pmatrix}. \tag{74}$$

It may be noted that there are situations here, as in the normal case, where solutions to the ε^1 problem do not exist. This occurs when the determinant in the systems (73) and (74) vanishes, that is, when

$$n^2\frac{\alpha^2}{m^2}\left[n^2\frac{\alpha^2}{m^2} - \frac{1}{c_{22}}\left(1 - \frac{c_{66}}{c_{11}} - \frac{a^2}{c_{11}c_{66}}\right)\rho\right] = 0. \tag{75}$$

This implies that a solution does not exist for frequencies

$$v = \frac{n}{h}\sqrt{\frac{c_{22}}{\rho}}\left\{1 - \frac{c_{66}}{c_{11}} - \frac{a^2}{c_{11}c_{66}}\right\}^{-1/2}. \tag{76}$$

Near these frequencies the determinant will be small implying that the specific a_n, b_n, α_n and β_n will be large. It is thus expected that the perturbation will not be good near these frequencies.

Thus the coefficients a_n and the b_n only depend on the ratio α/m, which measures the wavelength of the wave in comparison to the spacing of the layers of material. For large α/m, meaning low frequencies, the a_n and b_n are given approximately by

$$a_n \approx \frac{1}{n^2}\frac{m^2}{\alpha^2}\frac{1}{c_{66}}\left\{\rho\frac{d_{11}}{c_{11}}p_{11,n} - d_\rho p_{\rho,n} - a\frac{\rho}{c_{11}}\frac{d_{12}}{c_{22}}p_{12,n}\right\}, \tag{77}$$

$$b_n \approx -\sqrt{\frac{\rho}{c_{11}}}\frac{d_{12}}{c_{22}}\frac{m}{n\alpha}p_{12,n}. \tag{78}$$

The perturbation terms a_n and b_n are small when α/m is large, or when the wavelength of the wave is large compared to the spacing of layers.

The wave travels at the root mean square average speed in the material. To see this, recall that c_{11} is the average of the $c_{11}(y)$ in the layers, so that

$$\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \left(\frac{1}{\lambda}\right)^2 = \frac{c_{11}}{\rho} = \frac{1}{h}\int_0^h \frac{c_{11}(y)}{\rho}\,\mathrm{d}y. \tag{79}$$

The wave has a tumbling structure. It travels in the high speed layer and then tumbles into the low speed layer, which results in the average wave speed observed.

To justify the statement about the longitudinal waves corresponding to ellipses with the major axis parallel to the axis of propagation, recall that η_1^1 and μ_1^1 are small, so that

$$\eta_1(y) \approx 1 + \eta_1^1(y), \tag{80}$$

$$\mu_1(y) \approx \mu_1^1(y). \tag{81}$$

Since

$$\frac{u^2}{(\eta(y))^2} + \frac{v^2}{(\mu(y))^2} = 1 \tag{82}$$

it is seen that u is the major axis, and u corresponds to material displacement in the x direction.

PARALLEL TRANSVERSE WAVES

For the transverse wave, from eqns (63), (64) and (66), the first-order perturbation equation is

$$c_{66}(\eta_2^1)'' + m^2\rho\left(1 - \frac{c_{11}}{c_{66}}\right)\eta_2^1 + am\sqrt{\frac{\rho}{c_{66}}}(\mu_2^1)' = -m\sqrt{\frac{\rho}{c_{66}}}d_{66}p_{66}'(\alpha y), \tag{83}$$

$$-am\sqrt{\frac{\rho}{c_{66}}}(\eta_2^1)' + c_{22}(\mu_2^1)'' = \rho\frac{d_{66}}{c_{66}}m^2 p_{66}(\alpha y) - m^2 d_\rho p_\rho, \tag{84}$$

and one can proceed with η^1 and μ^1 as given in eqns (69) and (70).

The equations solved by the coefficients are (already dividing through by m^2)

$$\begin{pmatrix} -c_{66}\left(\frac{n\alpha}{m}\right)^2+\rho\left(1-\frac{c_{11}}{c_{66}}\right) & -a\sqrt{\frac{\rho}{c_{66}}}\left(\frac{n\alpha}{m}\right) \\ -a\sqrt{\frac{\rho}{c_{66}}}\left(\frac{n\alpha}{m}\right) & -c_{22}\left(\frac{n\alpha}{m}\right)^2 \end{pmatrix}\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} \left(\frac{n\alpha}{m}\right)\sqrt{\frac{\rho}{c_{66}}}d_{66}q_{66,n} \\ \rho\frac{d_{66}}{c_{66}}q_{66,n}-d_\rho q_{\rho,n} \end{pmatrix}. \tag{85}$$

$$\begin{pmatrix} -c_{66}n^2\frac{\alpha^2}{m^2}+\rho\left(1-\frac{c_{11}}{c_{66}}\right) & a\sqrt{\frac{\rho}{c_{66}}}n\frac{\alpha}{m} \\ a\sqrt{\frac{\rho}{c_{66}}}n\frac{\alpha}{m} & -c_{22}n^2\frac{\alpha^2}{m^2} \end{pmatrix}\begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{\rho}{c_{66}}}n\frac{\alpha}{m}d_{66}p_{66,n} \\ \rho\frac{d_{66}}{c_{66}}p_{66,n}-d_\rho p_{\rho,n} \end{pmatrix}. \tag{86}$$

As in the longitudinal case, the wave speed is the root mean square average of the wave speeds of transverse waves in the two layers.

Here, existence of the solution to the ε^1 problem does not occur if

$$n^2\frac{\alpha^2}{m^2}\left[n^2\frac{\alpha^2}{m^2}-\frac{1}{c_{66}}\left(1-\frac{c_{11}}{c_{66}}-\frac{a^2}{c_{22}}\frac{1}{c_{66}}\right)\rho\right]=0, \tag{87}$$

whose frequencies are

$$v=\frac{n}{h}\sqrt{\frac{c_{66}}{\rho}}\left\{1-\frac{c_{11}}{c_{66}}-\frac{a^2}{c_{22}c_{66}}\right\}^{-1/2}. \tag{88}$$

Here, with η_2^1 and μ_2^1 small and

$$\eta_2(y)\approx\eta_2^1(y) \tag{89}$$

$$\mu_2(y)\approx 1+\mu_2^1(y), \tag{90}$$

the major axis of displacement is v, or perpendicular to the direction of wave propagation, which is why these waves are called transverse waves.

Finally, supposing one had started with

$$u=\eta(y)\sin(m(\lambda x-t)) \tag{91}$$

$$v=\mu(y)\cos(m(\lambda x-t)), \tag{92}$$

it is seen that the above longitudinal results go through if one replaces

$$\sqrt{\frac{\rho}{c_{11}}}\quad\text{by}\quad-\sqrt{\frac{\rho}{c_{11}}}. \tag{93}$$

Similarly, the transverse results go through if one replaces

$$\sqrt{\frac{\rho}{c_{66}}}\quad\text{by}\quad-\sqrt{\frac{\rho}{c_{66}}}. \tag{94}$$

SOME NUMERICAL RESULTS FOR THE NORMAL WAVE

In order to verify the perturbation and to examine the solutions of eqn (27) in the regions where the perturbation fails, numerical solutions were obtained. These were for

Table 1. The numerical results for $\alpha/m = 0.0001$ ($\log_{10}(\alpha/m) = -4$)

		Two terms		Six terms	
z	$\eta_{comp}(z)$	$\eta_{pert}(z)$	% Error	$\eta_{pert}(z)$	% Error
0.3307	0.328356	0.325574	−0.8	0.325646	−0.8
0.6614	0.328817	0.326130	−0.8	0.326182	−0.8
0.9921	0.329152	0.326559	−0.8	0.326560	−0.8
1.3228	0.329335	0.326813	−0.8	0.326769	−0.8
1.6535	0.329384	0.326865	−0.8	0.326813	−0.8
1.9842	0.329279	0.326710	−0.8	0.326685	−0.8
2.3149	0.329043	0.326364	−0.8	0.326392	−0.8
2.6456	0.328652	0.325865	−0.8	0.325934	−0.8
2.9762	0.328142	0.325266	−0.9	0.325314	−0.9
3.3069	0.327396	0.324634	−0.8	0.324585	−0.9
3.6376	0.326614	0.324035	−0.8	0.323966	−0.8
3.9683	0.326053	0.323536	−0.8	0.323508	−0.8
4.2990	0.325675	0.323190	−0.8	0.323215	−0.8
4.6297	0.325516	0.323035	−0.8	0.323087	−0.7
4.9604	0.325554	0.323087	−0.8	0.323131	−0.7
5.2911	0.325810	0.323341	−0.8	0.323340	−0.8
5.6218	0.326261	0.323769	−0.8	0.323718	−0.8
5.9525	0.326920	0.324326	−0.8	0.324254	−0.8
6.2832	0.327724	0.324950	−0.8	0.324950	−0.8

L_2 norm of residuals = 0.1615644E-14.
Average phase speed = 5252 ft s^{-1}.

the specific case of the Hercules A.S. 4/3501-6 material previously mentioned, and with the ...0°/90°/0°/90°... configuration.

The method of choice was Galerkin's method, which changed the problem to that of the solution of a set of algebraic nonlinear equations of the form

$$A\vec{c} = \int_0^h \Phi^T(y) \frac{1}{(c_{66}+d_{66}p(y))(\Phi(y)\vec{c})^3}\,dy, \tag{95}$$

where A is a nonsingular matrix, $\Phi(y)$ a row vector of basis functions, and the approximate solution is $\eta(y) \approx \Phi(y)\vec{c}$.

Being a periodic equation, a Fourier sine and cosine basis was used, with 19 basis functions. MINPACK was used to solve the nonlinear algebraic equations and took approximately 1 min to converge on a VAX 8600 with

$$\eta^0 = (\rho c_{66})^{-1/4} \tag{96}$$

as the initial guess. For details the reader may consult Walker (1988).

Table 1 displays the results for a specific frequency, $\alpha/m = 0.0001$, or from eqn (15):

$$v = \frac{1}{h}\left(\frac{m}{\alpha}\right) = 0.5\ \text{MHz}. \tag{97}$$

Shown here is the comparison of the computed solution with the perturbation expansion truncated at two terms, η^0 and one term of η^1,

$$\eta \approx \eta^0+\eta^1 \approx (\rho c_{66})^{-1/4}+a_1 \sin(z), \tag{98}$$

and six terms, η^0 and five terms of η^1,

Table 2. The perturbation solution compared with the computed solution for various α/m

$\log_{10}(\alpha/m)$	Two terms % Error Ave.	Max.	Six terms % Error Ave.	Max.	Wave (phase) speed (ft s^{-1}) Ave.	L_2 norm of residuals
0.000	0.7	0.7	0.7	0.7	5413	0.8432D-19
−1.000	0.7	0.7	0.7	0.7	5413	0.8438D-19
−2.000	0.7	0.7	0.7	0.7	5413	0.7107D-19
−3.000	0.7	0.7	0.7	0.7	5413	0.7936D-19
−4.000	0.8	0.9	0.8	0.9	5426	0.1616D-14
−4.540	7.7	10.2	7.7	10.2	6130	0.2097D-13
−5.000	11.5	22.1	5.1	12.6	5557	0.1267D-13
−5.500	2.8	7.7	0.7	1.8	5323	0.6172D-13
−6.000	1.4	4.1	0.5	1.0	5316	0.1299D-13
−7.000	1.3	4.0	0.5	1.0	5317	0.1101D-13
−8.000	1.3	4.0	0.5	1.0	5317	0.1098D-13
−9.000	1.3	4.0	0.5	1.0	5317	0.1098D-13
−10.000	1.3	4.0	0.5	1.0	5317	0.1098D-13

$$\eta \approx \eta^0+\eta^1 \approx (\rho c_{66})^{-1/4}+\sum_{n=1}^{9} a_n \sin(nz). \tag{99}$$

There are six terms since the Fourier sine expansion of the periodic square function has p_n and therefore a_n equal to zero for even n. The percentage error is simply

$$\%\ \text{error} = 100 \times \frac{\eta_{\text{perturbation}}-\eta_{\text{computed}}}{\eta_{\text{computed}}}. \tag{100}$$

To get the phase speed, or how fast the front of the wave travels, consider how long it will take the wave to travel a distance h. One has that $1/\varphi'$ is the local phase speed by eqn (13). Thus, the time to travel the distance h is

$$\Delta t = \int_0^h \frac{dt}{dy}\,dy = \int_0^h \varphi'(y)\,dy = \int_0^h \frac{1}{(c_{66}+d_{66}p)\eta^2}\,dy. \tag{101}$$

This gives

$$\text{phase speed} = \frac{h}{\Delta t} = h\left\{\int_0^h \frac{1}{(c_{66}+d_{66}p)\eta^2}\,dy\right\}^{-1}. \tag{102}$$

This is also included in the table. The wave speed is near one mile per second.

Next, Table 2 shows some results for a range of α/m. The numerics are the same as above, and the average percentage error and maximum percentage error refer to the absolute values of the percentage error.

For $\log_{10}(\alpha/m) > -4$ the wave speed is constant, and the errors stay at the limiting values indicated in the table. This is also true for the region $\log_{10}(\alpha/m) < -5.5$, which is a result of a finite number of basis functions used in the numerical solution. For values of $\log_{10}(\alpha/m)$ the accuracy of the perturbation depends on how close one is to an eigenvalue of the homogeneous problem, as is demonstrated in Fig. 2. The relationship between the homogeneous problem and the numerical solution will be explored more fully in the next section.

Next, Fig. 3 is a phase speed plot with only the first order perturbation, for a range of α/m. The upper curve is the phase speed of the wave based upon the numerical solution of eqn (27), while the lower curve is the phase speed based upon 50 terms of the first-order perturbation solution to the same equation. Breaks occur where the numerical method does not converge. The phase speed is not continuous. The qualitative difference in these two graphs is what led to an examination of the second-order perturbation. The lack of peaks at higher frequencies in the numerical solution will be discussed in the section on existence of solutions.

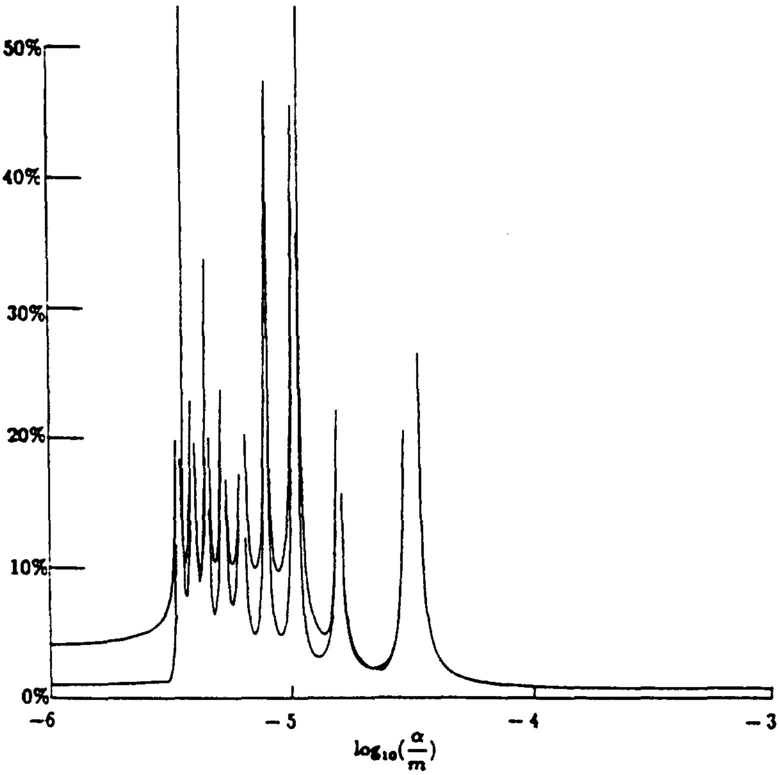

Fig. 2. A plot of the maximum percentage difference between the two perturbation solutions and the numerical solution. The upper curve is the two term error, and the lower curve is the six term error. Frequency increases from right to left (as m increases).

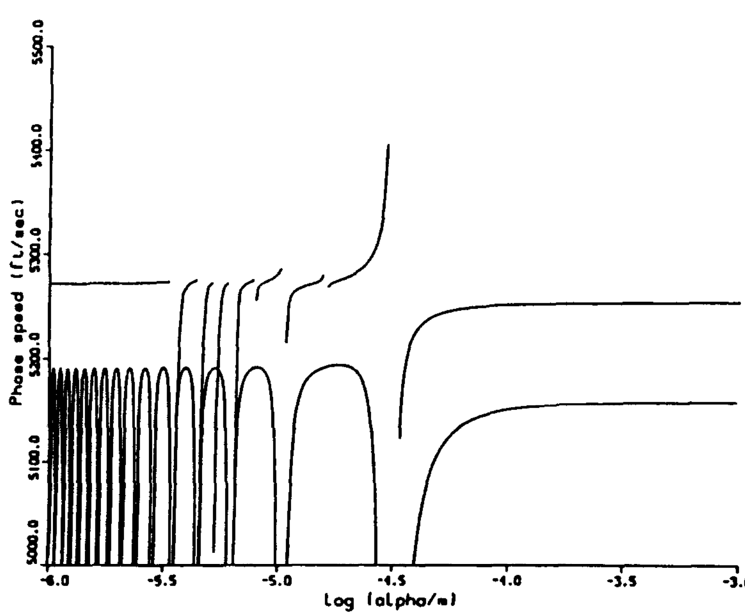

Fig. 3. A plot of the phase speed versus $\log_{10}(\alpha/m)$. The upper curve is the numerical result, while the lower curve is the perturbation result with 50 terms.

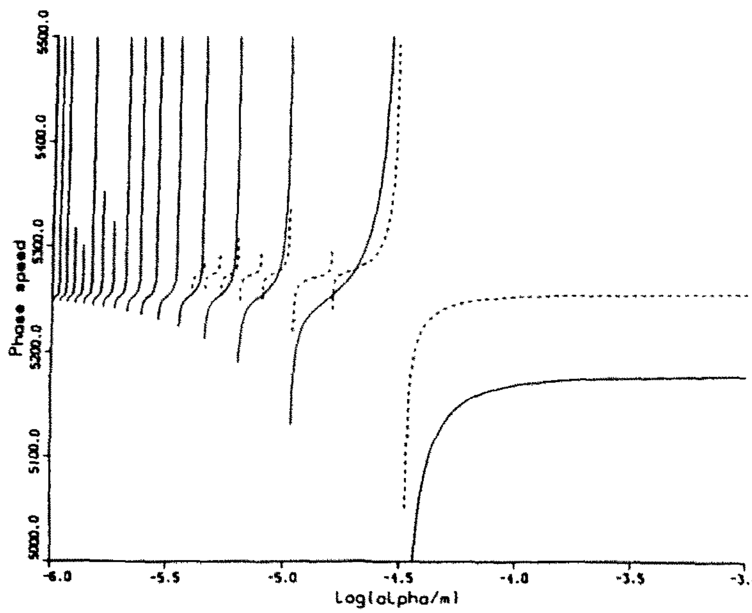

Fig. 4. A plot of the dispersion curve for the perturbation solution with second-order terms (solid curve), as well as a plot of the actual solution due to Rytov (dashed curve).

Figure 4 displays the dispersion curve for the perturbation solution including both first-order and approximate second-order terms (50 of each). With it is displayed the dispersion curve for the exact solution found in Rytov (1956) (and also found in quantum mechanics textbooks where the periodic square well potential is discussed). If

$$v_1 = \sqrt{\frac{c_{66}+d_{66}}{\rho}}, \quad v_2 = \sqrt{\frac{c_{66}-d_{66}}{\rho}}, \tag{103}$$

then this dispersion curve is given by

$$\cos\left(\frac{\pi}{v_1}\frac{m}{\alpha}\right)\cos\left(\frac{\pi}{v_2}\frac{m}{\alpha}\right)-\frac{1}{2}\left(\frac{v_1}{v_2}+\frac{v_2}{v_1}\right)\sin\left(\frac{\pi}{v_1}\frac{m}{\alpha}\right)\sin\left(\frac{\pi}{v_2}\frac{m}{\alpha}\right)=\cos\left(\frac{2\pi}{v}\frac{m}{\alpha}\right) \tag{104}$$

where v is the phase speed. Notice that the exact solution in Fig. 4 and the numerical solution displayed in Fig. 3 agree very well. Also, the perturbation solution including second-order terms agrees at least qualitatively with the actual dispersion curve. There are two apparent differences. First, the perturbation curve has only half as many discontinuities. This is due to the Fourier series expansion for $p(y)$ only having nonzero p_n for n odd ($p_n = 0$ if n is even). If the even np_n terms were nonzero, the perturbation-derived dispersion curve would have as many discontinuities as the actual dispersion curve does. The second difference is the low frequency limit of the phase speed. In the actual curve it is

$$2\left(\frac{1}{v_1}+\frac{1}{v_2}\right)^{-1} = 5273 \text{ ft s}^{-1}, \tag{105}$$

while in the perturbation case it is

$$\left(\frac{1}{h}\int_0^h \frac{1}{c(\eta_0)^2}\,dy\right)^{-1} = 2(\eta_0)^2\left(\frac{1}{c_{66}+d_{66}}+\frac{1}{c_{66}-d_{66}}\right)^{-1} = 5175 \text{ ft s}^{-1}. \tag{106}$$

Overall, however, agreement is good, and this demonstrates the validity of the perturbation approach.

A DISCUSSION OF EXISTENCE

This section discusses the existence of the nonlinear ordinary differential equation dealt with in the previous sections. The existence of a solution is not proved, but a relationship is demonstrated between the failure of the numerical method and the eigenvalues of the linear portion of the equation.

Equation (27) is, after letting $z = \alpha y$ and dividing through by $c_{66}\alpha^2/m^2$,

$$((1+rp(z))\eta')' - \frac{m^2/\alpha^2}{(1+rp(z))\eta^3} + \left(\frac{\rho}{c_{66}} \frac{m^2}{\alpha^2}\right)\eta = 0, \quad r = \frac{d_{66}}{c_{66}}, \tag{107}$$

with periodic boundary conditions of period 2π, and where the prime denotes differentiation with respect to z.

In a paper by Lazer and Solimini (1987), the existence of a periodic solution to the equation

$$u'' - \frac{1}{u^a} = g \tag{108}$$

is proven for $a \geqslant 1$ and

$$\int_{\text{period}} g < 0. \tag{109}$$

Unfortunately, the proof depends upon the existence of a lower bound on u obtained by the existence of an upper bound on g (which exists because g is piecewise continuous on a closed interval). So the method of proof does not apply to the equation considered. However, if a solution of eqn (107) did exist it would presumably be positive and so

$$\int_0^{2\pi} -\left(\frac{\rho}{c_{66}} \frac{m^2}{\alpha^2}\right)\eta \, \mathrm{d}z < 0 \tag{110}$$

and the corresponding $a = 3 > 1$. Comparing forms makes it seem reasonable for a solution to exist.

Next the Green's function will be formally developed. Let

$$(fu')' + \lambda u = g, \quad f \geqslant C > 0 \tag{111}$$

with C a constant. As the homogeneous part,

$$(fu')' + \lambda u = 0, \tag{112}$$

is a Sturm–Liouville problem, there exists a complete set of eigenvalues and eigenfunctions. Call these $\{\lambda_i\}$ and $\{\varphi_i\}$. If g is square integrable, it can be expressed in terms of the $\{\varphi_i\}$,

$$g = \sum_i g_i \varphi_i. \tag{113}$$

Assuming the solution u to be square integrable, one has

$$u = \sum_i u_i \varphi_i. \tag{114}$$

Using orthonormality of the eigenfunctions, the original equation yields

$$(\lambda - \lambda_i) u_i = g_i, \tag{115}$$

or

$$u = \sum_i u_i \varphi_i = \sum_i \frac{g_i}{\lambda - \lambda_i} \varphi_i = \sum_i \frac{\int g(\zeta) \overline{\varphi_i(\zeta)}\, \mathrm{d}\zeta}{\lambda - \lambda_i} \varphi_i(z). \tag{116}$$

In this, the integration is over the domain of u. If one exchanges the integration and the summation and lets

$$G(z, \zeta, \lambda) = \sum_i \frac{\varphi_i(z) \overline{\varphi_i(\zeta)}}{\lambda - \lambda_i} \tag{117}$$

then u can be written

$$u = \int G(z, \zeta, \lambda) g(\zeta)\, \mathrm{d}\zeta \tag{118}$$

where G is called the Green's function.

If the nonlinearity in eqn (108) is moved to the right-hand side, then the remaining left-hand side is a periodic Sturm–Liouville problem as described above, that is

$$((1 + rp(z))\eta')' + \left(\frac{\rho m^2}{c_{66} \alpha^2}\right) \eta = \frac{m^2/\alpha^2}{(1 + rp(z))\eta^3}. \tag{119}$$

Thus, a Green's function exists and the equation could be written as

$$\eta(z) = \int_0^{2\pi} G(z, \zeta, \lambda) \frac{m^2/\alpha^2}{(1 + rp(\zeta))\eta^3(\zeta)}\, \mathrm{d}\zeta. \tag{120}$$

This does not help a great deal in demonstrating existence, but it does indicate that when

$$\left(\frac{\rho}{c_{66}} \frac{m^2}{\alpha^2}\right)$$

equals an eigenvalue of

$$((1 + rp(z))u')' + \lambda u = 0 \tag{121}$$

a solution would not exist. The reasoning is that the Green's function does not exist there because of the $\lambda - \lambda_i$ term in the denominator, and so one should not expect a solution. However, as will be seen, there are twice as many values of α/m for which a solution does not exist than those indicated by this argument.

Table 3. A comparison of the eigenvalues of the homogeneous problem with some α/m where the numerical routine did not converge

$\log_{10}(\alpha/m)$	$2\sqrt{\rho/c_{66}}\,m/\alpha$	Squareroot of eigenvalues of $((1+rp)u')'+\lambda u=0$	Squareroot of eigenvalues of $u''+\lambda u=0$
		0.0000	0
−4.480	0.9439	0.9821	1
−4.500	0.9884	0.9962	1
−4.520	1.0350		
−4.800	1.9721	1.9646	2
		1.9919	2
−4.980	2.9849	2.9480	3
		2.9867	3
−5.105	3.9804	3.9327	4
		3.9801	4
−5.200	4.9537	4.9192	5
−5.210	5.0690	4.9718	5
−5.280	5.9556	5.9073	6
		5.9618	6
−5.350	6.9972	6.8974	7
		6.9499	7
−5.410	8.0339	7.8893	8
		7.9364	8
−5.470	9.2241	8.8828	9
		8.9211	9
		9.8777	10
		9.9050	10

Following the above argument, the job becomes obtaining the eigenvalues of eqn (118). These are expected to be very close to the eigenvalues of the periodic problem

$$u''+\lambda u = 0, \tag{122}$$

period 2π, which has eigenvalues 0, 1, 1, 4, 4, ...

It is simple to set up a numerical scheme to find them approximately. With

$$u = \Phi\vec{c} \tag{123}$$

the Galerkin method gives for eqn (121) a corresponding algebraic eigenvalue problem of

$$A\vec{c} = \lambda B\vec{c}, \tag{124}$$

where A and B are nonsingular matrices. This form of the eigenvalue problem can be solved by EISPACK, and was, with $N = 51$.

As was pointed out in the last section, the method did not converge for some regions of α/m. In Table 3 are displayed some values in each region for which convergence did not occur: the value

$$2\sqrt{\frac{\rho}{c_{66}}}\frac{m}{\alpha},$$

the squareroots of the numerically computed eigenvalues of eqn (121), and finally the squareroots of the eigenvalues of

$$u''+\lambda u = 0. \tag{125}$$

The correspondence is clear. In fact, the nonconvergence corresponding to $\lambda \approx 4$ was not found during the original phase speed calculation, where the method converged for $\log_{10}(\alpha/m) = -5.11$ and -5.12. Rather it was found by examining

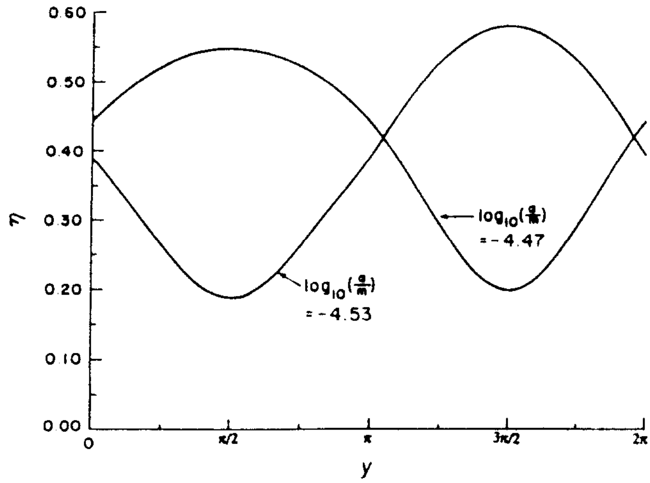

Fig. 5. A plot of $\eta(y)$ for frequencies above and below $\log_{10}(\alpha/m) = -4.5$.

$$\frac{\alpha}{m} = \frac{2}{\lambda}\sqrt{\frac{\rho}{c_{66}}} \tag{126}$$

for $\lambda \approx 4$, which gives $\log_{10}(\alpha/m) \approx -5.107$.

A surprising item is the factor 2 which appeared in the perturbation and now again in the numerical solution. There is more here than

$$\left(\frac{\rho}{c_{66}}\frac{m^2}{\alpha^2}\right)$$

being an eigenvalue of the homogeneous problem. So a conjecture: The nonlinear ordinary differential equation, eqn (107), has solutions except when

$$4\left(\frac{\rho}{c_{66}}\frac{m^2}{\alpha^2}\right)$$

equals an eigenvalue of the corresponding homogeneous equation, eqn (121).

There is even more evidence than presented above that solutions do not exist, and this gives us greater insight into the solutions η and what happens near the peaks.

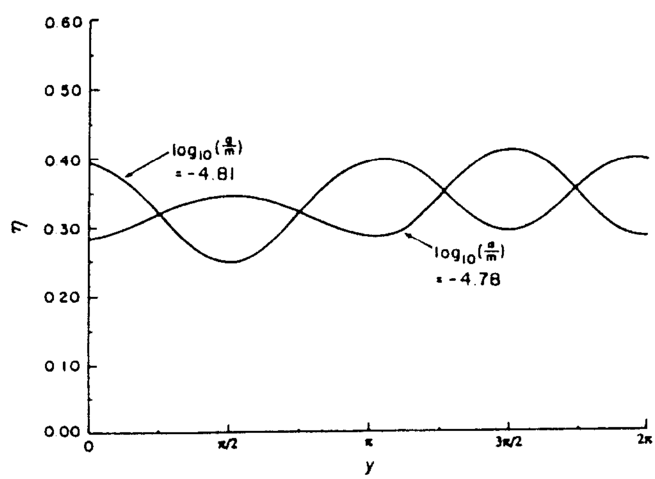

Fig. 6. A plot of $\eta(y)$ for frequencies above and below $\log_{10}(\alpha/m) = -4.8$.

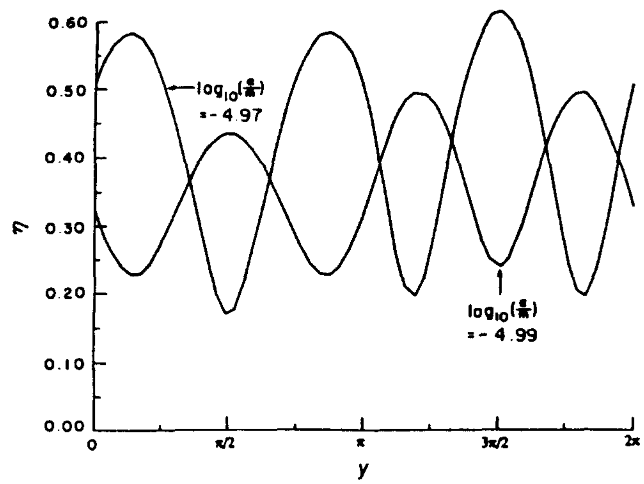

Fig. 7. A plot of $\eta(y)$ for frequencies above and below $\log_{10}(\alpha/m) = -4.98$.

Figures 5, 6 and 7 show the solutions to eqn (107) for

$$2\sqrt{\frac{\rho}{c_{66}}}\frac{m}{\alpha} \approx 1, 2 \text{ and } 3, \tag{127}$$

respectively. Each figure has two solution curves. One curve is for a frequency a little less than one that did not converge, and one for a frequency a little greater. The character of the solution changes in a substantial way, sort of flip-flopping. In the perturbation solution, the coefficient of the respective frequency changes sign at the transition point, and the numerical solution shows a similar effect.

Finally, some discussion of wavelengths is appropriate. Using the first term of the perturbation expansion,

$$\eta \approx (\rho c_{66})^{-1/4}, \tag{128}$$

one has

$$\varphi' \approx \frac{1}{c_{66}(\rho c_{66})^{-2/4}} = \sqrt{\frac{\rho}{c_{66}}}, \tag{129}$$

which leads to the wavelength:

$$y_\lambda \approx \frac{2\pi}{m\varphi'} \approx \frac{2\pi}{m}\sqrt{\frac{c_{66}}{\rho}}. \tag{130}$$

As $\alpha = 2\pi/h$ this gives

$$y_\lambda \approx h\frac{\alpha}{m}\sqrt{\frac{c_{66}}{\rho}} \tag{131}$$

and for $2\sqrt{\rho/c_{66}}\, m/\alpha \approx n$ one obtains

$$y_\lambda \approx \frac{2h}{n}. \tag{132}$$

The wavelength where solutions do not exist is twice one period of the material, for

$n = 1$, or *four* layers of the ...0/90/0/90 material. Comparing these wavelengths with those exhibited by the η in the previous figures, it is seen that the wavelength of these waves is twice the wavelength which the corresponding η would be, if it existed.

Equation (131) may be used to get an idea of wavelengths for various α/m. For $\alpha/m = 1$ the wavelength is about 33 meters, and for $\alpha/m = 10^{-6}$, the wavelength is about 0.0033 cm. For higher frequencies than this, though some were presented in Table 2, a model should be used which takes into account the microstructure of the layer, since the fibers are 0.001 cm in diameter.

Some comments seem in order. First, the factor of 2 which appears in this section is not a result of the method. If one chose to let h cover two complete periods of layering, all the extra Fourier coefficients would vanish and the same frequencies would lack solutions as before. Thus, there is something real about it. Second, although the perturbation was not very accurate numerically near the transition points in the solution, it did however predict the qualitative behavior as to how the solutions would change. Third, the reason why the numerical solution portrays no more peaks for higher frequencies in Figs 3 and 4 is that higher frequency Fourier terms in the numerical basis would be needed to pick up the higher frequency terms in the solutions, which terms lead to the peaks. Finally, for this material the frequencies where the solution does not exist are roughly

$$v \approx \frac{1}{h}\frac{m}{\alpha} = \frac{n}{2h}\sqrt{\frac{c_{66}}{\rho}} = 1.6n \text{ MHz}, \tag{133}$$

with n any positive integer.

As a practical matter, these are the frequencies which the designer should take into account in the use of these materials, for they lead to large strains.

SOME NUMERICAL RESULTS FOR THE PARALLEL WAVE

In a similar fashion, the parallel waves were examined numerically. These waves required more terms in the perturbation in order to show good agreement with the numerical results.

Using the Galerkin method, eqns (55) and (56) lead to a generalized eigenvalue problem of the form

$$(\lambda^2 A + \lambda B + C)\vec{c} = 0 \tag{134}$$

where A, B, and C are symmetric materials. This λ is the same inverse wavespeed which appears in eqns (65) and (66). If these waves were dispersive, then λ would depend upon m.

Equation (134) can be factored to give

$$\begin{pmatrix} -C & 0 \\ -B & I \end{pmatrix}\begin{pmatrix} \vec{c} \\ \vec{y} \end{pmatrix} = \lambda \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}\begin{pmatrix} \vec{c} \\ \vec{y} \end{pmatrix} \tag{135}$$

where $\vec{y}$ is a dummy vector. This is a form of an eigenvalue problem which can be solved by EISPACK. Using the double precision RGG path and 19 Fourier basis functions for both η and μ, some results were obtained.

Of the eigenvalues, four are found to be real. They came in pairs, a positive and a negative one for the longitudinal wave, and a positive and a negative one for the transverse wave. These correspond to waves travelling in the $+x$ and $-x$ directions.

Table 4. The parallel wave numerical results for $\alpha/m = 1$ ($\log_{10}(\alpha/m) = 0$)

z	$\mu_{comp}(z)$	Two terms $\mu_{pert}(z)$	% Error	Six terms $\mu_{pert}(z)$	% Error
0.3307	0.1122E-06	0.1086E-06	−3.231	0.1122E-06	−0.011
0.6614	0.8210E-07	0.9061E-07	10.361	0.8210E-07	−0.011
0.9921	0.5194E-07	0.6280E-07	20.897	0.5195E-07	0.002
1.3228	0.2270E-07	0.2819E-07	24.182	0.2270E-07	−0.002
1.6535	−0.7873E-08	−0.9482E-08	20.425	−0.7873E-08	−0.004
1.9842	−0.3674E-07	−0.4612E-07	25.522	−0.3674E-07	0.000
2.3149	−0.6781E-07	−0.7776E-07	14.673	−0.6782E-07	0.002
2.6456	−0.9585E-07	−0.1010E-06	5.351	−0.9584E-07	−0.007
2.9762	−0.1287E-06	−0.1133E-06	−11.974	−0.1286E-06	−0.013
3.3069	−0.1287E-06	−0.1133E-06	−11.973	−0.1286E-06	−0.012
3.6376	−0.9585E-07	−0.1010E-06	5.350	−0.9584E-07	−0.007
3.9683	−0.6782E-07	−0.7776E-07	14.667	−0.6782E-07	−0.004
4.2990	−0.3674E-07	−0.4612E-07	25.519	−0.3674E-07	−0.002
4.6297	−0.7873E-08	−0.9482E-08	20.430	−0.7873E-08	0.000
4.9604	0.2270E-07	0.2819E-07	24.184	0.2270E-07	−0.001
5.2911	0.5195E-07	0.6280E-07	20.891	0.5195E-07	−0.003
5.6218	0.8210E-07	0.9061E-07	10.357	0.8210E-07	−0.005
5.9525	0.1122E-06	0.1086E-06	−3.229	0.1122E-06	−0.010
6.2832	0.1359E-06	0.1148E-06	−15.542	0.1359E-06	−0.013

Computed wave speed = 22913 ft s^{-1}.
Perturbation wave speed = 22914 ft s^{-1}.

Table 4 displays results for $\alpha/m = 1$, or $v = 50$ Hz. Next, Table 5 shows a wide range of results, from $\alpha/m = 10^{-3}$ ($v = 50{,}000$ Hz) to $\alpha/m = 10^2$ ($v = \frac{1}{2}$ Hz). Here the wavelengths are given by

$$\text{wavelength} = \sqrt{\frac{c_{11}}{\rho}}\frac{h}{\left(\dfrac{m}{\alpha}\right)} \tag{136}$$

and with $h = 0.02$ for the case considered, $\alpha/m = 10^{-3}$ has a wavelength of 14 cm, and $\alpha/m = 10^2$ has a wavelength of 14 km. For α/m below 10^{-4} EISPACK had errors, and for $\alpha/m = 10^3$ and above the "longitudinal" eigenvalues it lost all relation to the wavespeed as their wavespeed rapidly increased. For small frequencies, the eigenvalues for the transverse wave were very numerically sensitive and EISPACK was unable to obtain them accurately, even in double precision.

It was noted that the ε^1 perturbation solutions did not exist for certain frequencies, and a calculation of eqn (76) for the longitudinal waves gives the frequencies as

$$v = n6.34\ \text{MHz}, \tag{137}$$

which correspond to $\alpha/m = 7.88 \times 10^{-6}$ which was beyond the range of the numerically considered values. It should be noted that all the results shown were for frequencies

Table 5. The parallel wave perturbation solution compared with the computed solution for various α/m. The perturbation wave speed is 22914 ft s^{-1} and 5332 ft s^{-1}.

$\log_{10}(\alpha/m)$	Wave speed (ft s^{-1}) Longitudinal	Transverse	μ Two terms Max. % error	Six terms Max. % error
−3.0	22911	5256	25.72	−0.43
−2.0	22938	5254	25.66	0.12
−1.0	22919	5251	25.59	0.06
0.0	22913		25.52	−0.01
1.0	22913		25.51	−0.02
2.0	22914		25.54	0.03

significantly below the first frequency for which the first-order perturbation did not exist. Based upon the richness of behavior displayed by the perpendicular case, we probably should not draw many conclusions about these higher frequencies. Also, for this material, the frequencies of nonexistence for the transverse waves turned out to be complex, and thus not physically realizable.

SUMMARY

A perturbation approach was presented and shown to give good results in wave propagation. It also predicted qualitative behavior in regions where solutions of the perturbation approach did not exist.

The analysis revealed that a discrete spectrum of frequencies (see eqns (132) and (133)) gives rise to large stresses and (elastic) strains. While it is well recognized that large stresses and strains are in violation of the linear hypothesis, nevertheless important physical information can be extracted from the results. For example, in the Linear Theory of Fracture, although the stresses close to the crack tip are very large, important physical results have been obtained on the bases of linear elasticity. Be that as it may, the authors believe that this phenomenon is a form of resonance attributed to the particular layered structure. These frequencies should be considered by the designer of composite structures for they may lead to failures and possibly the premature loss of a structure.

Finally, the above results may now be used to investigate the effects that this type of wave has on the mechanism of failure at pre-existing cracks. This study has recently been completed and the results will be reported in a follow-up paper.

Acknowledgements This work was supported by the University of Utah Department of Mathematics, and in part by Idaho National Engineering Labs with a contract from ONR, N 00014-87-F-0097. The authors are grateful for this support and would like to thank Dr V. Deason from INEL and Dr Y. Rajapakse from ONR. The authors would also like to thank Dr Jon Epstein for many helpful discussions during the progress of the project.

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