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SPATIAL BOUNDS ON THE EFFECTIVE COMPLEX PERMITTIVITY FOR TIME-HARMONIC WAVES IN RANDOM MEDIA*

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Abstract. We consider wave propagation in random cell materials when the wavelength is finite, so that scattering effects must be taken into account. An effective dielectric coefficient is introduced, which in general is a spatially dependent function, yet reduces, under the infinite wavelength assumptions, to the constant effective parameter in the quasistatic limit. We present an upper bound on the effective permittivity and a bound on its spatial variations that depends on the maximum volume of the inhomogeneities and the contrast of the medium. Numerical experiments illustrate the rigorous results.

Key words. random media, effective properties, electromagnetic scattering

AMS subject classifications. 78A48, 78A40, 78A45

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1. Background.

Usually, when one considers the propagation of an electromagnetic wave in a random medium, two parameters are of importance. The first, δ/λ, is the ratio of the length scales of the typical inhomogeneities in the medium to the wavelength of the electromagnetic wave probing the medium. The second is the contrast of the medium. Considerable effort over many decades has been applied to building effective medium theories that are applicable to wave propagation when the wavelengths associated with the fields are much larger than the microstructural scale. This limit where the ratio δ/λ goes to zero is called the quasistatic or infinite wavelength limit. In this case the heterogeneous material is replaced by a homogeneous, fictitious medium whose macroscopic characteristics are good approximations of the initial ones. The solutions of a boundary value partial differential equation describing the propagation of waves converge to the solution of a limit boundary value problem, which is explicitly described when the size of the heterogeneities goes to zero. Similarly, in the limit when the contrast goes to zero, convergence of the solution to the solution of a constant coefficient partial differential equation is obtained.

The problem of finding bounds on the effective properties of materials in the quasistatic limit has been investigated vigorously, and there have been significant advances not only in deriving optimal bounds, but also in describing the materials that attain these bounds. See [13] and the references within. Wellander and Kristensson [19] and Conca and Vanninathan [4] have both recently analyzed the homogenization of time-harmonic wave problems in periodic media, using entirely different methods. Their results are each applicable to problems in which the wavelength of the incident field is much larger than the microstructure.

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For waves in random media, Keller and Karal [11] and Papanicolaou [16] use averaging of random realizations of materials in order to describe the effective properties of the composites when interacting with electromagnetic waves. Both analyses assume that the random materials deviate slightly from a homogeneous material; i.e., the contrast of the random inclusions is small. Keller and Karal assume a priori that the effective dielectric coefficient is a constant. Using perturbation methods, they approximate the dielectric constant with a complex number, whose imaginary part accounts for the wave attenuation.

A comprehensive overview of the subject of wave propagation in random media is given in a book by Ishimaru [10]. Also, recent results in this field can be found in the AMS-IMS-SIAM proceedings edited by Kuchment [12].

The above methods that provide bounds and describe the behavior of the dielectric coefficients do not account for scattering effects that occur when the wavelength is no longer much larger than the inhomogeneities of the composite and when the contrast is large. Results for this problem are sparse. The problem is difficult and the techniques that come from the quasistatic regime cannot be applied directly to the scattering problem since the quasistatic methods utilize the condition that the size of the heterogeneities goes to zero.

Even the correct definition of "effective medium" is somewhat unclear outside the quasistatic regime. In this work, we assume that the purpose of the effective medium is to reproduce the average or expected wave field as the actual medium varies over a given set of random realizations.

For simplicity in this work we consider waves in two- or three-dimensional random cell materials (discussed in section 2.2) governed by the Helmholtz equation

$$\Delta u + \omega^2 \varepsilon u = f,$$

where realizations of the random permittivity function $\varepsilon(x)$ belong to some probability space. We average over all the possible material realizations to obtain the equation

$$\Delta\langle u\rangle + \omega^2 \langle \varepsilon u\rangle = f,$$

where $\langle\cdot\rangle$ denotes expected value, i.e., averaging over the set of realizations, and not a spatial average. The source f is assumed to be independent of the material. Problems like this arise, for example, in measurements of the properties of sea ice samples (usually through interrogation by electromagnetic fields), or of earth samples (by either acoustic or electromagnetic waves). We seek to find the dielectric coefficient ε^* that will solve the problem

$$\Delta\langle u\rangle + \omega^2 \varepsilon^* \langle u\rangle = f, \tag{1.1}$$

where $\langle u\rangle$ is the expected value of the solution u. From the above two equations, it is easy to see that the appropriate definition for ε^* is

$$\varepsilon^* = \frac{\langle \varepsilon u\rangle}{\langle u\rangle}. \tag{1.2}$$

Note that the definition of ε^* does not preclude spatial variations, $\varepsilon^* = \varepsilon^*(x)$.

The definition in (1.2) is similar to the definition of the effective dielectric coefficient of an isotropic medium in the quasistatic case. In this case, the effective permittivity ε^* is defined by

$$\varepsilon^* \langle E \rangle = \langle D \rangle = \langle \varepsilon E \rangle,$$

where the averaged electric field $\langle E \rangle = \bar{E}$ is a given constant, and the averaged dielectric displacement $\langle D \rangle$ is independent of x, which ensures that ε^* in the quasistatic case is a constant.

We can calculate the quasistatic effective dielectric constant by letting the wavelength λ go to infinity, or equivalently, by letting the frequency ω approach zero. Let $\varepsilon = \varepsilon_0 \chi + \varepsilon_1 (1 - \chi)$, where χ is a characteristic function of the material ε_0, and the expected value of χ when we sum over all possible material realizations is p; i.e., $\langle \chi \rangle = p$. Let G_{ω,ε_1} be the free-space Green's function for the operator $Lv = \Delta v + \omega^2 \varepsilon_1 v$ (with the outgoing wave condition). Our problem can be rewritten to yield the Lippmann–Schwinger equation

$$u(x) = \omega^2 (\varepsilon_1 - \varepsilon_0) \int_\Omega G_{\omega,\varepsilon_1}(|x-y|) \chi(y) u(y)\, dy + q(x), \tag{1.3}$$

where $q = G_{\omega,\varepsilon_1} \star f$. Define the operator $A_\omega : L^2(\Omega) \to L^2(\Omega)$ by

$$(A_{\omega,\varepsilon_1} v)(x) = \int_\Omega G_{\omega,\varepsilon_1}(|x-y|) v(y)\, dy, \qquad x \in \Omega. \tag{1.4}$$

In the case when $\omega^2 |\varepsilon_1 - \varepsilon_0| \|A_{\omega,\varepsilon_1}\| < 1$,

$$u = (I - \omega^2 (\varepsilon_1 - \varepsilon_0) A_{\omega,\varepsilon_1} \chi)^{-1} q, \tag{1.5}$$

and the Neumann series

$$u = q + \omega^2 (\varepsilon_1 - \varepsilon_0) A_{\omega,\varepsilon_1} \chi q + \cdots \tag{1.6}$$

converges absolutely. Take the average over all realizations to obtain

$$\begin{aligned} \langle u \rangle &= q + \omega^2 (\varepsilon_1 - \varepsilon_0) A_{\omega,\varepsilon_1} \langle \chi \rangle q + \cdots \\ &= q + \omega^2 (\varepsilon_1 - \varepsilon_0) p A_{\omega,\varepsilon_1} q + \cdots \end{aligned}$$

and

$$\langle \varepsilon u \rangle = \langle \varepsilon \rangle q + \omega^2 (\varepsilon_1 - \varepsilon_0) \langle \varepsilon A_{\omega,\varepsilon_1} \chi \rangle q + \cdots .$$

Thus, the quasistatic effective dielectric coefficient is

$$\lim_{\omega \to 0} \varepsilon^* = \frac{\lim_{\omega \to 0} \langle \varepsilon u \rangle}{\lim_{\omega \to 0} \langle u \rangle} = \frac{\langle \varepsilon \rangle q}{q} = \varepsilon_0 p + \varepsilon_1 (1 - p).$$

Note that only the arithmetic mean, and not the harmonic mean, appears since the material coefficients only appear in the lowest-order term in the equation. This is different from classical homogenization for the equation $\nabla \cdot \epsilon E = 0$.

Wave localization and cancellation must be accounted for when the wavelength is on the same order as the size of the heterogeneities, which means that the effective coefficients are no longer necessarily constants as in the quasistatic case, but functions of the spatial variable. We have illustrated in section 4 that as ω increases (which will decrease the wavelength), we begin to see spatial variations in the effective dielectric coefficient due to the presence of scattering effects. Nevertheless ε^* as defined in (1.2) is a "correct" definition of the effective dielectric coefficient, in that it reproduces the average field response through (1.1).

Since ε^* cannot be calculated explicitly in general, to be useful in applications it is important that we can bound both ε^* itself and some measure of the spatial variations in ε^*. The main result of this paper, presented in Theorem 3.1, is a bound on the magnitude of ε^* and a local bound on the total variation, $\|\varepsilon^*\|_{BV}$. The estimates hold for any fixed frequency $\omega > 0$ and show an explicit dependence on the feature size and contrast of the random medium.

The paper is organized as follows. We pose the model problem of electromagnetic wave propagation in a composite material in subsection 2.1. The two-component composite material is random, and its structure is defined in subsection 2.2 using random variables that describe its geometry and component dependence. In subsection 2.3 we obtain existence and uniqueness of solutions and uniform bounds on the solutions, as well as Lipschitz bounds with respect to the dielectric coefficients of the materials.

Both the uniform and Lipschitz bounds are instrumental in obtaining the results of the paper. Spatial variations due to scattering effects are allowed. Bounds on the effective dielectric coefficient and its spatial variations are obtained when certain conditions are satisfied. These results are stated in the theorem in section 3, which is proved using methods that incorporate both PDE analysis and probability arguments. In section 4 the effective dielectric coeffient is calculated numerically in one- and two-dimensional media, and the presence of spatial variations and their dependence on the size of the heterogeneities and the contrast in the material is confirmed.

We note that while the paper is focused on results in two- and three-dimensional spaces, simple modifications also provide one-dimensional results.

2. Model problem.

2.1. Electromagnetic wave propagation. Consider time-harmonic electromagnetic wave propagation through nonmagnetic ($\mu = 1$) heterogeneous media. Assuming that the electric field vector $E = (0, 0, u)$ and ε is independent of x_3, Maxwell's equations reduce to the Helmholtz equation

$$\Delta u + \omega^2 \varepsilon u = 0, \tag{2.1}$$

where ω represents the frequency, and $\varepsilon \in L^\infty(\mathbb{R}^n)$ is the dielectric coefficient. In media with heterogeneities in all three dimensions, (2.1) models time-harmonic acoustic wave propagation, where $\epsilon(x)$ is the squared slowness of an isotropic medium.

Let our bounded spatial domain be $\mathbf{\Omega} \subseteq \mathbb{R}^n$, where $n = 2, 3$. The region outside $\mathbf{\Omega}$ is filled with a homogeneous material. In particular, assume for $x \notin \mathbf{\Omega}$, we have $\varepsilon(x) = 1$. Let S_0 be the sphere of radius R_0, i.e., $S_0 = \{r = R_0\}$, and let $\mathbf{\Omega}_0 = \{|x| < R_0\}$, where R_0 is chosen such that $\mathbf{\Omega} \subset \mathbf{\Omega}_0$ (see Figure 2.1).

Outside the ball $\mathbf{\Omega}_0$, we separate the solution u to (2.1) into the incident and scattered field: $u = u_i + u_s$. The scattered field u_s can also be separated. Wellposedness of the problem requires imposing Sommerfeld's radiation condition as a boundary condition at infinity; i.e.,

$$\lim_{r\to\infty} r^{\frac{n-1}{2}} \left(\frac{\partial}{\partial r} - i\omega \right) u_s = 0,$$

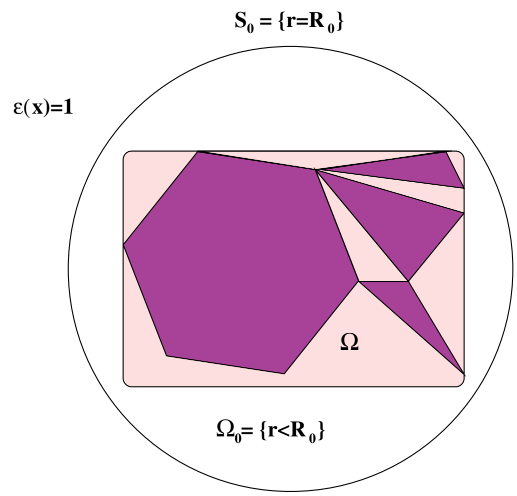

FIG. 2.1. *Bounded random medium* (Ω)*, enclosed in a sphere* S_0 *to form the domain* $\Omega_0 = \{|x| < R_0\}$.

uniformly in all directions, where $n = 2, 3$ is the spatial dimension. Here, it is assumed that the time-harmonic field is $e^{-i\omega t}u$.

The linear operator $T\colon H^{\frac{1}{2}}(S_0) \to H^{-\frac{1}{2}}(S_0)$ (Dirichlet-to-Neumann map) defines the relationship between the traces $u_s|_{\{r=R_0\}}$ and $\partial_r u_s|_{\{r=R_0\}}$; i.e., $T(u_s|_{\{r=R_0\}}) = (\partial_r u_s)|_{\{r=R_0\}}$. The Dirichlet-to-Neumann operator defines an exact nonreflecting boundary condition on the artificial boundary S_0; i.e., there are no spurious reflections of the scattered solution introduced at S_0. We write T explicitly for the two- and three-dimensional cases in the appendix. On the boundary $S_0 = \{r = R_0\}$, the solution $u = u_i + u_s$ should then satisfy

$$\partial_r u - Tu = \partial_r u_i - Tu_i + \partial_r u_s - Tu_s = \partial_r u_i - Tu_i \equiv c.$$

In this way the problem on $\mathbb{R}^n$ is equivalently replaced by

$$\Delta u + \omega^2 \varepsilon u = 0 \quad \text{in } \Omega_0 \supset \Omega,$$
$$(\partial_r u - Tu) = c \quad \text{on } S_0.$$

2.2. Random structure. We are interested in computing expected values of wave fields as the underlying medium ranges over some class of random materials. In this section, we define the probability space characterizing these materials.

We fill our bounded domain Ω by random cell materials (see, e.g., Milton [13]). Our two-phase random materials are constructed as follows. The first step is to divide Ω into a finite number of cells. The cells may vary in size and shape, but their volume is bounded by a parameter.

The second step is to randomly assign to each cell a material of permittivity ϵ_0 with probability p or ϵ_1 with probability $1 - p$ in a way that is uncorrelated both with the shape of the cell and with the phases assigned to the surrounding cells. We then have a probability space $(\Psi_\delta, \mathcal{J}_\delta, P_\delta)$, where Ψ_δ is a set of material realizations with a σ-algebra $\mathcal{J}_\delta$ of subsets of Ψ_δ, and a probability measure P_δ on $\mathcal{J}_\delta$ with $P_\delta(\Psi_\delta) = 1$. The parameter δ bounds the volume of each cell, and its precise definition is given later in the section.

Elements $\psi \in \mathbf{\Psi}_\delta$ are characterized by two random variables, $\psi = (m, g)$, where the variable m depends on the random variable g. The variable g describes the geometry of the material by partitioning the domain $\mathbf{\Omega}$ into N_g parts, each of which is filled either with material ε_0 or material ε_1, which is done by the random variable m. Thus, g describes the subdivision of our domain into subdomains; once the geometry g is fixed, the random variable m distributes the material in the subdomains. Denoting some set of partitions of $\mathbf{\Omega}$ by $\mathbf{\Gamma}_\delta$, the variable $g \in \mathbf{\Gamma}_\delta$, partitions the spatial domain $\mathbf{\Omega}$ into N_g disjoint subdomains $\{\mathbf{\Omega}_j\}_{j=1}^{N_g}$ such that $\cup\, \mathbf{\Omega}_j = \mathbf{\Omega}$. The variable $m_g = \{m_1, \ldots, m_{N_g}\}$ assigns zero for material ε_0 with probability p or one for material ε_1 with probability $1 - p$ in each spatial subdomain. The real part of the dielectric constant in the composite material is defined by

$$\varepsilon_{m,g}(x) = \begin{cases} \varepsilon_0 & \text{if } m_j = 0 \quad \text{and} \quad x \in \mathbf{\Omega}_j; \\ \varepsilon_1 & \text{if } m_j = 1 \quad \text{and} \quad x \in \mathbf{\Omega}_j. \end{cases}$$

We assume without loss of generality that $\varepsilon_1 > \varepsilon_0$.

Fix a geometry g. Denote the set of realizations for geometry g by R_g:

$$R_g = \{m_g = (m_1, \ldots, m_{N_g})\colon m_j = 0 \text{ or } m_j = 1, j = 1, \ldots, N_g\}.$$

The set R_g has 2^{N_g} elements. Thus the set of material realizations, $\mathbf{\Psi}_\delta$ is described as follows:

$$\mathbf{\Psi}_\delta = \{(g, m_g)\colon g \in \mathbf{\Gamma}_\delta, m_g \in R_g\}.$$

The probability measure is

$$P = \sum_{m_g \in R_g} \prod_{j=1}^{N_g} p^{1-m_j}(1-p)^{m_j} G_\delta, \tag{2.2}$$

where G_δ is the probability measure on the space of all geometries, $\mathbf{\Gamma}_\delta$. The product describes the multiplication of the probabilities of the materials in each subdomain $\mathbf{\Omega}_j$, which is summed over the set of all realizations for a particular geometry g.

$(\mathbf{\Psi}_\delta, \mathcal{J}_\delta, P_\delta)$ depends on a parameter $\delta > 0$. Let k be a whole number, independent of δ. We make the following assumptions on the subdomain partitions in $\mathbf{\Gamma}_\delta$ (see Figure 2.2):

- A1: The volume of each subdomain $\{\mathbf{\Omega}_j\}_{j=1}^{N_g}$ is bounded by δ; i.e., $|\mathbf{\Omega}_j| \leq \delta$. Note that since the volume of $\mathbf{\Omega}$ is fixed, as δ decreases, the set of realizations $\mathbf{\Psi}_\delta$ must change.
- A2: Let k be a fixed number. For each $\delta > 0$, there exists $\eta > 0$ such that any ball with volume η, $B_r(x)$ intersects at most k subdomains $\mathbf{\Omega}_j$ for all $x \in \mathbf{\Omega}$. This condition excludes from consideration materials with infinitely many subdomains interfacing at any $x \in \mathbf{\Omega}$. Here $B_r(x)$ denotes the ball of radius $r = \sqrt{\eta/\pi}$ in two dimensions and radius $r = (\frac{3\eta}{4\pi})^{1/3}$ in three dimensions, centered at x.
- A3: Using $B_r(x)$ from A2, define the set

$$S_{x,r} = \left(\bigcup \partial\mathbf{\Omega}_j\right) \bigcap B_r(x).$$

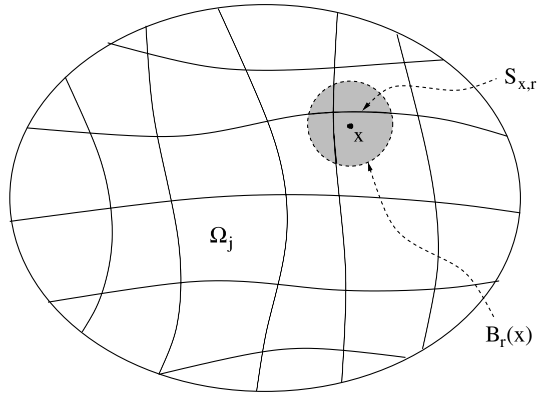

FIG. 2.2. *Example of a particular subdomain partition $\Omega = \bigcup_j \Omega_j$ in Γ_δ, illustrating assumptions A1–A3. For each such partition, all subdomains Ω_j must satisfy $|\Omega_j| \le \delta$, there can be only a finite number k of subdomain boundaries intersecting near any given point, and the local measure (arclength in the figure) of the subdomain boundaries $S_{x,r}$ must remain bounded for all δ.*

There exists a constant C_p (independent of δ) such that the Lebesgue measure of the set $S_{x,r}$ satisfies

$$\mathcal{L}^{n-1}(S_{x,r}) \le C_{p^{r^{n-1}}} \quad \text{for all } x \in \Omega.$$

This condition excludes from consideration materials containing subdomains with boundaries with infinite perimeter in $B_r(x)$.

One can readily check, for example, that a simple subdivision of Ω by a uniform grid of rectangles when $n = 2$, or rectangular solids when $n = 3$, satisfies A1–A3, where δ is the maximum volume of each subregion.

2.3. Existence and uniqueness of solutions and Lipschitz bounds. For a fixed dissipation constant $\epsilon_i > 0$, define a set

$$\mathcal{A} \coloneqq \{\varepsilon = \varepsilon_r + i\varepsilon_i \colon \varepsilon_r = \varepsilon_{m,g} \text{ for some } (m, g) \in \Psi_\delta\}.$$

Given an incident field u_i, we must solve the following problem:

$$\Delta u + \omega^2 \varepsilon_r u + i\omega^2 \varepsilon_i u = 0 \quad \text{in } \Omega_0, \tag{2.3}$$

$$\left(\frac{\partial u}{\partial r} - Tu\right) = c \quad \text{on } S_0. \tag{2.4}$$

Existence and uniqueness of weak solutions, with a uniform bound, may be obtained for materials with a little bit of absorption; i.e., $\varepsilon_i > 0$.

Throughout the remainder of the paper, in order to simplify estimates within proofs, C will denote a constant that is independent of (ε, u), whose value may change from line to line.

LEMMA 2.1. *For each $\varepsilon \in \mathcal{A}$, problem (2.3)–(2.4) admits a unique weak solution $u \in H^2(\Omega_0)$. Furthermore, there exists a constant C depending on $\mathcal{A}$ such that $\|u\|_{H^2(\Omega_0)} \le C$, independent of $\varepsilon \in \mathcal{A}$. Note that the constant C depends in particular on the fixed parameter $\epsilon_i > 0$.*

Proof. The ideas for the proof of the lemma come from the proof of a similar lemma in [5]. Define for u, $v \in H^1(\Omega_0)$

$$a(u,v) = \int_{\Omega_0} \nabla u \cdot \overline{\nabla v} - \omega^2 \int_{\Omega_0} \varepsilon u \bar{v} - \int_{S_0} (Tu)\bar{v},$$

and

$$b(v) = c \int_{S_0} \bar{v}.$$

Using bounds (6.3) and (6.6) in the appendix for the two- and three-dimensional problems, respectively, it is straightforward to show that $a(u, v)$ defines a bounded sesquilinear form over $H^1(\Omega_0) \times H^1(\Omega_0)$, and that $b(v)$ is a bounded linear functional on $H^1(\Omega_0)$. Weak solutions $u \in H^1(\Omega_0)$ of (2.3) solve the variational problem

$$a(u,v) = b(v) \quad \text{for all } v \in H^1(\Omega_0). \tag{2.5}$$

The sesquilinear form a uniquely defines a linear operator $A\colon H^1(\Omega_0) \to H^1(\Omega_0)$ such that $a(u, v) = \langle Au, v\rangle_{H^1(\Omega_0)}$, and the functional $b(v)$ is uniquely identified with an element $b \in H^1(\Omega_0)$ such that $b(v) = \langle b, v\rangle$. By reflexivity, problem (2.5) is then equivalently stated as

$$Au = b. \tag{2.6}$$

We intend to show that a is coercive by establishing a bound $|a(u, u)| \geq c > 0$ for all $u \in H^1(\Omega_0)$ with $\|u\|_{H^1(\Omega_0)} = 1$. We have

$$\begin{aligned} a(u,u) = &\int_{\Omega_0} |\nabla u|^2 - \omega^2 \int_{\Omega_0} \varepsilon_r |u|^2 - \mathfrak{R}\left(\int_{S_0} (Tu)\bar{u}\right) \\ &- i\mathfrak{I}\left(\int_{S_0} (Tu)\bar{u}\right) - i\omega^2 \varepsilon_i \int_{\Omega_0} |u|^2. \end{aligned} \tag{2.7}$$

For the two-dimensional problem, we have

$$\int_{S_0} (Tu)\bar{u} = \int_{S_0} \sum_{m=1}^{\infty} \gamma_m \hat{u}_m e^{im\theta} \bar{u} = \sum_{m=1}^{\infty} \gamma_m |\hat{u}_m|^2,$$

where $\hat{u}_m$ are the Fourier coefficients of the trace $u|_{S_0}$ (see appendix). $\mathfrak{R}(\gamma_m) < 0$ and $\mathfrak{I}(\gamma_m) > 0$ for every m. Thus,

$$\mathfrak{R}\left(\int_{S_0} (Tu)\bar{u}\right) < 0 \quad \text{and} \quad \mathfrak{I}\left(\int_{S_0} (Tu)\bar{u}\right) > 0.$$

Similarly, for the three-dimensional case

$$\int_{S_0} (Tu)\bar{u} = \int_{S_0} \sum_{l=0}^{\infty} \gamma_l \sum_{m=-l}^{l} \hat{u}_{lm} Y_{lm} \bar{u} = \sum_{l=0}^{\infty} \gamma_l \sum_{m=-l}^{l} |\hat{u}_{lm}|^2,$$

where $\hat{u}_{lm}$ are the coefficients in the spherical harmonics expansion of the trace $u|_{S_0}$ (see appendix). $\mathfrak{R}(\gamma_l) < 0$ and $\mathfrak{I}(\gamma_l) > 0$ for every l. Thus,

$$\mathfrak{R}\left(\int_{S_0} (Tu)\bar{u}\right) < 0 \quad \text{and} \quad \mathfrak{I}\left(\int_{S_0} (Tu)\bar{u}\right) > 0.$$

Assuming $\|u\|^2_{H^1(\Omega_0)} = \int_{\Omega_0}|\nabla u|^2 + \int_{\Omega_0}|u|^2 = 1$, and noticing that the first three terms on the right-hand side of (2.7) are purely real and the last two terms are purely imaginary, we find

$$2|a(u,u)| \geq \left|1 - \int_{\Omega_0}(1+\omega^2\varepsilon_r|u|^2) - \mathfrak{R}\left(\int_{S_0}(Tu)\bar{u}\right)\right| + \left| - \omega^2\varepsilon_i\int_{\Omega_0}|u|^2 - \mathfrak{I}\left(\int_{S_0}(Tu)\bar{u}\right)\right|.$$

For convenience, write $r = \int_{\Omega_0}(1+\omega^2\varepsilon_r)|u|^2$, $s = \int_{\Omega_0}|u|^2$, and

$$t = \begin{cases} -\sum_{m=1}^{\infty}\mathfrak{R}(\gamma_m)|\hat{u}_m|^2 & \text{in two dimensions;} \\ -\sum_{l=0}^{\infty}\mathfrak{R}(\gamma_l)\sum_{m=-l}^{l}|\hat{u}_{lm}|^2 & \text{in three dimensions.}\end{cases}$$

Obviously t, r, and s are nonnegative real numbers that depend on u (and ε in the case of r). Although t and s are essentially independent, r must satisfy

$$(1+\omega^2\varepsilon_0)s \leq r \leq (1+\omega^2\varepsilon_1)s. \tag{2.8}$$

With this notation,

$$2|a(u,u)| \geq |1+t-r| + \omega^2\varepsilon_i s.$$

Note that in the case $s \geq \frac{1}{2(1+\omega^2\varepsilon_1)}$, we have $|a(u,u)| \geq \frac{1}{2}\omega^2\varepsilon_i s \geq \frac{\omega^2\varepsilon_i}{4(1+\omega^2\varepsilon_1)}$. Otherwise, $s < \frac{1}{2(1+\omega^2\varepsilon_1)}$ so that $r < \frac{1}{2}$, and $|a(u,u)| \geq \frac{1}{2}|1+t-r| > \frac{1}{4}$. Hence, for all s, $t \geq 0$, and all r satisfying (2.8),

$$|a(u,u)| \geq c = \min\left\{\frac{\omega^2\varepsilon_i}{4(1+\omega^2\varepsilon_1)}, \frac{1}{4}\right\}.$$

The bound thus holds for every u with $\|u\|_{H^1(\Omega_0)} = 1$ and for every $\varepsilon \in \mathcal{A}$ with $\varepsilon_i > 0$. Given this coercivity bound, direct application of the Lax–Milgram theorem (see, e.g., Lemma 2.21, p. 20 in Monk [14]) yields existence of a bounded solution operator A^{-1} for problem (2.6). Since b is fixed and bounded, it follows that $\|u\|_{H^1(\Omega_0)} \leq C$.

Given the bound on $\|u\|_{H^1(\Omega_0)}$, a uniform $H^2(\Omega_0)$ bound follows easily, since $\Delta u = -\omega^2\varepsilon u$ is uniformly bounded in $L^2(\Omega_0)$. ◻

LEMMA 2.2. *There exists a constant K such that for every ε_s, $\varepsilon_t \in \mathcal{A}$, if $u_s(\varepsilon_s)$, $u_t(\varepsilon_t)$ are the corresponding solutions of the Helmholtz equation (2.3)–(2.4), then u_s and u_t satisfy the Lipschitz condition*

$$\|u_t - u_s\|_{H^2} \leq K\|\varepsilon_t - \varepsilon_s\|_{L^2}. \tag{2.9}$$

Moreover, there exists a constant C such that

$$\|u_t - u_s\|_{W^{1,\infty}} \leq CK\|\varepsilon_t - \varepsilon_s\|_{L^2}. \tag{2.10}$$

Proof. We subtract one of the Helmholtz equations from the other to obtain

$$\Delta u_t - \Delta u_s + \omega^2\varepsilon_t u_t - \omega^2\varepsilon_s u_s = 0.$$

Subtract $\omega^2\varepsilon_t u_s$ on both sides:

$$\Delta(u_t - u_s) + \omega^2 \varepsilon_t (u_t - u_s) = -\omega^2 (\varepsilon_t - \varepsilon_s) u_s.$$

Let $w = u_t - u_s$. Thus the above equation is written as

$$\Delta w + \omega^2 \varepsilon_t w = -\omega^2 (\varepsilon_t - \varepsilon_s) u_s. \tag{2.11}$$

The function $-\omega^2(\varepsilon_t - \varepsilon_s)u_s \in L^2(\Omega)$ and thus Lemma 2.1 applies and w is a solution to (2.11). Let us rewrite (2.11) using the operator L_{ε_t}:

$$L_{\varepsilon_t} w := \Delta w + \omega^2 \varepsilon_t w = -\omega^2 (\varepsilon_t - \varepsilon_s) u_s.$$

Lemma 2.1 ensures that the inverse operator $L_{\varepsilon_t}^{-1}: L^2(\Omega) \to H^2(\Omega)$ exists and is uniformly bounded with respect to $\varepsilon_t \in \mathcal{A}$. Thus,

$$w = -\omega^2 L_{\varepsilon_t}^{-1} (\varepsilon_t - \varepsilon_s) u_s.$$

For both two- and three- dimensional materials, the Sobolev imbedding theorem implies that $H^2(\Omega) \subset C_B^0(\Omega)$ [1] and hence $\|u_s\|_{L^\infty}$ is bounded, so

$$\|w\|_{H^2} \leq \|L_{\varepsilon_t}^{-1}\|_{L^2(\Omega), H^2(\Omega)} \|\varepsilon_t - \varepsilon_s\|_{L^2} \|u_s\|_{L^\infty} \leq K \|\varepsilon_t - \varepsilon_s\|_{L^2}.$$

To prove the second part of the lemma, we use the Sobolev imbedding theorem and interpolation inequalities. We prove that $w \in W^{2,q}$ for any q such that $3 < q < \infty$. Using the interpolation inequalities in [1] we see that for any solution u of (2.3)–(2.4)

$$\|\Delta u\|_{L^q} \leq \|\Delta u\|_{L^2}^{2/q} \|\Delta u\|_{L^\infty}^{1-2/q} \leq \omega^2 \|u\|_{H^2}^{2/q} \|\varepsilon u\|_{L^\infty}^{1-2/q} \leq \omega^2 \varepsilon_1^{1-2/q} \|u\|_{H^2}.$$

Thus $u \in W^{2,q}$. However, the Sobolev imbedding theorem [1] implies that $W^{2,q}(\Omega) \subset C_B^1(\Omega)$; i.e., there exists a constant C such that

$$\|u_t - u_s\|_{1,\infty} \leq C \|u_t - u_s\|_{W^{2,q}} \leq CK \|\varepsilon_t - \varepsilon_s\|_{L^2}, \tag{2.12}$$

where

$$\|u\|_{1,\infty} := \max_{0 \leq |\alpha| \leq 1} \sup_{x \in \Omega} |D^\alpha u(x)|.$$

We deduce the Lipschitz condition (2.10) from (2.12). ☐

We also obtain a Lipshitz-type bound that estimates the proximity of solutions u of the Helmholtz equation (2.3)–(2.4) and the solution $\tilde{u}$ of the constant coefficient Helmholtz equation, where the constant coefficient is the expected value of ε; i.e., $\tilde{\varepsilon} \equiv \langle \varepsilon \rangle = \varepsilon_0 p + \varepsilon_1 (1-p) + i\varepsilon_i$. The bound is in terms of the local proximity of the random medium ε and the homogeneous medium $\tilde{\varepsilon}$. For any subdomain $\tilde{\Omega} \subset \Omega$, we define the diameter

$$d(\tilde{\Omega}) = \sup_{x,y \in \tilde{\Omega}} |x - y|.$$

Lemma 2.3. *Let $\tilde{u}$ be the solution to the Helmholtz equation with constant coefficient $\tilde{\varepsilon} = \varepsilon_0 p + \varepsilon_1 (1-p) + i\varepsilon_i$, still satisfying the boundary condition* (2.4)

$$\Delta \tilde{u} + \omega^2 \tilde{\varepsilon}\, \tilde{u} = 0. \tag{2.13}$$

Let $\nu > 0$ and $3 < q < \infty$ be given. Then there exist constants K^* and K^*_∞, and $\gamma > 0$ such that if Ω is divided into N' nonoverlapping subdomains O_i such that $d(O_i) \le \gamma$ for all $i = 1, \dots, N'$, then

$$\|u - \tilde{u}\|_{L^2} \le K^* \left(\sum_{i=1}^{N'} \left| \int_{O_i} (\tilde{\varepsilon} - \varepsilon)\, dx \right| \right) + \nu \tag{2.14}$$

and

$$\|u - \tilde{u}\|_{L^\infty} \le K^*_\infty(q) \left(K^* \left(\sum_{i=1}^{N'} \left| \int_{O_i} (\tilde{\varepsilon} - \varepsilon)\, dx \right| \right) + C\nu \right)^{\frac{1}{q}} \tag{2.15}$$

for all realizations (u, ε) with $\varepsilon \in \mathcal{A}$, and u satisfying (2.3), (2.4).

For any given tolerance $\nu > 0$, the lemma gives the existence of a number $\gamma > 0$ (depending on ν) such that bounds (2.14), (2.15) hold for all realizations of the material $\varepsilon \in \mathcal{A}$, provided only that the diameter $d(O_i)$ of the covering subdomains O_i is less than γ. This lemma will be a key component in the proof of the main Theorem 3.1, allowing global control of the solutions u in terms of local averages of the coefficient ε over subdomains.

Proof. In the following proof, the difference between the solutions of (2.3) and (2.13) is written in terms of the solution operator $L^{-1}_{\tilde{\varepsilon}}$. This compact solution operator is approximated by a sequence of finite-rank operators L^{-1}_n, written in their canonical form in terms of orthonormal basis functions. These measurable functions are approximated outside of a set of small measure by continuous functions. The domain Ω is divided into N' nonoverlapping subdomains O_i of diameter at most γ such that the uniformly continuous functions are approximated by a sequence of step functions with characteristic functions of O_i. Hölder continuity of u is proven, and the difference between the solution u for every x in O_i and the maximum of u over the set O_i is bounded in terms of the diameter γ. All of these are combined to give the desired inequalities. The details of the proof follow.

Subtract the two equations (2.3) and (2.13) and manipulate them to get the equation

$$\Delta(u - \tilde{u}) + \omega^2 \tilde{\varepsilon}(u - \tilde{u}) = \omega^2 (\tilde{\varepsilon} - \varepsilon) u$$

for any realization (ε, u). Thus, we can apply the solution operator $L^{-1}_{\tilde{\varepsilon}}$ to obtain

$$u - \tilde{u} = \omega^2 L^{-1}_{\tilde{\varepsilon}}((\tilde{\varepsilon} - \varepsilon)u).$$

Now, $L^{-1}_{\tilde{\varepsilon}}$ is a bounded operator $L^{-1}_{\tilde{\varepsilon}}: L^2 \to H^2$ and a compact operator $L^{-1}_{\tilde{\varepsilon}}: L^2 \to L^2$. Since $L^{-1}_{\tilde{\varepsilon}}: L^2 \to L^2$ is compact, it can be approximated by a sequence of finite-rank operators L^{-1}_n, and for every given error $\nu_1 > 0$, there exists M_1 such that $\|L^{-1}_{\tilde{\varepsilon}} - L^{-1}_n\|_{L^2(\Omega), L^2(\Omega)} \le \nu_1$ for $n \ge M_1$ [6]. We apply the triangle inequality to obtain

$$\begin{aligned} \|u - \tilde{u}\|_{L^2} &= \omega^2 \|L^{-1}_{\tilde{\varepsilon}}(\tilde{\varepsilon} - \varepsilon)u\|_{L^2} \\ &\le \omega^2 \|L^{-1}_{\tilde{\varepsilon}} - L^{-1}_n\|_{L^2(\Omega), L^2(\Omega)} \|\tilde{\varepsilon} - \varepsilon\|_{L^\infty} \|u\|_{L^2} + \omega^2 \|L^{-1}_n(\tilde{\varepsilon} - \varepsilon)u\|_{L^2} \\ &\le C\nu_1 + \omega^2 \|L^{-1}_n(\tilde{\varepsilon} - \varepsilon)u\|_{L^2}, \end{aligned}$$

where C is independent of the material ε. Finite-rank operators can be decomposed

$$L_n^{-1}(\tilde{\varepsilon}-\varepsilon)u = \sum_{i=1}^{N} w_i^n \langle (\tilde{\varepsilon}-\varepsilon)u, g_i^n\rangle_{L^2},$$

where $g_i^n \in L^2(\mathbf{\Omega})$ and $w_i^n \in \mathrm{Range}(L_n^{-1})$. Thus,

$$\|L_n^{-1}(\tilde{\varepsilon}-\varepsilon)u\|_{L^2} = \left\|\sum_{i=1}^{N} w_i^n \int_{\mathbf{\Omega}} (\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx\right\|_{L^2} \le \sum_{i=1}^{N} \|w_i^n\|_{L^2}\left|\int_{\mathbf{\Omega}} (\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx\right|.$$

Fix $n \ge M_1$; g_i^n is a measurable function on $\mathbf{\Omega}$. Given $\nu_2 \ge 0$, there exist continuous functions v_i^n on $\mathbf{\Omega}$ such that $|S_{\nu_2}| = m\{x : g_i^n(x) \ne v_i^n(x)\} \le \nu_2$ for each $i = 1, \ldots, N$ [17]. Decompose the integral

$$\int_{\mathbf{\Omega}} (\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx = \int_{\mathbf{\Omega}\setminus S_{\nu_2}} (\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx + \int_{S_{\nu_2}} (\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx.$$

Using this we obtain the following bound for each $i = 1, \ldots, N$:

$$\left|\int_{\mathbf{\Omega}} (\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx\right| \le \left|\int_{\mathbf{\Omega}\setminus S_{\nu_2}} (\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx\right| + \left|\int_{S_{\nu_2}} (\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx\right|$$
$$\le \left|\int_{\mathbf{\Omega}\setminus S_{\nu_2}} (\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx\right| + \|\tilde{\varepsilon}-\varepsilon\|_{L^\infty}|S_{\nu_2}|^{\frac{1}{2}}\|u\|_{L^\infty}\|g_i^n\|_{L^2} \le \left|\int_{\mathbf{\Omega}\setminus S_{\nu_2}} (\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx\right| + C_2\nu_2^{\frac{1}{2}}.$$

The function v_i^n is continuous on the compact domain $\mathbf{\Omega}$ and thus it is uniformly continuous and can be approximated by a sequence of step functions $\psi_{N'}$. Divide $\mathbf{\Omega}$ into N' nonoverlapping subdomains O_i such that $d(O_i) \le \gamma$. Define $\psi_{N'} = \sum_{i=1}^{N'} a_i^{N'}\chi_{O_i}$, where χ_{O_i} is a characteristic function of the subdomain O_i. For every given error $\nu_3 > 0$, there exists $\gamma > 0$ such that $\|v_i^n - \psi_{N'}\|_{L^\infty} \le \nu_3$. Thus,

$$\left|\int_{\mathbf{\Omega}\setminus S_{\nu_2}} (\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx\right| = \left|\int_{\mathbf{\Omega}\setminus S_{\nu_2}} (\tilde{\varepsilon}-\varepsilon)uv_i^n\,dx\right| = \left|\int_{\mathbf{\Omega}} (\tilde{\varepsilon}-\varepsilon)uv_i^n\,dx - \int_{S_{\nu_2}} (\tilde{\varepsilon}-\varepsilon)uv_i^n\,dx\right|$$
$$\le \left|\int_{\mathbf{\Omega}} (\tilde{\varepsilon}-\varepsilon)uv_i^n\,dx\right| + \|\tilde{\varepsilon}-\varepsilon\|_{L^\infty}|S_{\nu_2}|\|u\|_{L^\infty}\|v_i^n\|_{L^\infty}$$
$$\le \left|\int_{\mathbf{\Omega}} (\tilde{\varepsilon}-\varepsilon)u(v_i^n - \psi_{n'})\,dx\right| + \left|\int_{\mathbf{\Omega}} (\tilde{\varepsilon}-\varepsilon)u\psi_{N'}\,dx\right| + C_2\nu_2^{\frac{1}{2}}$$
$$\le \left|\int_{\mathbf{\Omega}} (\tilde{\varepsilon}-\varepsilon)u\psi_{N'}\,dx\right| + \|v_i^n - \psi_{N'}\|_{L^\infty}\|\tilde{\varepsilon}-\varepsilon\|_{L^1}\|u\|_{L^\infty} + C_2\nu_2^{\frac{1}{2}}$$
$$\le \left|\int_{\mathbf{\Omega}} (\tilde{\varepsilon}-\varepsilon)u\sum_{i=1}^{N'} a_i^{N'}\chi_{O_i}\,dx\right| + C_3\nu_3 + C_2\nu_2^{\frac{1}{2}} \le \sum_{i=1}^{N'}|a_i^{N'}|\left|\int_{O_i}(\tilde{\varepsilon}-\varepsilon)u\,dx\right| + C_3\nu_3 + C_2\nu_2^{\frac{1}{2}}.$$

Lemma 2.2 implies there exists a constant K such that $\|u\|_{H^2} \le K$ for every realization u. Since H^2 imbeds in $C^{0,1/2}$, there exists a constant K_L such that

$$|u(x) - u(y)| \le K_L|x-y|^{1/2}$$

for all u and for all $x, y \in \mathbf{\Omega}$. Let

$$u_\gamma^i = \max_{x\in O_i} u(x)$$

and we have

$$|u(x) - u^i_\gamma| \leq K_L \gamma^{1/2}$$

for all $x \in O_i$. Thus,

$$\begin{aligned}
&\left| \int_{\Omega \backslash S_{\nu_2}} (\tilde{\varepsilon} - \varepsilon) u g_i^n \, dx \right| \\
&\quad \leq \sum_{i=1}^{N'} |a_i^{N'}| \left| \int_{O_i} (\tilde{\varepsilon} - \varepsilon)(u - u^i_\gamma) dx \right| + \sum_{i=1}^{N'} |a_i^{N'}| \left| \int_{O_i} (\tilde{\varepsilon} - \varepsilon)(u^i_\gamma) dx \right| + C_3 \nu_3 + C_2 \nu_2^{\frac{1}{2}} \\
&\quad \leq K_L \gamma^{1/2} \sum_{i=1}^{N'} |a_i^{N'}| \left| \int_{O_i} (\tilde{\varepsilon} - \varepsilon) dx \right| + \sum_{i=1}^{N'} |a_i^{N'}| \left| \int_{O_i} (\tilde{\varepsilon} - \varepsilon)(u^i_\gamma) dx \right| + C_3 \nu_3 + C_2 \nu_2^{\frac{1}{2}} \\
&\quad \leq C \gamma^{1/2} + \sum_{i=1}^{N'} |a_i^{N'}| |u^i_\gamma| \left| \int_{O_i} (\tilde{\varepsilon} - \varepsilon) dx \right| + C_3 \nu_3 + C_2 \nu_2^{\frac{1}{2}}.
\end{aligned}$$

We obtain the desired bound by taking γ, ν_2, and ν_3 sufficiently small. Let $C\gamma^{1/2} + C_2 \nu_2^{\frac{1}{2}} + C_3 \nu_3 < \nu$; hence

$$\|u - \tilde{u}\|_{L^2} \leq K^* \left(\sum_{i=1}^{N'} \left| \int_{O_i} (\tilde{\varepsilon} - \varepsilon) dx \right| \right) + \nu. \tag{2.16}$$

The interpolation inequality [1] states that there exists a constant K_I such that

$$\|u\|_{W^{1,q}} \leq K_I \|u\|^{\frac{1}{2}}_{W^{2,q}} \|u\|^{\frac{1}{2}}_{L^q}.$$

Since $W^{1,q}$ imbeds in L^∞ for $3 < q < \infty$ [1], there exists a constant C such that

$$\|u - \tilde{u}\|_{L^\infty} \leq C \|u - \tilde{u}\|_{W^{1,q}}.$$

Also, the interpolation inequality for L^p-spaces [8] states that when $3 < q < \infty$,

$$\|u\|_{L^q} \leq \|u\|^{\frac{2}{q}}_{L^2} \|u\|^{\frac{q-2}{q}}_{L^\infty}.$$

Combining the above inequalities and the bound (2.16), we prove the second bound in the statement of the lemma

$$\begin{aligned}
\|u - \tilde{u}\|_{L^\infty} &\leq C K_I \|u - \tilde{u}\|^{\frac{1}{2}}_{W^{2,q}} \|u - \tilde{u}\|^{\frac{1}{2}}_{L^q} \leq C K_I \|u - \tilde{u}\|^{\frac{1}{2}}_{W^{2,q}} \|u - \tilde{u}\|^{\frac{1}{q}}_{L^2} \|u - \tilde{u}\|^{\frac{q-2}{2q}}_{L^\infty} \\
&\leq K^*_\infty(q) \left(K^* \left(\sum_{i=1}^{N'} \left| \int_{O_i} (\tilde{\varepsilon} - \varepsilon) dx \right| \right) + \nu \right)^{\frac{1}{q}}. \qquad \square
\end{aligned}$$

3. Effective dielectric coefficient. The expected value $\langle u \rangle$ of the solution u of the Helmholtz equation (2.3)–(2.4), which depends on the random variables through its dependence on the composite material, is defined, recalling (2.2), as follows:

$$\langle u\rangle = \int_{\Psi_\delta} u\,dP = \int_{\Gamma_\delta} \sum_{m_g\in R_g} \prod_{j=1}^{N_g} p^{1-m_j}(1-p)^{m_j} u(\varepsilon_{m,g},x)\,dG_\delta. \tag{3.1}$$

Note that $\langle\rangle$ is an expectation over material realizations, not the spatial variables, so that $\langle u\rangle$ is in general still a function of x. Thus, the effective dielectric coefficient, defined in (1.2) as

$$\varepsilon^* = \frac{\langle \varepsilon u\rangle}{\langle u\rangle},$$

is a function of the spatial variable x.

Our main theorem gives a bound on the effective dielectric coefficient and its spatial variations provided we have a lower bound on the expected value of u. Such a bound is proven to exist for sufficiently small δ. The theorem shows that as the maximum volume δ of the subdomains decreases, so does the magnitude of the spatial variations, and as $\delta \to 0$, the effective coefficient equals the constant predicted by the quasistatic case.

Theorem 3.1. *Let $\varepsilon^*(x)$ be the effective dielectric coefficient of the medium defined by* (1.2). *There exist $\delta_0 > 0$ and a constant C^* such that for all $0 < \delta < \delta_0$ and any $x_0 \in \mathbf{\Omega}$, the local total variation of ε^* satisfies*

$$\int_{B_r(x_0)} |\nabla\varepsilon^*|\,dx \le C^*|\varepsilon_1 - \varepsilon_0|\delta,$$

where r is determined as in assumption A2. *As the size of the inhomogeneities goes to* 0, *the spatial variations decrease in magnitude, and $\varepsilon^*(x) \to p\varepsilon_0 + (1-p)\varepsilon_1$.*

Thus, $|\varepsilon^*(x)|$ is uniformly bounded above for all x, and the spatial variations of ε^* are bounded in terms of the size of the inhomogeneities δ and the contrast of the medium $|\varepsilon_1 - \varepsilon_0|$.

Proof. The proof applies to one-, two-, and three-dimensional random media. In order to obtain a bound on $|\varepsilon^*| = \frac{|\langle\varepsilon u\rangle|}{|\langle u\rangle|}$, we must obtain a lower bound on the denominator $|\langle u\rangle|$. We show that a uniform bound exists provided δ is chosen sufficiently small; i.e., $|\langle u\rangle| \ge c > 0$ for all $x \in \mathbf{\Omega}$. The proof is based on a probability argument that shows that the probability that the solutions u will be within a certain radius α from the solution of the constant boundary value problem with dielectric constant $\tilde\varepsilon = p\varepsilon_0 + (1-p)\varepsilon_1$ goes to one as the maximum volume δ or the contrast $|\varepsilon_1 - \varepsilon_0|$ goes to zero. The probability β that a solution u lies outside the circle with radius α depends on the parameter δ, and $\beta \to 0$ as $\delta \to 0$. This prevents $\langle u\rangle$ from equaling 0 and gives a lower bound on $|\langle u\rangle| \ge c > 0$. The numerical experiment in Figure 3.1 illustrates this argument, and the proof follows.

We let α and β be arbitrary constants such that $\beta \le 1$ and $\alpha \le K_1$. We want to prove that for every such α and β, one can find $\delta > 0$ such that

$$|\langle u\rangle| \ge (1-\beta)(A-\alpha) - \beta K_1,$$

where $\|\tilde u\|_{L^\infty} = A$ and $\|u\|_{L^\infty} \le K_1$.

We use Lemma 2.3. There our domain $\mathbf{\Omega}$ was divided into N' nonoverlapping subdomains O_i such that $d(O_i) \le \gamma$ for all $i = 1, \dots, N'$. Note that the subdomain partition O_i is independent of the material partitions $\mathbf{\Omega} = \cup\, \mathbf{\Omega}_j$, which vary randomly over the set of all realizations. The partition O_i allows (through Lemma 2.3) the computation of local ensemble averages of the material coefficients, which tend toward a constant as

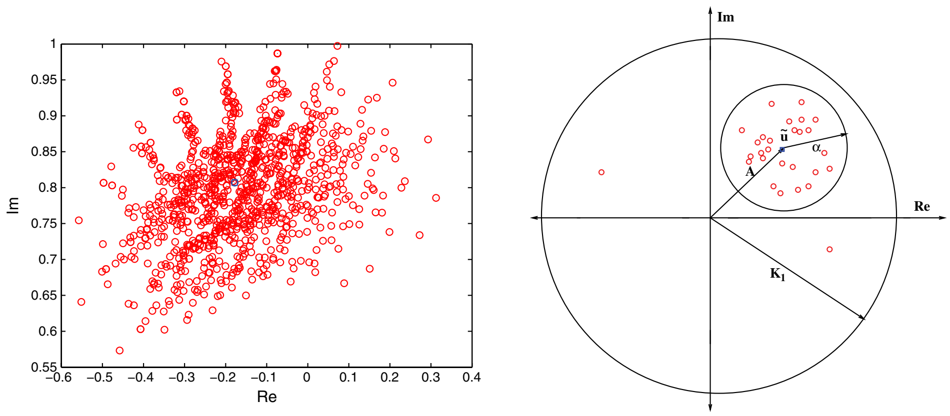

Fig. 3.1. *Proximity to the constant coefficient solution. Left: from numerical experiments, solutions* u *for a medium with* 10 *layers at* $x = 0.5$ *(dots) and the solution to the constant coefficient problem* $\tilde{u}(0.5)$ *(square); right: for an appropriate parameter* δ*, the probability that solutions* u *cluster within a circle with center* $\tilde{u}$ *and radius* α *is* $1-\beta$*. The probability* β *that solutions lie outside this circle depends on* δ*, and* $\beta \to 0$ *as* $\delta \to 0$*. All solutions are contained in the circle with radius* K_1*, since* $\|u\|_{L^\infty} \le K_1$*.*

the scale δ of the material partition decreases. Each O_i contains at most $\tilde{N}$ subdomains $\mathbf{\Omega}_j$ and subdomains $\mathbf{\Omega}_j \cap O_i$. We are guaranteed that any subdomain $\mathbf{\Omega}_j$ coming from material realizations has volume less than or equal to δ; hence $|\mathbf{\Omega}_j \cap O_i| \le \delta$. Denote by χ the indicator function assigning 1 if we have material ε_0 or 0 if we have material ε_1 in a given domain. Given the radius α and using Chebyshev's inequality [7] and estimate (2.15), we obtain

$$\begin{aligned}
P(\|u-\tilde{u}\|_{L^\infty} \le \alpha) &\ge P\left(K^*_\infty\left(K^*\left(\sum_{i=1}^{N'}\left|\int_{O_i}(\tilde{\varepsilon}-\varepsilon)\,dx\right|\right)+\nu\right)^{\frac{1}{q}} \le \alpha\right)\\
&\ge P\left(\max_i\left|\int_{O_i}(\tilde{\varepsilon}-\varepsilon)\,dx\right| \le \frac{\left(\frac{\alpha}{K^*_\infty}\right)^q-\nu}{K^*N'}\right)\\
&= P\left(\left|(\varepsilon_1(1-p)+\varepsilon_0 p)|O^M| - \left(\varepsilon_0\sum_{j=1}^{\tilde{N}}\chi_j|O^M_j| + \varepsilon_1\left(|O^M| - \sum_{j=1}^{\tilde{N}}\chi_j|O^M_j|\right)\right)\right|\right.\\
&\left.\le \frac{\left(\frac{\alpha}{K^*_\infty}\right)^q-\nu}{K^*N'}\right) \quad \ge P\left(\left|\sum_{j=1}^{\tilde{N}}\chi_j|O^M_j| - p|O^M|\right| \le \frac{\left(\frac{\alpha}{K^*_\infty}\right)^q-\nu}{K^*N'(\varepsilon_1-\varepsilon_0)}\right)\\
&= 1 - P\left(\left|\sum_{j=1}^{\tilde{N}}\chi_j|O^M_j| - p|O^M|\right| \ge \frac{\left(\frac{\alpha}{K^*_\infty}\right)^q-\nu}{K^*N'(\varepsilon_1-\varepsilon_0)}\right)\\
&\ge 1 - \left(\frac{(K^*_\infty)^q K^* N'(\varepsilon_1-\varepsilon_0)}{\alpha^q-\nu(K^*_\infty)^q}\right)^2 \mathrm{Var}\left(\sum_{j=1}^{\tilde{N}}\chi_j|O^M_j|\right) \equiv 1-\beta.
\end{aligned} \tag{3.2}$$

Here O^M is the set O_i over which the quantity $|\int_{O_i}(\tilde{\varepsilon}-\varepsilon)\,dx|$ is maximized and the sets $O^M_j \equiv \mathbf{\Omega}_j \cap O^M$, and χ_j is the indicator function of the set O^M_j. We have also used the fact that $\langle \sum_{j=1}^{\tilde N} \chi_j |O_j^M|\rangle = p|O^M|$. We notice that the random variables χ_j are independent and calculate the variance

$$\mathrm{Var}\left(\sum_{j=1}^{\tilde N} \chi_j |O_j^M|\right) = \sum_{j=1}^{\tilde N} |O_j^M|^2 \mathrm{Var}(\chi_j) = p(1-p)\sum_{j=1}^{\tilde N} |O_j^M|^2 \le p(1-p)\tilde N \delta^2.$$

Thus,

$$\begin{aligned}\beta &\equiv \left(\frac{(K_\infty^*)^q K^* N'(\varepsilon_1-\varepsilon_0)}{\alpha^q - \nu(K_\infty^*)^q}\right)^2 \mathrm{Var}\left(\sum_{j=1}^{\tilde N} \chi_j |O_j^M|\right)\\ &\le \left(\frac{(K_\infty^*)^q K^* N'(\varepsilon_1-\varepsilon_0)}{\alpha^q - \nu(K_\infty^*)^q}\right)^2 p(1-p)\tilde N \delta^2.\end{aligned}$$

We have shown that the probability that solutions u are within a radius α of the constant coefficient solution $\tilde u$ goes to one as either δ or the contrast in the medium $|\varepsilon_1 - \varepsilon_0|$ goes to 0.

Let us call $\|u - \tilde u\|_{L^\infty} \le \alpha$ condition L and call the complement condition L^c. Define the conditional expectations

$$\langle u|L\rangle \equiv \frac{\int_{\Psi_\delta(L)} u dP}{P(L)} \quad \text{and} \quad \langle u|L^c\rangle \equiv \frac{\int_{\Psi_\delta(L^c)} u dP}{P(L^c)},$$

and note that $P(L) \ge 1-\beta$ and $P(L^c) \le \beta$. The expected value $\langle u\rangle$ is given by

$$\langle u\rangle = P(L)\langle u|L\rangle + P(L^c)\langle u|L^c\rangle,$$

and using estimate (3.2), we obtain

$$|\langle u\rangle| \ge (1-\beta)|\langle u|L\rangle| - \beta|\langle u|L^c\rangle|.$$

If u satisfies condition L, then u satisfies the inequality

$$\|u\|_{L^\infty} \ge \|\tilde u\|_{L^\infty} - \alpha \ge A - \alpha.$$

Now using the uniform upper bound $\|u\|_{L^\infty} \le K_1$, we obtain the desired result:

$$|\langle u\rangle| \ge (1-\beta)(A-\alpha) - \beta K_1,$$

where the constant β depends on δ, the maximum volume of the subdomains, and on the contrast $|\varepsilon_1 - \varepsilon_0|$, and $\beta \to 0$ as δ or $|\varepsilon_1 - \varepsilon_0| \to 0$. Thus by picking the appropriate α and β, where β is controlled by the parameter δ, we obtain the lower bound $|\langle u\rangle| \ge c > 0$ for all $x \in \Omega$. This provides a bound on the effective dielectric coefficient

$$|\varepsilon^*| \le \frac{\tilde\varepsilon K_1}{c}.$$

The uniform lower bound on $|\langle u\rangle|$ is utilized in proving that $\|\varepsilon^*\|_{BV} \le C^*|\varepsilon_1 - \varepsilon_0|\delta$, as follows. Formally, the gradient $\nabla\varepsilon^*$ is given by

$$\nabla \varepsilon^* = \frac{\langle u\rangle\langle(\nabla\varepsilon)u\rangle + \langle u\rangle\langle\varepsilon\nabla u\rangle - \langle\nabla u\rangle\langle\varepsilon u\rangle}{\langle u\rangle^2}, \tag{3.3}$$

where $\nabla\varepsilon$ is understood in the sense of a distribution. Now choose δ such that $|\langle u\rangle| \geq c > 0$. We want to bound the numerator in terms of this δ and the contrast $|\varepsilon_1 - \varepsilon_0|$. First we bound

$$|\langle u\rangle\langle\varepsilon\nabla u\rangle - \langle\nabla u\rangle\langle\varepsilon u\rangle| \leq C_1\delta|\varepsilon_1 - \varepsilon_0| \tag{3.4}$$

pointwise, where C_1 is a constant. In the proof we use the Lipschitz bound (2.10) from Lemma 2.2.

The bound (3.4) is obtained by looking at material realizations that differ only in one particular subdomain $\mathbf{\Omega}_j$ and realizing that the pointwise difference in solutions propagating through two such material realizations can be bounded in terms of the L^2-norm of the difference in the two materials, where the two materials differ only on the subdomain $\mathbf{\Omega}_j$ with $|\mathbf{\Omega}_j| \leq \delta$.

Fix x. Divide the set of material realizations $\mathbf{\Psi}_\delta$ into two subsets $\mathbf{\Psi}_\delta = \mathbf{\Psi}_\delta^0 \cup \mathbf{\Psi}_\delta^1$, where $\mathbf{\Psi}_\delta^0$ is the subset of realizations such that $\varepsilon(x) = \varepsilon_0$ and $\mathbf{\Psi}_\delta^1$ is the subset of realizations such that $\varepsilon(x) = \varepsilon_1$. Representative elements of the subsets $\mathbf{\Psi}_\delta^0$ and $\mathbf{\Psi}_\delta^1$ are shown in Figure 3.2. For each geometry g, let R_g^0 and R_g^1 be subsets of the set of material assignments R_g such that

$$R_g^0 = \{m_g = (m_1, \ldots, m_{N_g})\colon m_j = 0 \text{ for } x \in \mathbf{\Omega}_j\},$$

and

$$R_g^1 = \{m_g = (m_1, \ldots, m_{N_g})\colon m_j = 1 \text{ for } x \in \mathbf{\Omega}_j\}.$$

Thus, $R_g = R_g^0 \cup R_g^1$. The expected value of u is given by

$$\begin{aligned}\langle u\rangle(x) &= \int_{\mathbf{\Psi}_\delta} u\,dP = \int_{\Gamma_\delta}\sum_{m_g\in R_g}\prod_{l=1}^{N_g} p^{1-m_l}(1-p)^{m_l}u(\varepsilon_{m,g},x)\,dG_\delta \\ &= p\int_{\Gamma_\delta}\sum_{m_g\in R_g^0}\prod_{\substack{l=1\\ l\neq j}}^{N_g} p^{1-m_l}(1-p)^{m_l}u\,dG_\delta + (1-p)\int_{\Gamma_\delta}\sum_{m_g\in R_g^1}\prod_{\substack{l=1\\ l\neq j}}^{N_g} p^{1-m_l}(1-p)^{m_l}u\,dG_\delta \\ &= p\langle u\rangle_{\mathbf{\Psi}_\delta^0} + (1-p)\langle u\rangle_{\mathbf{\Psi}_\delta^1},\end{aligned}$$

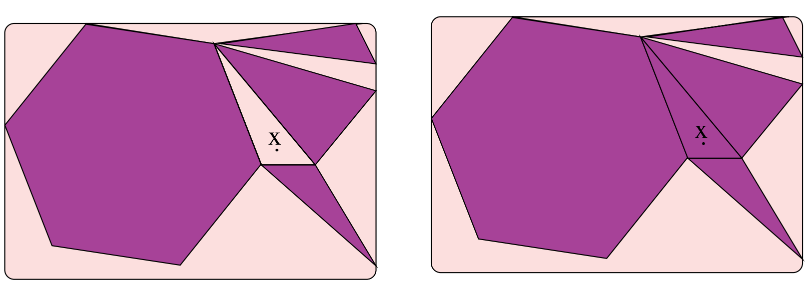

FIG. 3.2. *Sample materials in $\mathbf{\Psi}_\delta^0$ and $\mathbf{\Psi}_\delta^1$ for fixed x. Left: material realization ψ_0; right: corresponding material realization ψ_1 obtained by switching material ε_0 with material ε_1 in the domain containing x.*

where $\langle u\rangle_{\Psi_\delta^0} = \langle u|\varepsilon(x)=\varepsilon_0\rangle$ and $\langle u\rangle_{\Psi_\delta^1} = \langle u|\varepsilon(x)=\varepsilon_1\rangle$. Using this notation we can rewrite

$$\begin{aligned}
&\langle u\rangle\langle\varepsilon\nabla u\rangle - \langle\nabla u\rangle\langle\varepsilon u\rangle \\
&= \varepsilon_1 p(1-p)(\langle u\rangle_{\Psi_\delta^0}\langle\nabla u\rangle_{\Psi_\delta^1} - \langle u\rangle_{\Psi_\delta^1}\langle\nabla u\rangle_{\Psi_\delta^0}) \\
&\quad + \varepsilon_0 p(1-p)(\langle u\rangle_{\Psi_\delta^1}\langle\nabla u\rangle_{\Psi_\delta^0} - \langle u\rangle_{\Psi_\delta^0}\langle\nabla u\rangle_{\Psi_\delta^1}).
\end{aligned}$$

For every material described by Ψ_δ^0, there exists a material described by Ψ_δ^1 such that the two materials differ only in a subdomain $\Omega_j \ni x$. Let us call u_{ψ_0} the solution of the Helmholtz equation when the material realization belongs to Ψ_δ^0 and u_{ψ_1} the corresponding solution of the Helmholtz equation when the material realization, differing only in m_j, belongs to Ψ_δ^1. We have

$$\begin{aligned}
\left|\int_{\Psi_\delta^1} u_{\psi_1}(x)dP - \int_{\Psi_\delta^0} u_{\psi_0}(x)dP\right| &\leq \int_{\Gamma_\delta}\sum_{i=1}^{2^{N_g-1}}\prod_{\substack{l=1\\ l\neq j}}^{N_g} p^{1-m_l^i}(1-p)^{m_l^i}|u_{\psi_1}-u_{\psi_0}|(x)dG_\delta \\
&\leq \sup_{\substack{g\in\Gamma_\delta\\ m_1\in R_g^1\\ m_0\in R_g^0}}\|u_{\psi_1}(m_1,g)-u_{\psi_0}(m_0,g)\|_{L^\infty} \\
&\leq CK\sup_{\substack{g\in\Gamma_\delta\\ m_1\in R_g^1\\ m_0\in R_g^0}}\|\varepsilon_{\psi_1}(m_1,g)-\varepsilon_{\psi_0}(m_0,g)\|_{L^2} \leq CK\delta|\varepsilon_1-\varepsilon_0|.
\end{aligned}$$

The preceding comes from the fact that for any material realization in Ψ_δ^1, there exists a material realization in Ψ_δ^0. The application of Lemma 2.2 yields the second-to-last inequality. Thus, we have that $|\langle u\rangle_{\Psi_\delta^1} - \langle u\rangle_{\Psi_\delta^0}| \to 0$ pointwise as $\delta\to 0$. By a similar argument, $|\langle\nabla u\rangle_{\Psi_\delta^1} - \langle\nabla u\rangle_{\Psi_\delta^0}| \leq CK\delta|\varepsilon_1-\varepsilon_0|$, and $|\langle\nabla u\rangle_{\Psi_\delta^1} - \langle\nabla u\rangle_{\Psi_\delta^0}|\to 0$ pointwise as $\delta\to 0$. Now,

$$\begin{aligned}
&|\langle u\rangle_{\Psi_\delta^0}\langle\nabla u\rangle_{\Psi_\delta^1} - \langle u\rangle_{\Psi_\delta^1}\langle\nabla u\rangle_{\Psi_\delta^0}| \\
&\leq |\langle u\rangle_{\Psi_\delta^0}||\langle\nabla u\rangle_{\Psi_\delta^1} - \langle\nabla u\rangle_{\Psi_\delta^0}| + |\langle\nabla u\rangle_{\Psi_\delta^0}||\langle u\rangle_{\Psi_\delta^1} - \langle u\rangle_{\Psi_\delta^0}|. \qquad (3.5)
\end{aligned}$$

Referring to Lemmas 2.2 and 2.1, we know that $u\in C_B^1(\Omega)$, and that there exist constants K_1 and K_2 such that $\|u\|_{L^\infty}\leq K_1$ and $\|\nabla u\|_{L^\infty}\leq K_2$ for every u. Then

$$|\langle u\rangle_{\Psi_\delta^0}\langle\nabla u\rangle_{\Psi_\delta^1} - \langle u\rangle_{\Psi_\delta^1}\langle\nabla u\rangle_{\Psi_\delta^0}| \leq KC|\varepsilon_1-\varepsilon_0|\delta(K_1+K_2)\to 0 \quad \text{as} \quad \delta\to 0$$

and similarly for the second term in (3.5). Thus, we obtain the following bound:

$$\begin{aligned}
&|\langle u\rangle\langle\varepsilon\nabla u\rangle - \langle\nabla u\rangle\langle\varepsilon u\rangle| \\
&\leq \varepsilon_1 p(1-p)|\langle u\rangle_{\Psi_\delta^0}\langle\nabla u\rangle_{\Psi_\delta^1} - \langle u\rangle_{\Psi_\delta^1}\langle\nabla u\rangle_{\Psi_\delta^0}| \\
&\quad + \varepsilon_0 p(1-p)|\langle u\rangle_{\Psi_\delta^1}\langle\nabla u\rangle_{\Psi_\delta^0} - \langle u\rangle_{\Psi_\delta^0}\langle\nabla u\rangle_{\Psi_\delta^1}| \\
&\leq KCp(1-p)(\varepsilon_1+\varepsilon_0)|\varepsilon_1-\varepsilon_0|(K_1+K_2)\delta. \qquad (3.6)
\end{aligned}$$

Looking back at (3.3) to get an upper bound on $|\nabla\varepsilon^*|$, we now want to prove that $|\langle(\nabla\varepsilon)u\rangle| \leq C_2\delta|\varepsilon_1-\varepsilon_0|$ in the distributional sense.

Since $\varepsilon(x)$ equals a constant in every subdomain $\mathbf{\Omega}_j$, $\nabla\varepsilon = 0$ there, and the only problem occurs at the interface between two or more subdomains with different materials, where ε is discontinuous and $\nabla\varepsilon$ is defined only in the distributional sense.

Fix a realization ψ_α such that x_0 is at the interface between k subdomains $\mathbf{\Omega}_j$, $j = 1 \dots k$ with alternating materials ε_0 and ε_1 in them. This assumption will pose no loss of generality since the other cases are attained at material realizations satisfying our assumptions. Call ψ_β the realization that has the same geometry as realization ψ_α, but with the materials in the k subdomains interfacing at x_0 switched, e.g., Figure 3.3. Without loss of generality, let realization ψ_α have material ε_0 in $\mathbf{\Omega}_1$; thus realization ψ_β has material ε_1 in the same subdomain $\mathbf{\Omega}_1$. Let $\boldsymbol{\phi}$ be a test function $\boldsymbol{\phi} \in C_0^\infty(\mathbf{\Omega}, \mathbb{R}^n)$ such that supp$\boldsymbol{\phi} \subset B_r(x_0)$. We can find $\nabla(\varepsilon_\alpha)u_\alpha$ at x_0 in the generalized sense:

$$\begin{aligned}
&\int_{B_r(x_0)} u_\alpha \nabla(\varepsilon_\alpha)\boldsymbol{\phi}\, dx \\
&\quad = (\varepsilon_1 - \varepsilon_0)\int_{\partial(\mathbf{\Omega}_1\cap\mathbf{\Omega}_2)} u_\alpha \boldsymbol{\phi}\nu_{\partial(\mathbf{\Omega}_1\cap\mathbf{\Omega}_2)}\, dx + (\varepsilon_1 - \varepsilon_0)\int_{\partial(\mathbf{\Omega}_2\cap\mathbf{\Omega}_3)} u_\alpha \boldsymbol{\phi}\nu_{\partial(\mathbf{\Omega}_2\cap\mathbf{\Omega}_3)}\, dx + \cdots \\
&\qquad + (\varepsilon_1 - \varepsilon_0)\int_{\partial(\mathbf{\Omega}_{k-1}\cap\mathbf{\Omega}_k)} u_\alpha \boldsymbol{\phi}\nu_{\partial(\mathbf{\Omega}_{k-1}\cap\mathbf{\Omega}_k)}\, dx + (\varepsilon_1 - \varepsilon_0)\int_{\partial(\mathbf{\Omega}_1\cap\mathbf{\Omega}_k)} u_\alpha \boldsymbol{\phi}\nu_{\partial(\mathbf{\Omega}_1\cap\mathbf{\Omega}_k)}\, dx,
\end{aligned}$$

where $\partial(\mathbf{\Omega}_1 \cap \mathbf{\Omega}_2)$ is the interface between subdomains $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ and $\nu_{\partial(\mathbf{\Omega}_1\cap\mathbf{\Omega}_2)}$ is the unit normal vector to $\mathbf{\Omega}_1$ on the interface with $\mathbf{\Omega}_2$. Note that $\nu_{\partial(\mathbf{\Omega}_1\cap\mathbf{\Omega}_2)} = -\nu_{\partial(\mathbf{\Omega}_2\cap\mathbf{\Omega}_1)}$.

Similarly, we find that $\nabla(\varepsilon_\beta)u_\beta$ at x_0 in the generalized sense is

$$\begin{aligned}
&\int_{B_r(x_0)} u_\beta \nabla(\varepsilon_\beta)\boldsymbol{\phi}\, dx \\
&\quad = -(\varepsilon_1 - \varepsilon_0)\int_{\partial(\mathbf{\Omega}_1\cap\mathbf{\Omega}_2)} u_\beta \boldsymbol{\phi}\nu_{\partial(\mathbf{\Omega}_1\cap\mathbf{\Omega}_2)}\, dx - (\varepsilon_1 - \varepsilon_0)\int_{\partial(\mathbf{\Omega}_2\cap\mathbf{\Omega}_3)} u_\beta \boldsymbol{\phi}\nu_{\partial(\mathbf{\Omega}_2\cap\mathbf{\Omega}_3)}\, dx - \cdots \\
&\qquad - (\varepsilon_1 - \varepsilon_0)\int_{\partial(\mathbf{\Omega}_{k-1}\cap\mathbf{\Omega}_k)} u_\beta \boldsymbol{\phi}\nu_{\partial(\mathbf{\Omega}_{k-1}\cap\mathbf{\Omega}_k)}\, dx - (\varepsilon_1 - \varepsilon_0)\int_{\partial(\mathbf{\Omega}_1\cap\mathbf{\Omega}_k)} u_\beta \boldsymbol{\phi}\nu_{\partial(\mathbf{\Omega}_1\cap\mathbf{\Omega}_k)}\, dx.
\end{aligned}$$

Divide again $\mathbf{\Psi}_\delta$ into three subsets $\mathbf{\Psi}_\delta = \mathbf{\Psi}_\delta^c \cup \mathbf{\Psi}_\delta^\alpha \cup \mathbf{\Psi}_\delta^\beta$: $\mathbf{\Psi}_\delta^c$ is the subset of realizations such that x_0 is inside some subdomain; $\mathbf{\Psi}_\delta^\alpha$ is the subset of realizations such that x_0 is at the interface between k subdomains $\mathbf{\Omega}_j$, $j = 1 \dots k$ for any integer k with alternating

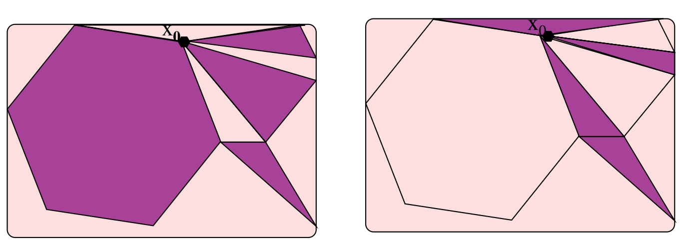

FIG. 3.3. *Sample materials in* $\mathbf{\Psi}_\delta^\alpha$ *and* $\mathbf{\Psi}_\delta^\beta$ *for fixed* x *on the boundary between several materials. Left: material realization* ψ_α*; right: corresponding material realization* ψ_β *obtained by interchanging the materials at domains interfacing at* x.

materials ε_0 and ε_1 in them and material ε_0 in $\mathbf{\Omega}_1$; $\mathbf{\Psi}_\delta^\beta$ is the subset of realizations such that x_0 is at the interface between k subdomains $\mathbf{\Omega}_j$, $j = 1 \ldots k$ for any integer k with alternating materials ε_1 and ε_0 in them and material ε_1 in $\mathbf{\Omega}_1$. Note that $\langle \nabla\varepsilon \rangle_{\mathbf{\Psi}_\delta^c} = 0$. Utilizing assumptions A2 and A3, we obtain

$$\begin{aligned} &\left| \left\langle \int_{B_r(x_0)} (u \nabla \varepsilon) \phi \, dx \right\rangle \right| \\ &\quad \leq |\varepsilon_1 - \varepsilon_0| \|\phi\|_{L^\infty} \sum_{j=1}^{k} \|\chi_{\mathbf{\Omega}_j}\|_{BV} \int_{G_\delta} \sum_{i=1}^{2^{N_g - 1}} p^{\frac{k}{2}} (1-p)^{\frac{k}{2}} \prod_{\substack{l=1 \\ l \neq j+1 \ldots \\ j+k}}^{N_g} p^{1 - m_l^i} (1-p)^{m_l^i} \|u_\alpha - u_\beta\|_{L^\infty} \, dG_\delta \\ &\quad \leq kKCC_p p(1-p) \|\phi\|_{L^\infty} |\varepsilon_1 - \varepsilon_0|^2 \delta. \end{aligned} \tag{3.7}$$

Note that the inequality

$$\|u_\alpha - u_\beta\|_{L^\infty} \leq kKC|\varepsilon_1 - \varepsilon_0|\delta$$

comes from Lemma 2.2 and the fact that for any material in $\mathbf{\Psi}_\delta^\alpha$, one can find a material in $\mathbf{\Psi}_\delta^\beta$, which differs only on the subdomains $\mathbf{\Omega}_j$ through $\mathbf{\Omega}_{j+k}$, each with volume less than or equal to δ.

Choose δ small enough that $|\langle u \rangle| \geq c > 0$. Using the lower bound $|\langle u \rangle| \geq c > 0$, (3.6), and (3.7), we obtain

$$\int_{B_r(x_0)} |\nabla \varepsilon^*| \, dx \leq \frac{C|\varepsilon_1 - \varepsilon_0|\delta \|\phi\|_{L^\infty}}{c^2} \leq C^* |\varepsilon_1 - \varepsilon_0| \delta, \tag{3.8}$$

where $\nabla\varepsilon^*$ is defined in the generalized sense. This will ensure that $\varepsilon^* \in BV(\mathbf{\Omega})$, and thus, we can bound the spatial variations of ε^*

$$\begin{aligned} V(\varepsilon^*, \mathbf{\Omega}) &:= \sup \left\{ \int_{\mathbf{\Omega}} \varepsilon^* \operatorname{div} \phi \, dx : \phi \in C_0^1(\mathbf{\Omega}, \mathbb{R}^n), \|\phi\|_{L^\infty(\mathbf{\Omega})} \leq 1 \right\} \\ &\leq C \int_{\mathbf{\Omega}} |\nabla \varepsilon^*| \, dx \to 0 \quad \text{as } \delta \quad \text{or} \quad |\varepsilon_1 - \varepsilon_0| \to 0. \end{aligned}$$

The formula that prescribes the appropriate δ takes into account the contrast $|\varepsilon_1 - \varepsilon_0|$ in the medium (Theorem 3.1, (3.6), and (3.8)).

Note that

$$\varepsilon^* = \frac{\langle \varepsilon u \rangle}{\langle u \rangle} = \frac{p\varepsilon_0 \langle u \rangle_{\mathbf{\Psi}_\delta^0} + (1-p)\varepsilon_1 \langle u \rangle_{\mathbf{\Psi}_\delta^1}}{p \langle u \rangle_{\mathbf{\Psi}_\delta^0} + (1-p) \langle u \rangle_{\mathbf{\Psi}_\delta^1}}.$$

Since $|\langle u \rangle_{\mathbf{\Psi}_\delta^1} - \langle u \rangle_{\mathbf{\Psi}_\delta^0}| \to 0$ pointwise as $\delta \to 0$, we obtain that $\varepsilon^* \to p\varepsilon_0 + (1-p)\varepsilon_1$ as $\delta \to 0$, which is consistent with the quasistatic case since by letting $\delta \to 0$, we are effectively operating in the quasistatic limit. ∎

We can obtain an estimate of how much ε^* differs from the expected value $\tilde{\varepsilon}$:

$$\begin{aligned}|\varepsilon^* - \tilde{\varepsilon}| &= \frac{|\langle \varepsilon u\rangle - \tilde{\varepsilon}\langle u\rangle|}{|\langle u\rangle|} \\ &\leq \frac{|p\varepsilon_0\langle u\rangle_{\Psi_0} + (1-p)\varepsilon_1\langle u\rangle_{\Psi_1} - (p\varepsilon_0 + (1-p)\varepsilon_1)(p\langle u\rangle_{\Psi_0} + (1-p)\langle u\rangle_{\Psi_1})|}{c} \\ &\leq \frac{p(1-p)|\varepsilon_1 - \varepsilon_0||\langle u\rangle_{\Psi_1} - \langle u\rangle_{\Psi_0}|}{c} \\ &\leq p(1-p)C|\varepsilon_1 - \varepsilon_0|\delta.\end{aligned}$$

4. Numerical experiments. Without loss of generality, assume that the dielectric coefficient of the medium is

$$\varepsilon(x) = 1 + z\chi(x,\psi) + i\varepsilon_i, \tag{4.1}$$

where the function $\chi(x,\psi)$ is a random characteristic function in x, and z is the contrast in the medium. The main Theorem 3.1 showed that the spatial variations in the effective coefficient are bounded by the contrast in the medium z (or as appears in the theorem, $z \equiv |\varepsilon_1 - \varepsilon_0|$). Although our analysis in the previous sections required $\epsilon_i > 0$ to guarantee stability, we found the results of the numerical experiments were insensitive to small ϵ_i. All of the results in this section take $\epsilon_i = 0$.

We observe the spatial dependence of the effective dielectric coefficient by numerically calculating ε^* and graphing it as a function of x. In these numerical experiments, ε^* is calculated by dividing the interval (0,1) into the corresponding number of intervals m, each layer of length $\frac{1}{m}$, and going through all possible realizations by assigning in each layer either material of type one or material of type two, both with probability $\frac{1}{2}$. The solution u for each particular layered material is computed by the transfer matrix method [18]. Sample realizations in the case of a six-layer medium are given in Figure 4.1. In these numerical experiments $\omega = 53$, corresponding to a free-space wavelength $\lambda = \frac{2\pi}{\omega} \approx 0.118$. The graph on the left shows the sample six-layer medium, composed of material of type one ($\varepsilon_0 = 1$) in the first, second, and fifth layers, and material of type two ($\varepsilon_1 = 2$) in the third, fourth, and sixth layers (above), and the real part of the solution u (below). In the interest of space, in all of the figures that follow, only

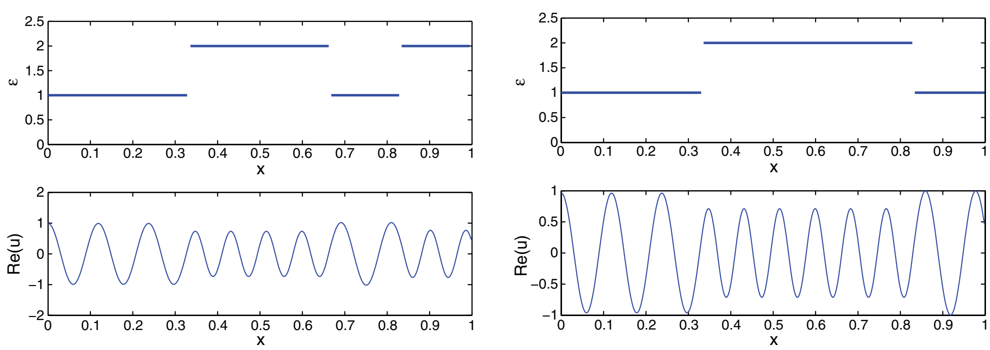

FIG. 4.1. *Sample realizations in a six-layer medium: ε (top) and corresponding real part of u (bottom).*

the real part of the solution will be graphed. The imaginary part generally looks qualitatively similar. The graph on the right shows a six-layer sample medium, composed of material of type one ($\varepsilon_0 = 1$) in the first, second, and sixth layers, and material of type two ($\varepsilon_1 = 2$) in the third, fourth, and fifth layers (above), and the real part of the solution u.

The expected $\langle u \rangle$ is obtained by evaluating the solution u for each realization and multiplying it by the probability of the particular realization; i.e.,

$$\langle u \rangle = \sum_{m_g \in R_g} u(x, m_g) \prod_{j=1}^{N_g} p^{1-m_j}(1-p)^{m_j}.$$

In the case when both materials are assigned according to probability $\frac{1}{2}$, each solution u is multiplied by $(\frac{1}{2})^m$. The expected $\langle \varepsilon u \rangle$ is computed similarly. We observe that when the length of the layers is $1/6$, the spatial variations of ε^* are more pronounced than in the case when the length of the layer is $1/16$ (Figure 4.2).

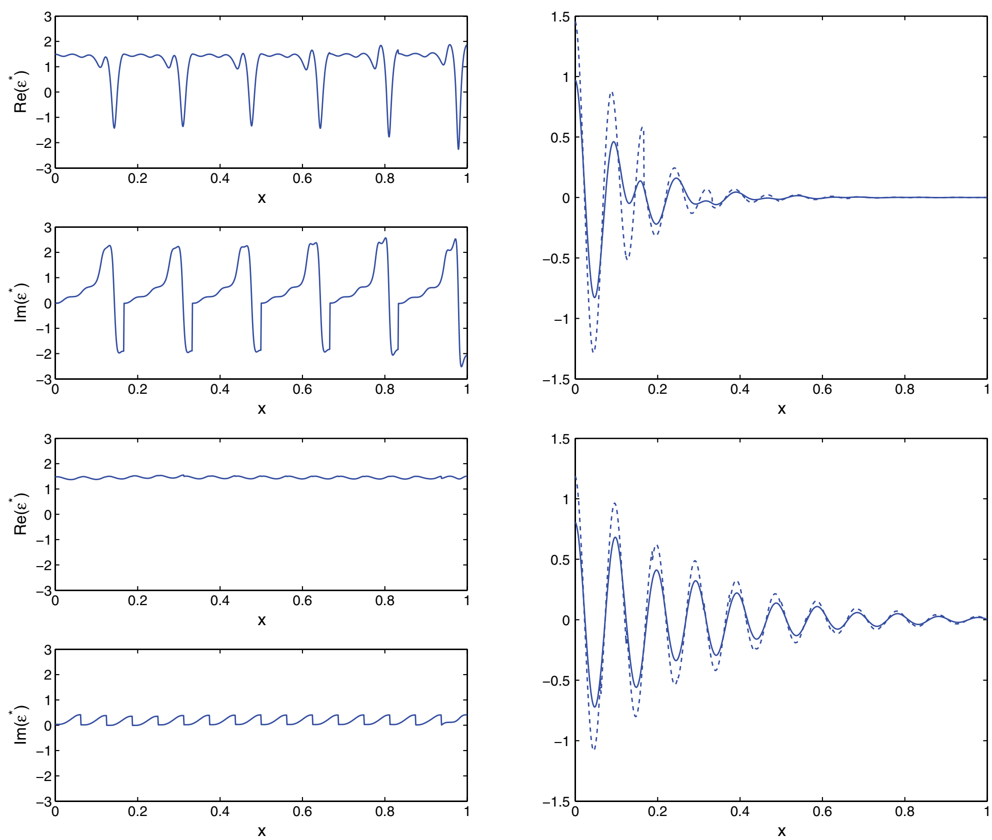

Fig. 4.2. *Spatial variations. Upper left: real and imaginary ε^* in a medium of six layers; upper right: real part of $\langle \varepsilon u \rangle$ (dashed line) and $\langle u \rangle$ (solid line) in a medium of six layers; lower left: real and imaginary ε^* in a medium of sixteen layers; lower right: real part of $\langle \varepsilon u \rangle$ (dashed line) and $\langle u \rangle$ (solid line) in a medium of sixteen layers.*

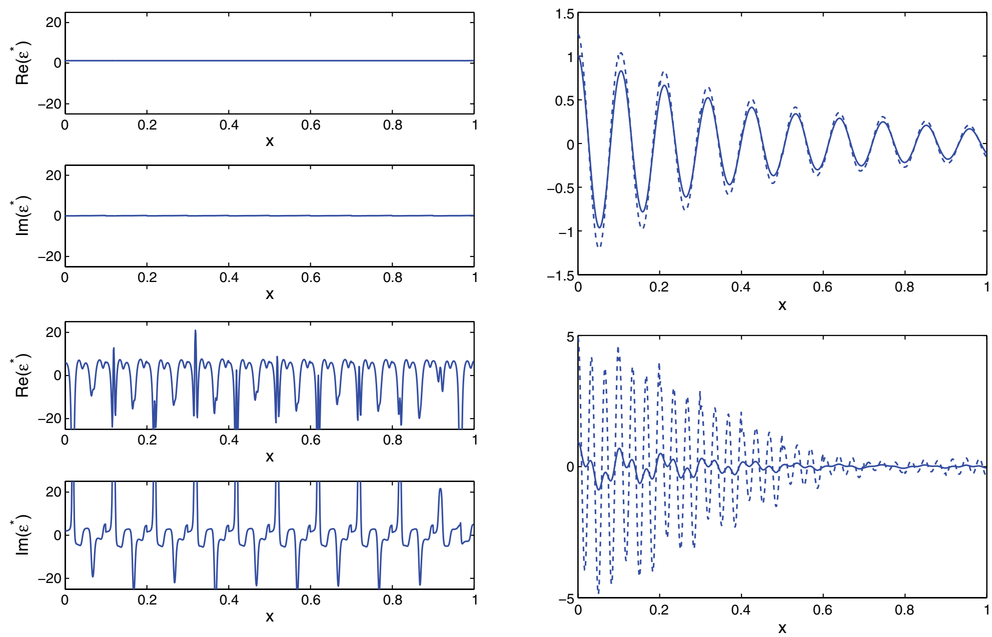

FIG. 4.3. *Spatial variations. Upper left: real and imaginary ε^* in a medium of ten layers and contrast $z = 0.5$; upper right: real part of $\langle \varepsilon u\rangle$ (dashed line) and $\langle u\rangle$ (solid line) in a medium of ten layers and contrast $z = 0.5$; lower left: real and imaginary ε^* in a medium of ten layers and contrast $z = 12$; lower right: real part of $\langle \varepsilon u\rangle$ (dashed line) and $\langle u\rangle$ (solid line) in a medium of ten layers and contrast $z = 12$.*

The numerical experiments also show that the spatial variations decrease in magnitude when the contrast z between the two materials is small (Figure 4.3). In these experiments we are looking at a ten-layer medium and $\omega = 53$. We vary the contrast. In the first experiment, we assign material of type one ($\varepsilon_0 = 1$) or material of type two ($\varepsilon_1 = 1.5$), both with probability $\frac{1}{2}$. In the second experiment, we assign material of type one ($\varepsilon_0 = 1$) or material of type two ($\varepsilon_1 = 13$), both with probability $\frac{1}{2}$. The dependence of the magnitude of the spatial variations on the contrast in the medium is obvious.

An important feature of these results is that even for real material coefficients ϵ_0, ϵ_1, the resulting effective ϵ^* can contain a substantial imaginary part, which accounts for damping of the expected $\langle u\rangle$ as it propagates into the medium. As the numerical experiments show, the amplitude of $\langle u\rangle$ generally does in fact decay as it propagates into the medium, and the effect is accentuated for higher contrast and higher frequencies. This is due to two phenomena. First, for higher contrast and higher frequencies, scattering increases for each realization u, and less energy propagates into the medium. Second, the phases of the waves for individual realizations u become less correlated as one moves deeper into the medium, so that phase cancellation tends to reduce the amplitude of the averaged wave $\langle u\rangle$. The imaginary part of ϵ^* accounts for these effects, without directly modeling the scattering and phase cancellation.

A question may arise as to the practical utility of modeling with an effective parameter ϵ^* with spatial variation as large as the one shown in the lower left of Figure 4.3. We think in fact that there is probably little use for such a parameter, and the point of this paper is not to advocate for its practicality. Instead, these results are to quantify the spatial variation of $\epsilon*$ as a function of contrast and length scale, so that as one moves away from the quasistatic parameter regime, one can have some understanding of the viability of modeling ensemble average wave behavior with an effective material parameter.

Numerical experiments are performed in a two-dimensional random medium, which is periodic in the x direction. The medium is obtained by randomly picking points in a square cell with sides equal to 2π and drawing circles of random radii around the randomly selected points. The coordinates of the points and the values of the radii are drawn from a normal distribution. After the cell is divided into subdomains, either material ε_0 or material ε_1 is assigned to each subdomain, both with probability $1/2$. The variational problem (2.6) was discretized with a first-order finite element method, using piecewise bilinear elements on a uniform, rectangular grid. The design variable ε was approximated by a piecewise constant function on the same uniform grid. The nonlocal boundary operators T defined by (6.1) in the appendix were approximated by explicitly calculating the Fourier coefficients of the traces of the finite element basis, then truncating the sum in (6.1). The resulting finite element scheme can be shown to converge and to conserve energy, provided all the propagating terms are included in the sum [2]. This discretization leads to a large, sparse (except for the boundary terms),

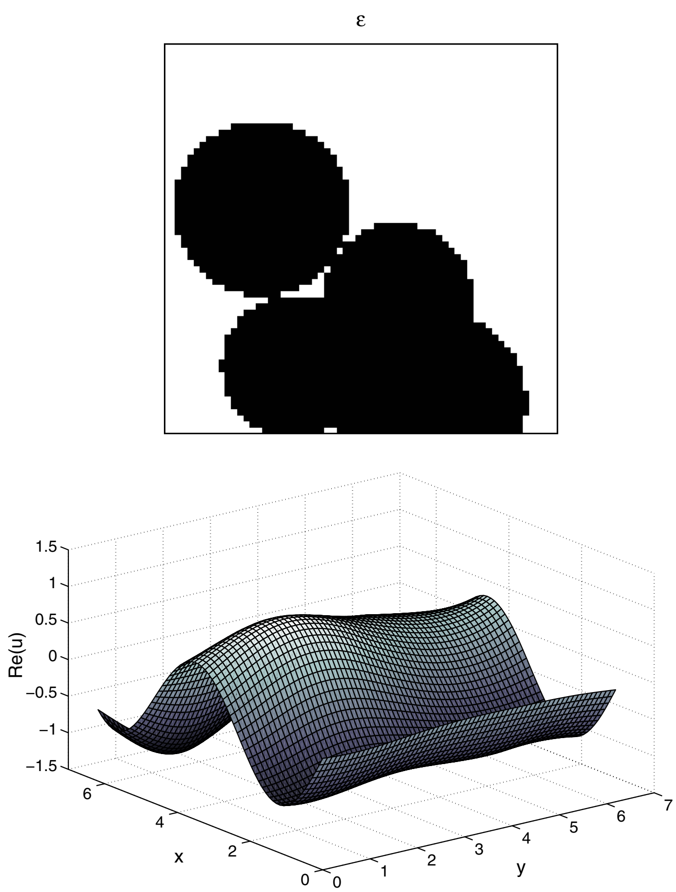

FIG. 4.4. *Sample material* I*: constitutive materials* $\varepsilon_0 = 1$ *and* $\varepsilon_1 = 1.5$ *(top). Contributions from sample material* I *to the real part of solution u (bottom).*

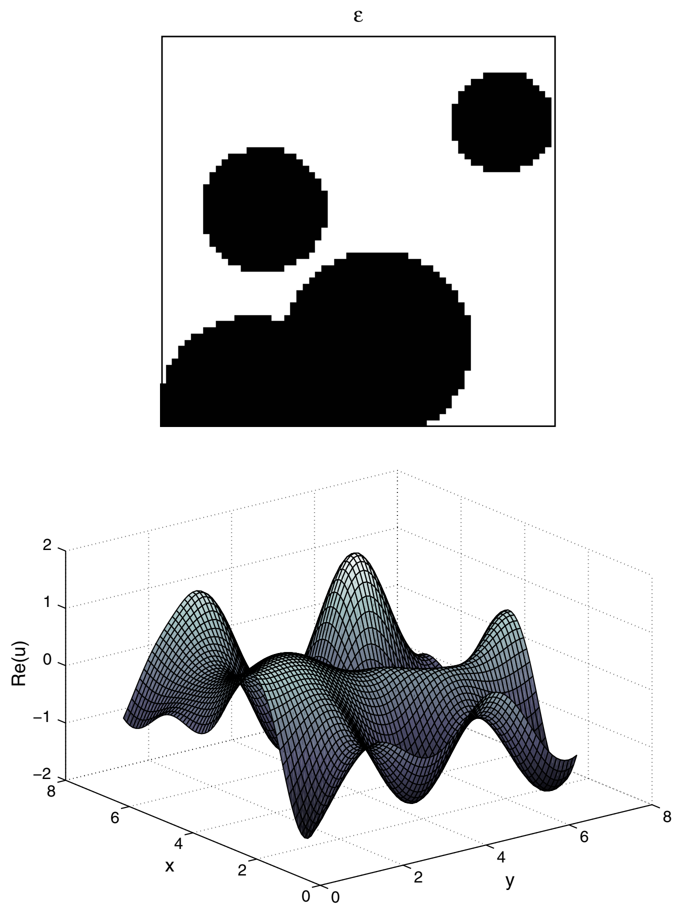

FIG. 4.5. *Sample material* II*: constitutive materials* $\varepsilon_0 = 1$ *and* $\varepsilon_1 = 4$ *(top). Contributions from sample material* II *to the real part of solution u (bottom).*

non-Hermitian matrix problem, which for simplicity is solved using the direct sparse solver in MATLAB.

In all two-dimensional numerical experiments, the frequency $\omega = 1.2$. In Figure 4.4 a single material realization (top) and the real part of the corresponding solution u (bottom) for a medium with contrast $z = 0.5$ (as defined in (4.1) are displayed. In Figure 4.5 another material realization (top) and the real part of the corresponding solution u (bottom) for a medium with contrast $z = 3$ are shown. The average $\langle \varepsilon u \rangle$ is obtained by calculating εu for each material realization, summing up over realizations, and dividing the sum by the number of realizations. In our experiments the number of material realizations is 75000. The expectation $\langle u \rangle$ is calculated similarly. The effective coefficient ε^* is the quotient of these quantities: $\varepsilon^* = \frac{\langle \varepsilon u \rangle}{\langle u \rangle}$.

In Figure 4.6 the expectations $\langle u \rangle$, $\langle \varepsilon u \rangle$, and the effective dielectric coefficient for the random medium with contrast $z = 0.5$ are shown. Let us investigate the effect of increasing the contrast z in the medium on the magnitude of the spatial variation in ε^*. In Figure 4.7 we have shown the averaged quantities $\langle u \rangle$ and $\langle \varepsilon u \rangle$ for a random medium with contrast $z = 3$. The spatial variations of the effective coefficient

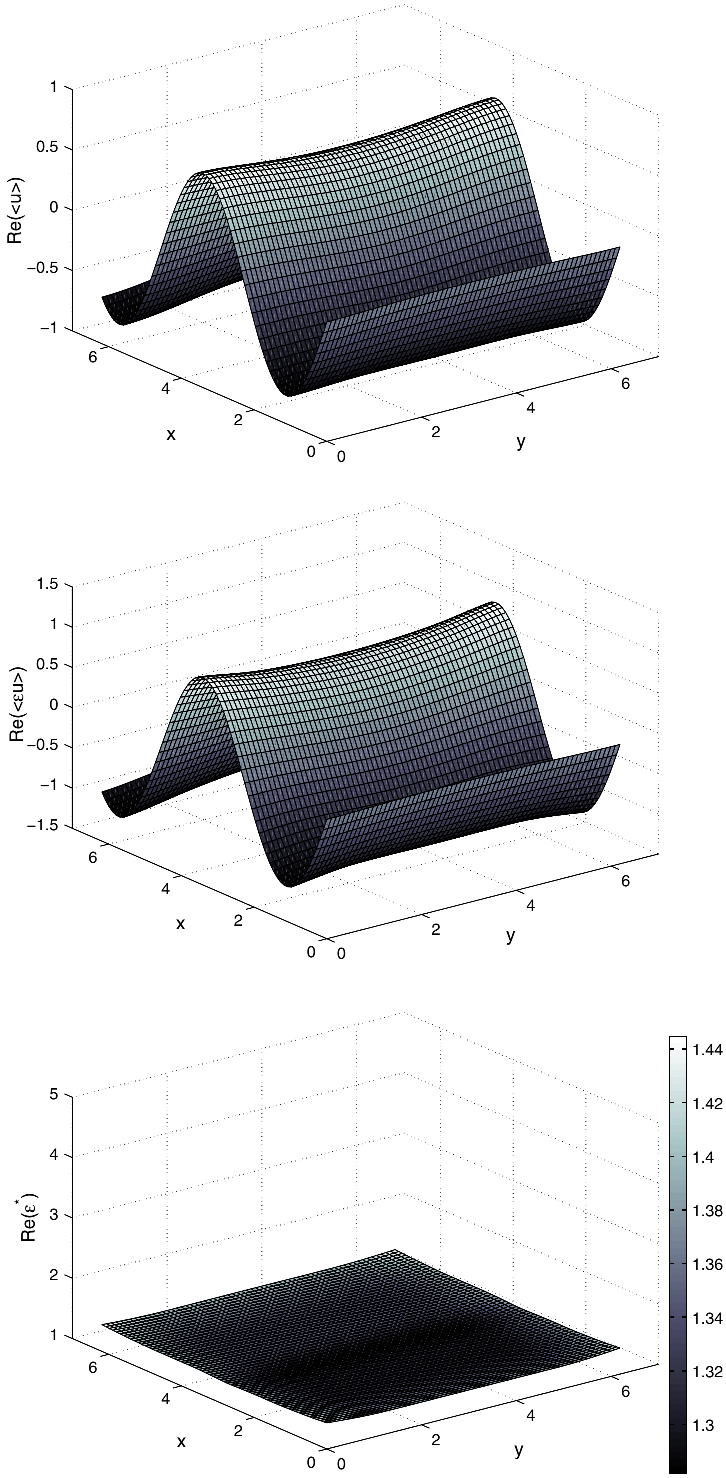

FIG. 4.6. *Averaged quantities of a medium with contrast* $z = 0.5$*: real part of* $\langle u \rangle$ *(top); real part of* $\langle \varepsilon u \rangle$ *(middle); real part of* ε^* *(bottom).*

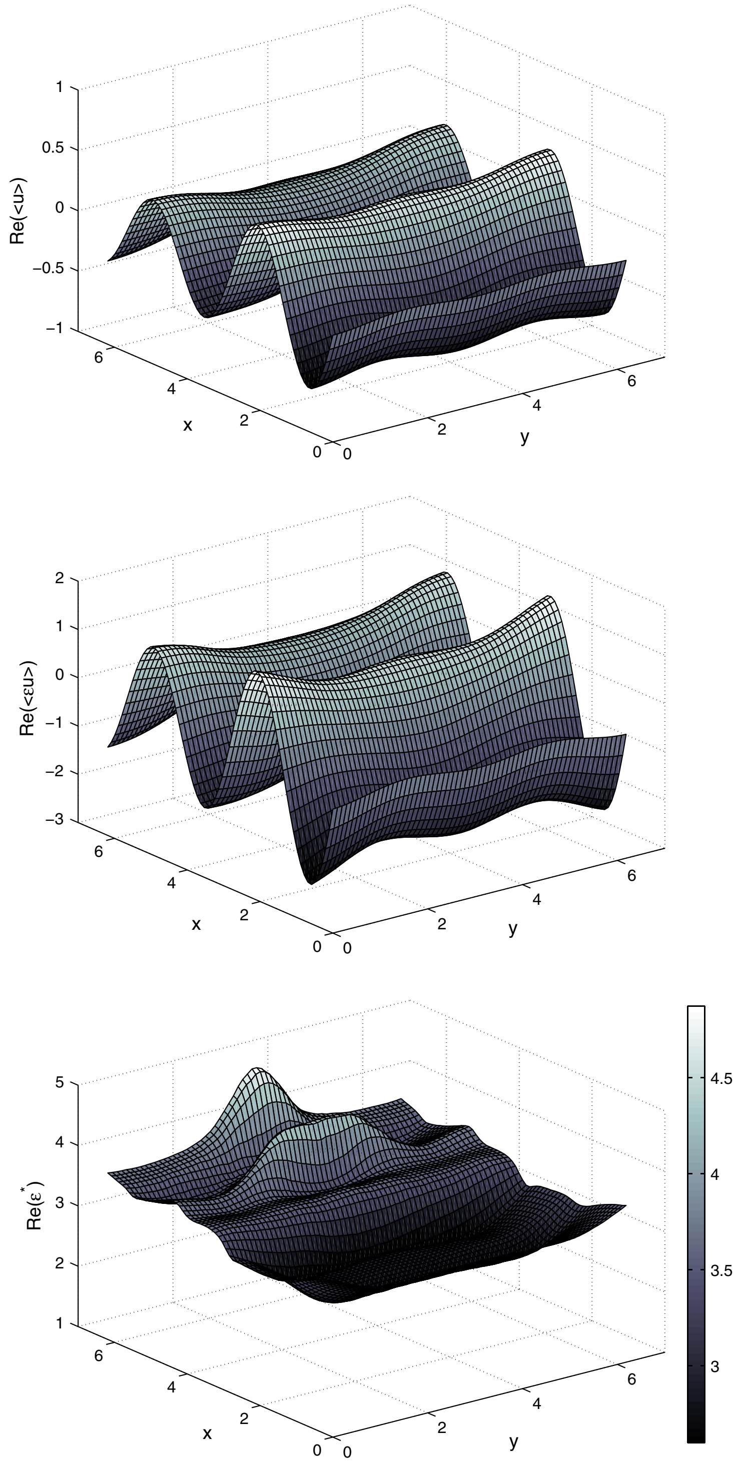

Fig. 4.7. *Averaged quantities of a medium with contrast $z = 3$: real part of $\langle u \rangle$ (top); real part of $\langle \varepsilon u \rangle$ (middle); real part of ε^* (bottom).*

(Figure 4.7) are much larger in magnitude for the medium with the greater contrast.

5. Conclusions. When we consider wave propagation in a medium for which the size of the inhomogeneities is of the same order as the wave length, scattering effects must be accounted for and the effective dielectric coefficient is no longer a constant, but a spatially dependent function. In this paper we study the spatial variations of the effective permittivity, obtaining an estimate that shows the dependence on material contrast and length scale. Numerical experiments confirm the presence of spatial variations and their dependence on the size of the inhomogeneities and the magnitude of the contrast. The purpose of this study is to gain some understanding of the viability of modeling ensemble average wave behavior with an effective material parameter, as one moves away from the well-studied low-contrast, low-frequency parameter regimes.

Appendix. In two dimensions using polar coordinates (r, θ) and assuming no incoming waves, the exterior scattered solution is

$$u_{ex}(r, \theta) = \sum_{m=1}^{\infty} A_m H_m^1(\omega r) e^{im\theta},$$

where $H_m^1(\omega r)$ are Hankel functions of first kind. Suppose that the Dirichlet data u_{in} is given on the circle. The interior solution $u_{in} \in L^2(S_0)$, and thus it has a Fourier series representation

$$u_{in}(\theta) = \sum_{m=1}^{\infty} \hat{u}_m e^{im\theta},$$

where

$$\hat{u}_m = \frac{1}{2\pi} \int_0^{2\pi} u(\omega R_0, \theta') e^{-im\theta'} d\theta'.$$

The constants A_m are found from the Dirichlet condition to be

$$A_m = \frac{\hat{u}_m}{H_m^1(\omega R_0)}.$$

Thus the radiating solution is given by

$$u_s(r, \theta) = \sum_{m=1}^{\infty} \frac{H_m^1(\omega r)}{H_m^1(\omega R_0)} \hat{u}_m e^{im\theta}.$$

Differentiating in the radial direction and setting $r = R_0$ leads to

$$\frac{\partial u_s}{\partial r}(R_0, \theta) = \omega \sum_{m=1}^{\infty} \frac{\frac{\partial H_m^1}{\partial r}(\omega R_0)}{H_m^1(\omega R_0)} \hat{u}_m e^{im\theta} \equiv (T u_s)(\theta).$$

Thus, we see that

$$(Tv)(\theta) = \omega \sum_{m=1}^{\infty} \left(\frac{\frac{\partial H_m^1}{\partial r}(\omega R_0)}{H_m^1(\omega R_0)} \right) \hat{v}_m e^{im\theta}, \tag{6.1}$$

where $\hat{v}_m$ are the Fourier coefficients of v, where v satisfies the Helmholtz equation (2.1).

Let

$$\gamma_m \equiv \omega \frac{\frac{\partial H_m^1}{\partial r}(\omega R_0)}{H_m^1(\omega R_0)}. \tag{6.2}$$

By using the properties and identities of Hankel functions, it can be shown that $\mathfrak{I}(\gamma_m) > 0$ and $\mathfrak{R}(\gamma_m) < 0$ for all m.

For $m \geq 0$ and r in compact subsets of $(0, \infty)$, we have [3]

$$|H_m^1(\omega r)| \leq C \frac{2^m m!}{(\omega r)^m}.$$

The derivative of the Hankel function is

$$\frac{\partial H_m^1}{\partial r}(\omega r) = \frac{m H_m^1(\omega r)}{r} - \omega H_{m+1}^1(\omega r).$$

This way we can bound the ratio

$$\left| \frac{\frac{\partial H_m^1}{\partial r}(\omega R_0)}{H_m^1(\omega R_0)} \right| \leq Cm.$$

We obtain the bound

$$\begin{aligned} \| Tv \|^2_{H^{-\frac{1}{2}}(S_0)} &\leq \sum_{m=1}^{\infty} (1+m^2)^{-\frac{1}{2}} \left| \frac{\frac{\partial H_m^1}{\partial r}(\omega R_0)}{H_m^1(\omega R_0)} \right|^2 |\hat{v}_m|^2 \\ &\leq \sum_{m=1}^{\infty} C(1+m^2)^{-\frac{1}{2}} m^2 |\hat{v}_m|^2 \\ &\leq \sum_{m=1}^{\infty} C(1+m^2)^{\frac{1}{2}} |\hat{v}_m|^2 \leq C\|v\|^2_{H^{\frac{1}{2}}(\Gamma_0)} \leq C\|v\|^2_{H^1(\Omega_0)}, \end{aligned} \tag{6.3}$$

where we have used the trace imbedding theorem [1].

In three dimensions using spherical coordinates (r, θ, ϕ) assuming $\varepsilon(x) = 1$ and no incoming waves, the scattered solution is

$$u_{ex}(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_{lm} h_l^1(\omega r) Y_{lm}(\theta, \phi),$$

where $h_l^1(\omega r)$ are spherical Hankel functions of first kind and $Y_{lm}(\theta, \phi)$ are the normalized spherical harmonics. The latter form an orthonormal complete set of $L^2(S_0)$ [15]. Suppose that the Dirichlet data is given on the sphere. Since $u_{in} \in L^2(S_0)$, it can be expanded into spherical harmonics as

$$u_{in}(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{u}_{lm} Y_{lm}(\theta,\phi)$$

with

$$\hat{u}_{lm} = \int_{S_0} u(R_0,\theta',\phi') \bar{Y}_{lm}(\theta',\phi') dS'.$$

The constants B_{lm} are found from the Dirichlet condition to be

$$B_{lm} = \frac{\hat{u}_{lm}}{h_l(wR_0)}.$$

Thus,

$$u_s(r,\theta,\phi) = \sum_{l=0}^{\infty} \frac{h_l^1(\omega r)}{h_l(\omega R_0)} \sum_{m=-l}^{l} \hat{u}_{lm} Y_{lm}(\theta,\phi).$$

Differentiating in the radial direction and setting $r = R_0$ gives

$$\frac{\partial u_s}{\partial r}(R_0,\theta,\phi) = \sum_{l=0}^{\infty} \omega \frac{\frac{\partial h_l^1}{\partial r}(\omega R_0)}{h_l^1(\omega R_0)} \sum_{m=-l}^{l} \hat{u}_{lm} Y_{lm}(\theta,\phi) \equiv (Tu_s)(\theta,\phi).$$

We see that

$$(Tv)(\theta,\phi) = \sum_{l=0}^{\infty} \omega \left(\frac{\frac{\partial h_l^1}{\partial r}(\omega R_0)}{h_l^1(\omega R_0)} \right) \sum_{m=-l}^{l} \hat{v}_{lm} Y_{lm}(\theta,\phi), \tag{6.4}$$

where $\hat{v}_{lm}$ are the coefficients in the spherical harmonics expansion of v, where v satisfies the Helmholtz equation (2.1).

Let

$$\gamma_l \equiv \omega \frac{\frac{\partial h_l^1}{\partial r}(\omega R_0)}{h_l^1(\omega R_0)}. \tag{6.5}$$

The following is obtained by very slight modification of the analysis of the exterior scattering problem discussed in [9]: for all l, $\mathfrak{I}\gamma_l > 0$ and $\mathfrak{R}\gamma_l < 0$.

The Sobolev space $H^s(S_0)$ with real parameter s consists of all distributions f such that

$$\|f\|^2_{H^s(S_0)} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (1+\lambda_l)^s |\hat{f}_{lm}|^2 < \infty,$$

where $\hat{f}_{lm}$ are the spherical harmonics Fourier coefficients and $\lambda_l = l(l+1)$, $l \geq 0$ is the eigenvalue of the Laplace–Beltrami operator on S_0. For $l \geq 0$ and r in compact subsets of $(0,\infty)$, we have

$$|h_l^1(\omega r)| \leq C \frac{2^l l!}{(\omega r)^{l+1}}.$$

The derivative of the spherical Hankel function is

$$\frac{\partial h_l^1}{\partial r}(\omega r) = \frac{1}{2}\left(\omega h_{l-1}^1(\omega r) - \frac{h_l^1(\omega r) + \omega r h_{l+1}^1(\omega r)}{r}\right).$$

This way we can bound the ratio

$$\left|\frac{\frac{\partial h_l^1}{\partial r}(\omega R_0)}{h_l^1(\omega R_0)}\right| \le Cl.$$

We then obtain the bound

$$\begin{aligned}
\| Tv\|^2_{H^{-\frac{1}{2}}(\Gamma_0)} &\le \omega \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (1 + l(l+1))^{-\frac{1}{2}} \left|\frac{\frac{\partial H_l^1}{\partial r}(\omega R_0)}{H_l^1(\omega R_0)}\right|^2 |\hat{v}_{l,m}|^2 \\
&\le \omega \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C(1 + l(l+1))^{\frac{1}{2}} |\hat{v}_{l,m}|^2 \\
&\le C\|v\|^2_{H^{\frac{1}{2}}(\Gamma_0)} \le C\|v\|^2_{H^1(\Omega_0)},
\end{aligned} \tag{6.6}$$

where we have used the trace imbedding theorem [1].

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