MATH 7800: BOUNDEDNESS IN HIGHER DIMENSIONAL BIRATIONAL GEOMETRY

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1. Introduction

One of the most natural goals in algebraic geometry is to classify projective varieties over an algebraically closed field k up to birational isomorphism. Recall that two varieties are birational if they have isomorphic open subsets or equivalently isomorphic function fields. In this note we will assume that $k = \mathbb{C}$ is the field of complex numbers. The strategy behind this classification is as follows. Starting with a (irreducible and reduced) variety $X \subset \mathbb{P}^N_{\mathbb{C}}$ of dimension $d := \dim X$, by Hironaka's theorem, we may assume that X is smooth and consider the pluricanonical ring $R(K_X) := \bigoplus_{m \geq 0} H^0(mK_X)$. Note that if X and Y are smooth and birational, then their pluricanonical rings are isomorphic $R(K_X) \cong R(K_Y)$.

Recall that if X is smooth, then the top exterior power of the cotangent bundle $\omega_X = \Lambda^d T_X^\vee$ is a line bundle and the global sections of $H^0(mK_X)$ correspond to the global sections of $\omega_X^{\otimes m}$ so that in local coordinates $x_1, \ldots, x_n$, they can be written as $f(x_1, \ldots, x_n)(dx_1 \wedge \ldots \wedge dx_n)^{\otimes m}$ where $f(x_1, \ldots, x_n)$ is a holomorphic function.

By [BCHM10], it is known that $R(K_X)$ is finitely generated and so one may consider the corresponding projective variety $Z = \mathrm{Proj} R(K_X)$. The integer

$$\kappa(X) := \mathrm{tr.deg.}_{\mathbb{C}} R(K_X) - 1 \in \{-1, 0, 1, \ldots, \dim X\}$$

is the *Kodaira dimension* of X and the rational map $X \dashrightarrow Z$ is the *Iitaka fibration*. Note that $\kappa(X) = \dim Z$. Often we replace X by an appropriate birational model so that $X \to Z$ is a morphism. There are three main cases to consider, namely varieties of maximal Kodaira dimension $\kappa(X) = \dim X$ also known as varieties of *general type*, varieties of intermediate Kodaira dimension $\dim X > \kappa(X) \geq 0$, and varieties of negative Kodaira dimension $0 > \kappa(X)$.

Exercise 1.1. *Show that if X and Y are birational smooth projective varieties, then $H^0(\omega_X^{\otimes m}) \cong H^0(\omega_Y^{\otimes m})$ for all $m \in \mathbb{N}$.*

Exercise 1.2. *Give an example of a singular projective variety X such that $R(K_X) \not\cong R(K_Y)$ where Y is birational to X.*

1.1. **Curves.** If $d = 1$, then we say that X is a curve. In this case ω_X is a line bundle of degree $2g - 2$ where g is the genus of X so that $2g = B_1(X)$ is the first Betti number.

The case $\kappa(X) < 0$ corresponds to $g = 0$. In this case $X \cong \mathbb{P}^1$ and $\omega_X = \mathcal{O}_{\mathbb{P}^1}(-2)$ so that $H^0(\omega_X^{\otimes m}) = H^0(\mathcal{O}_{\mathbb{P}^1}(-2m)) = 0$ for all $m \geq 1$. Thus $R(K_X) \cong \mathbb{C}$ is clearly finitely generated.

The case $\kappa(X) = 0$ corresponds to $g = 1$. In this case we say that X is an elliptic curve. We have that $\omega_X \cong \mathcal{O}_X$ so that $H^0(\omega_X^{\otimes m}) \cong H^0(\mathcal{O}_X) \cong \mathbb{C}$ and hence $R(K_X) \cong \mathbb{C}[t]$ which is also finitely generated. Note however that there is a one parameter family of elliptic curves (X_t defined by $y^2 = x(x-1)(x-t)$) and the canonical ring does not identify the isomorphism class of X.

The case $\kappa(X) = 1$ corresponds to $g \geq 2$. In this case we say that X is a curve of general type. The line bundle ω_X has degree $2g-2 > 0$ and hence is ample. Recall that this means that some tensor power $\omega_X^{\otimes m}$ for some $m > 0$ is very ample (i.e. the sections of $H^0(\omega_X^{\otimes m})$ define an embedding in to $\mathbb{P}^N$). It is not hard to see that $\omega_X^{\otimes 3}$ is very ample and so every curve of general type can be embedded in $|3K_X| \cong \mathbb{P}^{5g-6}$ as a curve of degree $6g-6$. In particular, by a Hilbert scheme argument, for fixed $g \geq 2$, these curves belong to a bounded family. In fact it is known that these curves are parametrized by an irreducible quasi-projective variety of dimension $3g - 3$.

Exercise 1.3. *Show that if X is a smooth curve of genus $g \geq 2$ then $\omega_X^{\otimes m}$ is very ample for $m \geq 3$.*

Exercise 1.4. *Show that if X is a smooth curve of genus $g \geq 2$ then $h^0(\omega_X^{\otimes m}) = (2m-1)(g-1)$ for $m \geq 2$.*

Exercise 1.5. *Show that if X is a smooth curve of genus $g \geq 2$ then $h^1(T_X) = 3g - 3$.*

1.2. **Surfaces.** If $d = 2$, then we say that X is a *surface*. In this case $\kappa(X) \in \{-1, 0, 1, 2\}$. Note that there are many birational but not isomorphic smooth surfaces. To see this, just take a smooth surface X and blow up a point $x \in X$ to obtain $f : X' \to X$ such that f is an isomorphism over $X \setminus x$ and $f^{-1}(x) = E$ is a rational curve $E \cong \mathbb{P}^1$ and $K_{X'} \cdot E = E \cdot E = -1$. Recall that by Castelnuovo's Criterion, if a smooth surface X contains a rational curve $E \cong \mathbb{P}^1$ such that $E \cdot K_X = -1$ (or equivalently $E \cdot E = -1$), then there exists a morphism $f : X \to Y$ such that $f(E) = y$ is a point on Y and X is isomorphic to the blow up of Y at y. If X contains no *minus 1 curves* (i.e. smooth rational curves E such that $E^2 = -1$), then we say that X is *minimal*. By Castelnuovo's Criterion, it follows easily that every smooth surface is birational to a (smooth) minimal surface. It is well known that if $\kappa(X) \geq 0$, then the corresponding minimal surface is unique. This is not the case if $\kappa(X) < 0$.

If $\kappa(X) < 0$, then X is birational to a ruled surface $X' = \mathbb{P}_C(\mathcal{E})$ where C is a curve and $\mathcal{E}$ is a rank 2 vector bundle on C.

If $\kappa(X) = 0$, then X is birational to one of the following minimal surfaces:

(1) Abelian surface: $h^0(\Omega^1_X) = 2$ and $h^0(\omega_X) = 1$. We have $T_X \cong \mathcal{O}_X^{\oplus 2}$ and so $K_X = 0$. For example $X = E \times F$ where E, F are elliptic curves.
(2) K3 surface: $h^0(\Omega^1_X) = 0$ and $h^0(\omega_X) = 1$. We have $K_X = 0$. For example a hypersurface of degree 4 in $\mathbb{P}^3$ or a complete intersection of degree $2, 3$ (resp. $2, 2, 2$) in $\mathbb{P}^4$ (resp. $\mathbb{P}^5$).
(3) Bielliptic surfaces: $h^0(\Omega^1_X) = 1$ and $h^0(\omega_X) = 0$. For these surfaces $K_X \neq 0$ but $12K_X = 0$. Let k be the smallest positive integer such that $kK_X = 0$, then there is an étale cover $X' \to X$ of degree k such that $X' = E \times F$ is an abelian surface given by the product of two elliptic curves. We may assume that X is the quotient of X' by a finite abelian group of order 2, 3, 4, or 6.
(4) Enriques surfaces: $h^0(\Omega^1_X) = 0$ and $h^0(\omega_X) = 0$. For these surfaces $K_X \neq 0$ but $2K_X = 0$. The corresponding double cover is a K3 surface.

If $\kappa(X) = 1$, then X is birational to a surface X' with a morphism $f : X' \to C$ such that the general fiber F of f is an elliptic curve. For example consider the product of an elliptic curve F with a curve C of genus $g \geq 2$.

If $\kappa(X) = 2$ and X is minimal, then we say that X is a minimal model (which is unique). In this case K_X is nef which means that $K_X \cdot C \geq 0$ for any curve $C \subset X$. By a result of Bombieri, it is known that the linear series $|5K_X|$ is base point free so that for any $x \in X$ there exists a divisor $D \in |5K_X|$ whose support does not contain x. It follows that $|5K_X|$ defines a morphism $f : X \to \mathbb{P}^N = |5K_X|$. It is well known that if D is nef and big (so that $D \cdot C \geq 0$ for any curve $C \subset X$ and $D^2 > 0$), then $h^i(K_X + D) = 0$ for $i > 0$. In particular $h^i(mK_X) = 0$ for $i > 0$ and $m \geq 2$ so that $h^0(mK_X) = \chi(mK_X) = \frac{m(m-1)}{2}K_X^2 + \chi(\mathcal{O}_X)$. Let $X_{\text{can}} = f(X)$, then $f : X \to X_{\text{can}}$ is a morphism that contracts all curves $C \subset X$ such that $K_X \cdot C =$

0. It is not the case that X_{can} is smooth, but its singularities are well understood: they are du Val singularities (also known as rational double points orcanonical singularities). In particular $\omega_{X_{\mathrm{can}}}$ is an ample line bundle and in fact $\omega_{X_{\mathrm{can}}} \cong \mathcal{O}_{X_{\mathrm{can}}}(1)$ is very ample. It follows easily that $R(K_X) \cong R(K_{X_{\mathrm{can}}})$ is finitely generated and hence we have $X_{\mathrm{can}} \cong \mathrm{Proj} R(K_{X_{\mathrm{can}}})$. We say that X_{can} is the *canonical model*. Notice that the canonical model uniquely determines the minimal model (which is obtained by taking the minimal desingularization). The upshot is that if we fix $v = K_X^2 = K_{X_{\mathrm{can}}}^2$, then X_{can} is a subvariety of $\mathbb{P}^N$ of degree $25v$ and so by a Hilbert scheme type argument, it belongs to a bounded family. In particular for fixed v there are only finitely many topological types for the corresponding minimal/canonical models.

Exercise 1.6. *Show that if $f : X' \to X$ is the blow up of X at x, then $E = f^{-1}(x)$ is a smooth rational curve with $K_X \cdot E = E \cdot E = -1$. Note that $B_2(X') = B_2(X) + 1$.*

Exercise 1.7. *Using Castelnuovo's Criterion, show that every smooth surface is birational to a minimal surface.*

Exercise 1.8. *Show that if X is ruled then $|mK_X| = \emptyset$ for all $m > 0$.*

Exercise 1.9. *Show that if X is ruled then $h^0(\Omega^1_X) = h^0(\omega_C)$.*

Exercise 1.10. *Show that $\mathbb{P}^2$ is birational to a ruled surface.*

Exercise 1.11. *Let $F_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$, then F_n and F_m are birational but not isomorphic for any $n \neq m \geq 0$.*

Exercise 1.12. *Show that F_n is minimal iff $n \neq 1$.*

Exercise 1.13. *Show that a hypersurface of degree 4 in $\mathbb{P}^3$ or a complete intersection of degree $2, 3$ (resp. $2, 2, 2$) in $\mathbb{P}^4$ (resp. $\mathbb{P}^5$) are K3 surfaces.*

Exercise 1.14. *Show that if X is birational to a surface X' with a morphism $f : X' \to C$ such that the general fiber F of f is an elliptic curve, then $\kappa(X) \leq 1$ give examples where $\kappa(X) = 0, -1$. Give examples where not all fibers are isomorphic.*

Exercise 1.15. *Show that if a divisor D is semiample, then it is also nef. Give an example of a divisor D which is nef but not semiample.*

Exercise 1.16. *Show that $\chi(\mathcal{O}_X)$ is a birational invariant.*

Exercise 1.17. *Show that $\chi(mK_X) = \frac{m(m-1)}{2} K_X^2 + \chi(\mathcal{O}_X)$.*

1.3. **n-folds.** Not surprisingly, in dimension $d \geq 3$ the situation is much more complicated. We begin by recalling the following fundamental result of [BCHM10].

Theorem 1.18. *Let X be a smooth complex projective variety, then $R(K_X)$ is finitely generated.*

An immediate consequence of this result, is that if X is of general type, then it is birational to a unique canonical model $X_{\text{can}} = \text{Proj}R(K_X)$. In fact, for any $m \gg 0$, the closure of the image of the induced rational map $\phi_m : X \dashrightarrow |mK_X|$ is $\overline{\phi_m(X)} \cong X_{\text{can}}$. We make the following observations:

(1) X_{can} is not necessarily smooth, but its singularities are canonical and in particular rational and we have $H^0(\omega_{X_{\text{can}}}^{\otimes m}) \cong H^0(\omega_X^{\otimes m})$ for any smooth (or with canonical singularities) variety X birational to X_{can}. It follows that
$$K^d_{X_{\text{can}}} = \text{vol}(K_X) := \lim \frac{d!h^0(mK_X)}{m^d}.$$
(2) If X is a variety of general type (smooth or with terminal singularities), then it admits a minimal model. The minimal model is not necessarily unique, but any two minimal models are related by a finite sequence of flops.
(3) $\omega_{X_{\text{can}}}^{\otimes m}$ is a line bundle for all $m > 0$ sufficiently divisible, however $\omega_{X_{\text{can}}}$ may not be Cartier so we could have $K^d_{X_{\text{can}}} < 1$. In fact, by an example of [Iano-Fletcher00], if X is a weighted complete intersection of degree 46 in weighted projective space $\mathbb{P}(4, 5, 6, 7, 23)$, then $\text{vol}(K_X) = 1/420$ and $|mK_X|$ is birational if and only if $m = 23$ or $m \geq 27$.

Following ideas of H. Tsuji in [HM06], [Takayama06] and [Tsuji07], the following remarkable result is proven.

Theorem 1.19. *Let $d \in \mathbb{N}$ and $\mathcal{P}_d$ be the set of all smooth projective d-dimensional varieties of general type.*

(1) *There exists an integer $m = m_d$ depending only on d such if $X \in \mathcal{P}_d$, then $|mK_X|$ is birational.*
(2) *The set $\mathcal{V}_d = \{\text{vol}(K_X)|X \in \mathcal{P}_d\}$ is discrete and in particular there is a minimal element $0 < v_d \in \mathcal{V}_d$.*
(3) *For any $v \in \mathcal{V}_d$, there exists a projective morphism of quasi-projective varieties $\mathcal{X} \to T$ such that if X is a d-dimensional canonical model, then $X \cong \mathcal{X}_t$ for some $t \in T$.*

The above theorem shows that $\text{vol}(K_X)$ is the correct higher dimensional generalization of the genus of a curve. It is a discrete birational invariant. Once this invariant is fixed, the corresponding canonical varieties of general type are parametrized by a quasi-projective variety.

If we wish to use these ideas to construct a proper moduli space, we must also consider the corresponding problem for slc models (X, B), i.e. the higher dimensional generalization of stable curves.

If $0 \leq \kappa(X) < d$, then one considers the Iitaka fibration $X \dashrightarrow Z := \mathrm{Proj}(R(K_X))$. This is defined by $|mK_X|$ for $m > 0$ sufficiently divisible. The fibers of $X \dashrightarrow Z$ are varieties F with $\kappa(F) = 0$ and $R(K_X) \cong R(K_Z + B)$ where (Z, B) is a klt pair and $K_Z + B$ is of general type. Therefore the geometry of of X can be described in terms of lower dimensional pairs of general type and varieties of Kodaira dimension 0. Note that the case of dimension 0 is particularly hard to study. In particular it is not even clear if there are finitely many topologies for minimal threefolds of Kodaira dimension 0.

Finally, if $\kappa(X) < 0$, then it is known that X is birational to a Mori fiber space $f : X' \to Z$. We have

(1) $\rho(X'/Z) = 1$,
(2) X' has terminal singularities, and
(3) $K_{X'}$ is ample over Z.

In particular, the fibers of f are Fano varieties (F is terminal and $-K_F$ is ample). Therefore we view terminal Fano varieties as building blocks for Fano varieties. A famous conjecture of Alexander Borisov, Lev Borisov, and Valery Alexeev (known as the BAB conjecture) states that terminal Fano varieties of dimension d are bounded. This conjecture was recentluy solved in [Birkar16b].

Theorem 1.20. *Fix $d \in \mathbb{N}$, then the set of all terminal projective varieties such that $-K_X$ is ample is bounded.*

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2. PRELIMINARIES

2.1. **Singularities of the MMP.** In this section we recal the standard notions of singularities of the minimal model program. Let X be a normal quasi-projective variety and $D = \sum d_i D_i$ be an $\mathbb{R}$-divisor. Here we assume that the D_i are distinct prime divisors and $d_i \in \mathbb{R}$. The *support* of D is $\mathrm{Supp}(D) = \cup_{d_i \neq 0} D_i$. Recall that by definition a divisor G is $\mathbb{R}$ Cartier if locally we may write $G = \sum r_i(g_i)$ where $r_i \in \mathbb{R}$ and (g_i) is the divisor associated to a rational function $g_i \in \mathbb{C}(X)$. If $r \in \mathbb{R}$, then we let

$$\lceil r \rceil = \min\{n \in \mathbb{Z} | n \geq r\}, \qquad \lfloor r \rfloor = \max\{n \in \mathbb{Z} | n \leq r\}$$

be the *round up* and *round down* of r. We also define the *fractional part* $\{r\} = r - \lfloor r \rfloor \in [0,1)$. For any $\mathbb{R}$-divisor $D = \sum d_i D_i$, we define the *round up* $\lceil D \rceil$, *round down* $\lfloor D \rfloor$ and *fractional part* by

$$\lceil D \rceil = \sum \lceil d_i \rceil D_i, \qquad \lfloor D \rfloor = \sum \lfloor d_i \rfloor D_i, \qquad \{D\} = \sum \{d_i\} D_i.$$

Recall that by definition

$$\Gamma(\mathcal{O}_X(D)) = \{f \in \mathbb{C}(X) | (f) + D \geq 0\} = \Gamma(\mathcal{O}_X(\lfloor D \rfloor)).$$

If X is normal, then the nonsingular locus X_{reg} is a *big open subset* so that $X_{\mathrm{sing}} := X \setminus X_{\mathrm{reg}}$ is a closed subset of codimension ≥ 2 in X. Therefore, we let $K_X = i_* K_{X_{\mathrm{reg}}}$ where $K_{X_{\mathrm{reg}}}$ is a Canonical divisor on $K_{X_{\mathrm{reg}}}$ (equivalently $\omega_{K_{X_{\mathrm{reg}}}} \cong \mathcal{O}_{X_{\mathrm{reg}}}(K_{X_{\mathrm{reg}}})$). We define $\omega_X = i_* \omega_{K_{X_{\mathrm{reg}}}} = \mathcal{O}_X(K_X)$. Note that K_X is a Weil divisor, but it may not be $\mathbb{R}$-Cartier. If $B = \sum b_i B_i$ is an $\mathbb{R}$-divisor such that $K_X + B$ is $\mathbb{R}$-Cartier, then we say that (X, B) is a pair. If (X, B) is a *pair*, then since $K_X + B$ is $\mathbb{R}$-Cartier, the pull-back $f^*(K_X + B)$ is defined for any morphism of normal varieties $f : X' \to X$. Suppose now that f is proper and birational. We write

$$K_{X'} = f^*(K_X + B) + A_{X'}(X, B).$$

Note that we chose $K_{X'}$ so that $f_* K_{X'} = K_X$.

Lemma 2.1. *The divisor $A_{X'}$ does not depend on the choice of K_X (here for simplicity we supress (X, B) from the notation so that $A_{X'} := A_{X'}(X, B)$).*

Proof. Suppose that we have $K_{X'} = f^*(K_X + B) + A$ and $K'_{X'} = f^*(K'_X + B) + A'$ where $K_X, K'_X, K_{X'}, K'_{X'}$ are canonical divisors chosen so that $f_*(K_{X'}) = K_X$ and $f_*(K'_{X'}) = K'_X$, then $f_*(A - A') = 0$ and $A - A' \sim_{\mathbb{R}} 0$ so that by the Negativity Lemma (see below), $A - A' = 0$. ☐

Lemma 2.2 (Negativity Lemma)**.** *Let $f : X \to Y$ be a proper birational morphism of normal varieties and B a $\mathbb{Q}$-Cartier divisor such that $-B$ is f nef, then B is effective if and only if so is $f_* B$. In this case, i.e. if $B \geq 0$, then for any $y \in Y$, either $f^{-1}(y) \subset \mathrm{Supp}(B)$ or $f^{-1}(y) \cap \mathrm{Supp}(B) = \emptyset$.*

Proof. See [KM98, 3.39] ☐

The $\mathbb{R}$-divisor $A_{X'}(X, B) = \sum a(E; X, B) E$ defined by the above equation is the *discrepancy divisor* and the numbers $a(E; X, B)$ are the discrepancies of (X, B) along the divisors E. Notice that $a(E; X, B) = 0$ unless E is contained in the support of the strict transform of B or E is exceptional. In particular $A_{X'}(X, B)$ is a finite sum. If $E = f^{-1}_* B_i$, then $a(E,X,B) = -b_i$. The discrepancies $a(E,X,B)$ do not depend on the choice of the birational model X'. In fact if $\nu : X'' \to X'$ is another birational morphism, then $\nu_* A_{X''}(X,B) = A_{X'}(X,B)$. In particular we have defined a b-divisor $\mathbf{A}$ by letting $\mathbf{A}_{X'} = A_{X'} := K_{X'} - f^*(K_X + B)$ for any birational model $X' \to X$. (Recall that a b-divisor $\mathbf{D}$ is specified by the choice of a divisor $\mathbf{D}_{X'}$ for any birational model $f : X' \to X$ which is compatible with respect to push-forwards; the set of all b-divisors is then identified with $\varprojlim \mathrm{Div}(X')$ where $f : X' \to X$ belongs to the partially ordered set of all birational models of X.)

The *total discrepancy* and the *discrepancy* of a pair (X,B) is given by

$$\text{total discrepancy}(X,B) := \inf\{a(E;X,B) | E \text{ is a prime divisor over } X\},$$

$$\text{discrepancy}(X,B) := \inf\{a(E;X,B) | E \text{ is an exceptional prime divisor over } X\}.$$

It is easy to see that if total discrepancy$(X,B) < -1$, then total discrepancy$(X,B) = -\infty$. We will also write $K_{X'} + B_{X'} = f^*(K_X + B)$ and so we define a b-divisor $\mathbf{B} = \mathbf{B}(X,B)$ by letting

$$\mathbf{B}_{X'} = B_{X'} := f^*(K_X + B) - K_{X'}$$

so that $\mathbf{B} = -\mathbf{A}$.

Definition 2.3. *If $B \geq 0$, then we say that a pair (X,B) is:*

1. *klt (Kawamata log terminal) if* total discrepancy$(X,B) > -1$*,*
2. *lc (log canonical) if* total discrepancy$(X,B) \geq -1$*,*
3. *canonical if* discrepancy$(X,B) \geq 0$*,*
4. *terminal if* discrepancy$(X,B) > 0$*,*
5. *plt if* discrepancy$(X,B) > -1$*,*
6. *dlt (divisorially log terminal) if $b_i \leq 1$ and there is a closed subset $Z \subset X$ such that $(X \setminus Z, B|_{X \setminus Z})$ has simple normal crossings and for any prime divisor E over X with center contained in Z, we have $a(E;X,B) > -1$.*

Recall that a pair (X,B) has *simple normal crossings* if X is smooth and the support of B is a union of smooth divisors meeting transversely. A log resolution of a pair (X,B) is a proper birational morphism $f : X' \to X$ such that the exceptional set $\mathrm{Ex}(f)$ is a divisor and $(X, f_*^{-1}B \cup \mathrm{Ex}(f))$ has simple normal crossings. To check whether (X,B) is klt/lc, it suffices to consider a log resolution $f : X' \to X$ and to check if $a(E;X,B) > -1$ (resp. ≥ -1) for all prime divisors E on X'.

Remark 2.4. *it is known that klt singularities are rational i.e. if $f : X' \to X$ is a log resolution then $f_*\omega_{X'} = \omega_X$ and $R^i f_* \mathcal{O}_{X'} = 0$ for $i > 0$* [KM98, 5.22].

Definition 2.5. *If (X, B) is a pair and E is a divisor over X such that $a(E; X, B) < -1$ (resp. $a(E; X, B) \leq -1$), then we say that E is a non-lc (resp. non-klt) place of (X, B), and the image of E on X is a non-lc (resp. a non-klt) center. The union of all non-lc (resp. non-klt) centers is the non-lc locus* $\mathrm{nlc}(X, B)$, *(resp. the non-klt locus* $\mathrm{nklt}(X, B)$*).*

Definition 2.6. *Let (X, B) be a lc pair and $D \geq 0$ an effective divisor. The log canonical threshold of (X, B) with respect to D is*

$$\mathrm{lct}(X, B; D) = \sup\{t \geq 0 | (X, B + tD) \text{ is lc}\}.$$

Exercise 2.7. *Let X be the cone over a rational curve of degree n (resp. an elliptic curve or a curve of genus $g \geq 2$) and $f : X' \to X$ the blow up of the vertex with exceptional curve E. Compute the discrepancy $a(E, X, 0)$.*

Exercise 2.8. *Show that if (X, B) is a lc snc pair then* $\mathrm{discrep}(X, B)$ *is the minimum of 1, $1 - b_i$, and $1 - b_i - b_j$ where $B_i \cap B_j \neq \emptyset$.*

Exercise 2.9. *Show that if* total discrepancy$(X, B) < -1$*, then* total discrepancy$(X, B) = -\infty$*.*

Exercise 2.10. *Let (X, B) be log smooth of dimension d. Assume that $B = \sum b_i B_i$ and $x \in B_i$ for each i. Let $X' \to X$ be the blow up of $x \in X$ with exceptional divisor E. Show that $a(E; X, B) = d - 1 - \sum b_i$.*

Exercise 2.11. *Suppose that $B \leq B'$, where (X, B) and (X, B') are log pairs. Show that $a(E; X, B) \geq a(E; X, B')$ for any divisor E over X.*

Exercise 2.12. *Show that if X is canonical and $X' \to X$ is a resolution, then $R(K_X) \cong R(K_{X'})$.*

Exercise 2.13. *Show that terminal singularities are canonical but not viceversa.*

Exercise 2.14. *Show that klt singularities are lc but not viceversa.*

Exercise 2.15. *Let X be the cone over an abelian surface. Show that X is lc but its singularities are not rational.*

Exercise 2.16. *If (X, B) is plt, show that $\lfloor B \rfloor$ is a disjoint union of prime divisors.*

Exercise 2.17. *Find a log resolution for 3 lines meeting at a point in the plane and for the cusp $y^2 - x^3 = 0$.*

Exercise 2.18. *Compute* $\mathrm{lct}(\mathbb{C}^2, 0; D)$ *where D is the cusp $y^2 - x^3 = 0$.*

2.2. **Kawamata-Viehweg vanishing.** Let A be an ample line bundle on a projective variety X and $\mathcal{F}$ a coherent sheaf, then by Serre vanishing $H^i(\mathcal{F}^{\otimes m}) = 0$ for $i > 0$ and $m \gg 0$. Notice that m here depends both on X and $\mathcal{F}$. In the case when X is a smooth projective variety and $\mathcal{F} = \omega_X$, then by Kodaira vanishing we have

$$H^i(\omega_X \otimes A) = 0, \qquad \forall i > 0.$$

Kodaira vanishing was vastly generalized by Kawamata and Viehweg. Recall that an $\mathbb{R}$-Cartier divisor D on a normal projective variety X is nef if $D \cdot C \geq 0$ for any curve $C \subset X$ and D is nef and big if D is nef and the top self intersection is positive $D^d > 0$.

Remark 2.19. *If D is nef then $D^d > 0$ if and only if* $\mathrm{vol}(D) > 0$*.*

Theorem 2.20. *[Kawamata-Viehweg vanishing] Let X be a smooth projective variety and B an $\mathbb{R}$-divisor with simple normal crossings support such that $\lfloor B \rfloor = 0$ (so that (X, B) is klt). If N is a Cartier divisor such that $N - K_X - B$ is nef and big, then $H^i(N) = 0$ for all $i > 0$.*

It is not hard to generalize this result to the relative setting.

Theorem 2.21. *[Relative Kawamata-Viehweg vanishing] Let X be a smooth quasi projective variety and B an $\mathbb{R}$-divisor with simple normal crossings support such that $\lfloor B \rfloor = 0$ (so that (X, B) is klt). If $f : X \to Y$ is a projective morphism and N is a Cartier divisor such that $N - K_X - B$ is f-nef and f-big, then $R^i f_*(N) = 0$ for all $i > 0$.*

Recall that an $\mathbb{R}$-Cartier divisor D is f-nef if $D \cdot C \geq 0$ for any curve contained in a fiber of f and f-big if $D|_F$ is nef and big where F is a general fiber of f.

Proof (Theorem 2.20 implies Theorem 2.21). Let A be an ample divisor on Y. Assuming for simplicity that X is projective, then (possibly replacing A by a multiple), we may assume that $N + f^*A - K_X - B$ is nef and big. By Theorem 2.20, we have $H^i(N + f^*A) = 0$ for all $i > 0$. By Serre vanishing, we may assume that $H^j(R^k f_* N \otimes \mathcal{O}_Y(A)) = 0$ for $j > 0$ and $R^k f_* N \otimes \mathcal{O}_Y(A)$ is generated for $k \in \mathbb{N}$. By the Leray spectral sequence, it follows that $0 = H^i(N + f^*A) \cong H^0(R^i f_* N \otimes \mathcal{O}_Y(A))$ for $i > 0$. Since $R^i f_* N \otimes \mathcal{O}_Y(A)$ is generated, $R^i f_* N = 0$ for $i > 0$. $\square$

It is also easy to generalize Theorem 2.20 to the klt setting.

Theorem 2.22. *[Kawamata-Viehweg vanishing for klt pairs] Let (X, B) be a projective klt pair. If N is a Cartier divisor such that $N - K_X - B$ is nef and big, then $H^i(N) = 0$ for all $i > 0$.*

Proof. Consider $f : X' \to X$ a log resolution. If $K_{X'} + B' = f^*(K_X + B)$, then $(X', \{B'\})$ is klt and $\lfloor B' \rfloor = -G$ where G is effective and exceptional (as (X, B) is klt). But then $f^*N + G - K_{X'} - \{B'\} = f^*(N - K_X - B)$ is nef and big. Therefore, by Theorem 2.20, $H^i(f^*N + G) = 0$ for $i > 0$ and by Theorem 2.21 $R^i f_*(f^*N + G) = 0$ for $i > 0$. But then

$$0 = H^i(f^*N + G) = H^i(N \otimes f_*G) = H^i(N) \qquad \text{for } i > 0,$$

where we used the fact that $f_*(f^*N + G) = N \otimes f_*G$ by the projection formula, and $f_*\mathcal{O}_X(G) = \mathcal{O}_X$ as G is effective and exceptional. $\square$

Putting these statements together, one obtains the following result.

Theorem 2.23. *Let $f : Y \to X$ be a proper morphism of quasi-projective varieties with (Y, D) a klt pair, N a $\mathbb{Q}$-Cartier Weil divisor on Y such that $N \equiv K_Y + M + D$ where M is f-big and f-nef $\mathbb{R}$-Cartier, then $R^i f_*\mathcal{O}_Y(N) = 0$ for $i > 0$*

Theorem 2.24. *[Kollár-Shokurov Connectedness Lemma] Let $f : X \to Y$ be a proper morphism of normal varieties with connected fibers and B a $\mathbb{Q}$-divisor such that $-(K_X + B)$ is $\mathbb{Q}$-Cartier, f-nef and f-big. We write $B = B^+ - B^-$ where B^+, B^- are effective and B^- is f-exceptional. Then* $\operatorname{nklt}(X, B) \cap f^{-1}(y)$ *is connected for every $y \in Y$.*

Proof. Let $\nu : X' \to X$ be a log resolution and write $K_{X'} + B' = \nu^*(K_X + B)$, then $\operatorname{nklt}(X, B) = \nu(\operatorname{nklt}(X', B'))$ and so it suffices to show that $\operatorname{nklt}(X', B') \cap (f \circ \nu)^{-1}(y)$ is connected for every $y \in Y$. Since $-(K_{X'} + B')$ is $\mathbb{Q}$-Cartier, $(f \circ \nu)$-nef and $(f \circ \nu)$-big, we may replace (X, B) by (X', B') and so we may assume that (X, B) is log smooth.

Let $B^{<1} = \sum_{b_i < 1} b_i B_i$ and $S = \lfloor B \rfloor - \lfloor B^{<1} \rfloor$. Then $\operatorname{Supp}(S) = \operatorname{nklt}(X, B)$. We have a short exact sequence

$$0 \to \mathcal{O}_X(-\lfloor B \rfloor) \to \mathcal{O}_X(-\lfloor B^{<1} \rfloor) \to \mathcal{O}_S(-\lfloor B^{<1} \rfloor) \to 0.$$

Notice that

$$-\lfloor B \rfloor = K_X - \lfloor K_X + B \rfloor = K_X - (K_X + B) + \{K_X + B\}.$$

Since $-(K_X + B)$ is f-nef and $(X, \{K_X + B\} = \{B\})$ is klt, by Theorem 2.21, we have $f_*\mathcal{O}_X(-\lfloor B \rfloor) = 0$ and hence there is a surjection

$$f_*\mathcal{O}_X(-\lfloor B^{<1} \rfloor) \to f_*\mathcal{O}_S(-\lfloor B^{<1} \rfloor).$$

Note that $-\lfloor B^{<1} \rfloor$ is effective and f-exceptional so that $f_*\mathcal{O}_X(-\lfloor B^{<1} \rfloor) = \mathcal{O}_Y$. Thus, the composition $\mathcal{O}_Y \to f_*\mathcal{O}_S(-\lfloor B^{<1} \rfloor)$ is surjective and factors through $f_*\mathcal{O}_S \to \mathcal{O}_{f(S)}$. But then $f_*\mathcal{O}_S \to \mathcal{O}_{f(S)}$ is also surjective so that $S \to f(S)$ has connected fibers. $\square$

Remark 2.25. *If (X,B) is a pair such that $B \geq 0$ and $-(K_X+B)$ is nef and big, then the above result states that* $\operatorname{nklt}(X,B)$ *is connected.*

Remark 2.26. *If (X,B) is a pair such that $B \geq 0$ and $f : X' \to X$ is a proper birational morphism, then the above result states that the fibers of* $\operatorname{nklt}(X',B') \to X$ *are connected.*

Theorem 2.27. *Let (X,B) be an lc pair and (X,B_0) a klt pair. If W_1 and W_2 are non-klt centers of (X,B), then so is every irreducible component of $W_1 \cap W_2$. It follows that for ony $x \in X$ there is a minimal non-klt center W of (X,B) containing x.*

Proof. We may assume that X is affine and $W = W_1 \cap W_2$. For simplicity, we will also assume that B is $\mathbb{R}$-Cartier and $(X,0)$ is klt. Let B_i be a general divisor containing W_i and $\mu : X' \to X$ a log resolution of $(X, B+B_0+B_1+B_2)$. There are divisors $E_i \subset X'$ that correspond to non-klt centers of (X,B) with centers W_i. Therefore, $\operatorname{mult}_{E_i}\mathbf{B}(X,B) = 1$. Let $e_i = \operatorname{mult}_{E_i}(\mu^*B)$ and $e'_i = \operatorname{mult}_{E_i}(\mu^*B_i)$. Note that as $(X,0)$ is klt, $e_i > 0$ and by our assumptions $e'_i > 0$ (whereas $\operatorname{mult}_{E_{3-i}}\mu^*B_i = 0$). Let $a_i = e_i/e'_i$, then E_i are non-klt places for $(X,(1-\epsilon)B + \epsilon(a_1B_1 + a_2B_2))$ for $0 < \epsilon \ll 1$ and

$$\operatorname{nklt}(X,(1-\epsilon)B + \epsilon(a_1B_1 + a_2B_2)) = W_1 \cup W_2.$$

By the Connectedness Theorem 2.24, for any $w \in W$ there are non-klt places $F_i(\epsilon)$ corresponding to divisors on X' with centers contained in W_i such that $F_1(\epsilon) \cap F_2(\epsilon) \cap \mu^{-1}(w) \neq \emptyset$ for $0 < \epsilon \ll 1$. By finiteness of μ-exceptional divisors, we may assume that $F_i = F_i(\epsilon)$ does not depend on ϵ. By continuity, we may assume that the F_i are also non-klt centers of (X,B) . But then $w \in f(F_1 \cap F_2)$ is also a non-klt center of (X,B). If $w \in W$ is general, we may assume that $f(F_1 \cap F_2) = W$. □

Exercise 2.28. *Prove Theorem 2.27. (Hint, consider pairs of the form $(X,(1-\epsilon)B + \epsilon(B_0 + a_1B_1 + a_2B_2))$.)*

Theorem 2.29. *Let (X,B) be an lc pair and (X,B_0) a klt pair. If W is a minimal non-klt center of (X,B), then W is normal.*

Proof. We may assume that X is affine. Let $\mu : X' \to X$ be a log resolution of $(X, B+B_0)$ such that there is a divisor E on X' which is a non-klt place of (X,B) with center W. Let B' be a general divisor on X containing W and $e' = \operatorname{mult}_E(\mu^*D')$. If $e = \operatorname{mult}_E(\mu^*B)$ and $a = e/e'$, then E is a non-klt place of $(X,(1-\epsilon)B + \epsilon B')$. $(X, \{K_{X'} + D\})$ is klt and $-(K_{X'}+D) \sim_{\mathbb{Q},X} 0$ is μ-nef and μ-big, it follows by Theorem 2.21 that $R^1\mu_*\mathcal{O}_{X'}(-\lfloor D \rfloor) = 0$ and hence that $\mu_*\mathcal{O}_{X'}(-\lfloor D^{<1} \rfloor) \to \mu_*\mathcal{O}_E(-\lfloor D \rfloor^{<1})$ is surjective. Since $-\lfloor D^{<1} \rfloor$ is effective and exceptional,

$\mu_*\mathcal{O}_{X'}(-\lfloor D^{<1}\rfloor) = \mathcal{O}_{X'}$ and so $\mathcal{O}_{X'} \to \mu_*\mathcal{O}_W$ is surjective. Thus $E \to W$ has connected fibers and W is normal. $\square$

Exercise 2.30. *Fix $d \in \mathbb{N}$. Show that for any $m > 0$ there exists a projective variety X of dimension d and an ample line bundle A on X such that $H^i(X, A^{\otimes m}) \neq 0$. (Hint: Let $d = 1$.)*

Exercise 2.31. *Show that if $f : X \to Y$ is a morphism of projective varieties, N is a nef $\mathbb{R}$-Cartier divisor on Y, then f^*N is nef on X. Moreover if N is nef and big, then f^*N is nef and big iff f is generically finite (i.e. f is dominant and $\dim X = \dim Y$).*

2.3. **Multiplier ideal sheaves.** Let X be a smooth projective variety and $B \geq 0$ an $\mathbb{R}$-divisor on X, then we define the *multiplier ideal sheaf* $\mathcal{J}(X, B)$ as follows: let $f : X' \to X$ be a log resolution, then

$$\mathcal{J}(X, B) := f_*\mathcal{O}_{X'}(K_{X'/X} - \lfloor f^*B \rfloor)$$

where $K_{X'/X} = K_{X'} - f^*K_X$.

Remark 2.32. *We have the following facts:*

1. *Since $K_{X'/X}$ is effective and exceptional,*
$$\mathcal{J}(X, B) \subset f_*\mathcal{O}_{X'}(K_{X'/X}) = \mathcal{O}_X$$
is an ideal sheaf.
2. *If B is Cartier, then by the projection formula,*
$$\mathcal{J}(X, B) = f_*\mathcal{O}_{X'}(K_{X'/X} - f^*B)$$
$$= \mathcal{O}_X(-B) \otimes f_*\mathcal{O}_{X'}(K_{X'/X}) = \mathcal{O}_X(-B).$$
3. *More generally, $\mathcal{J}(X, B) = \mathcal{J}(X, \{B\}) \otimes \mathcal{O}_X(-\lfloor B \rfloor)$.*
4. *If (X, B) is snc, then $\mathcal{J}(X, B) = \mathcal{O}_X(-\lfloor B \rfloor)$ for any choice of log resolution $X' \to X$.*

   By the previous point it suffices to show that $\mathcal{J}(X, \{B\}) = \mathcal{O}_X$ i.e. we may assume that $\lfloor B \rfloor = 0$. we must then show that $K_{X'/X} - \lfloor f^*B \rfloor \geq 0$ or equivalently $\operatorname{mult}_E(K_{X'/X} - f^*B) > -1$. This can be checked by a computation in local coordinates say $x_1, \ldots, x_d$ on X and $y_1, \ldots, y_d$ near the generic point of a divisor E on X'. If $c_i = \operatorname{mult}_E(f^*B_i)$, then $x_i = y_1^{c_i} \cdot b_i$ where b_i is a regular function and, B_i is defined by $x_i = 0$ and E by $y_1 = 0$. Since $dx_i = y_1^{c_i - 1} c_i b_i dy_1 + y_1^{c_i} db_i$, we have
$$dx_1 \wedge \ldots \wedge dx_n = y_1^{\gamma - 1} g dy_1 \wedge \ldots \wedge dy_n$$
   where g is a regular function and $\gamma = \sum c_i$. But then
$$\operatorname{mult}_E(K_{X'/X}) \geq \sum c_i - 1 > \operatorname{mult}_E(f^*B) - 1.$$

(5) *The definition of $\mathcal{J}(X,B)$ is independent of the choice of the log resolution $X' \to X$.*

To see this, suppose that $\nu : X'' \to X'$ is a birational morphism. By the previous point,

$$\nu_* \mathcal{O}_{X''}(K_{X''/X'} - \lfloor \nu^* f^* B \rfloor) = \mathcal{J}(X', f^*B) = \mathcal{O}_{X'}(-\lfloor f^* B \rfloor).$$

But then by the projection formula

$$(f \circ \nu)_* \mathcal{O}_{X''}(K_{X''/X} - \lfloor \nu^* f^* B \rfloor) = f_*(K_{X'/X} \otimes \nu_* \mathcal{O}_{X''}(K_{X''/X'} - \lfloor \nu^* f^* B \rfloor)) =$$

$$f_*(K_{X'/X} \otimes \mathcal{O}_{X'}(-\lfloor f^* B \rfloor)).$$

(6) *If $\operatorname{mult}_x(B) \geq \dim X$, then $\mathcal{J}(X,B) \subset \mathfrak{m}_x$. To see this, let $f : X' \to X$ be a log resolution and let E be the divisor on X' corresponding to the blow up of $x \in X$. Then $\operatorname{mult}_E(K_{X'/X} - f^*B) \leq -1$ and so $K_{X'/X} - \lfloor f^* B \rfloor \leq -E$. But then $\mathcal{J}(X,B) = f_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor f^* B \rfloor) \subset f_* \mathcal{O}_{X'}(-E) = \mathfrak{m}_x$.*

(7) *If $\operatorname{mult}_x(B) < 1$, then $\mathcal{J}(X,B) = \mathcal{O}_X$ (near $x \in X$). See Corollary 2.38*

Theorem 2.33 (Nadel vanishing)**.** *Let X be a smooth projective variety and $B \geq 0$ an effective $\mathbb{R}$-divisor and $f : X \to Z$ a projective morphism. If N is a Cartier divisor on X such that $N - B$ is f-nef and f-big, then*

$$R^i f_* \left(\mathcal{O}_X(K_X + N) \otimes \mathcal{J}(X,B) \right) = 0, \qquad \text{for } i > 0.$$

Proof. Let $\nu : X' \to X$ be a log resolution of (X,B), then $\nu^*(N-B)$ is f'-nef and f'-big where $f' = f \circ \nu$. By Theorem 2.21, we have $R^i \nu_* \mathcal{O}_{X'}(K_{X'} + \lceil \nu^*(N-B) \rceil) = 0$ for $i > 0$ and $R^i f'_* \mathcal{O}_{X'}(K_{X'} + \lceil \nu^*(N-B) \rceil) = 0$ for $i > 0$. Note that

$$R^i f'_* \mathcal{O}_{X'}(K_{X'} + \lceil \nu^*(N-B) \rceil) = R^i f_* (\mathcal{O}_X(K_X + N) \otimes \mathcal{J}(X,B))$$

vanishes for $i > 0$. □

Corollary 2.34. *Let X be a smooth projective variety, $B \geq 0$ an effective $\mathbb{R}$-divisor N a Cartier divisor such that $N - B$ is nef and big and H a very ample divisor, then $\mathcal{O}_X(K_X + N + nH) \otimes \mathcal{J}(D)$ is generated for any $n \geq \dim X$.*

Proof. This follows immediately from Lemma 2.35 below. □

Lemma 2.35. *Let $\mathcal{F}$ be a coherent sheaf on a smooth projective variety and H a very ample line bundle such that $H^i(\mathcal{F} \otimes \mathcal{O}_X(jH)) = 0$ for $i > 0$ and $0 \leq j \leq d = \dim X$, then $\mathcal{F}$ is globally generated.*

Proof. We must show that $\mathcal{F}$ is generated at $x \in X$. Let $\mathcal{F}' \subset \mathcal{F}$ be the biggest subsheaf with 0 dimensional support. There is a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

where $\mathcal{F}''$ has no sections supported at points. Since $H^i(\mathcal{F}' \otimes \mathcal{O}_X(jH)) = 0$ for $i > 0$, it follows that $H^i(\mathcal{F}'' \otimes \mathcal{O}_X(jH)) = 0$ for $i > 0$ and $0 \leq j \leq d = \dim X$. It is easy to see that $\mathcal{F}$ is generated iff so is $\mathcal{F}''$. We may therefore replace $\mathcal{F}$ by $\mathcal{F}''$ and hence assume that $\mathcal{F}$ has no sections supported at points. Pick $Y \in |H|$ a general element containing $x \in X$ and consider the induced short exact sequence

$$0 \to \mathcal{F} \otimes \mathcal{O}_X(-Y) \to \mathcal{F} \to \mathcal{F}|_Y \to 0.$$

It is easy to see that $H^i(\mathcal{F}|_Y \otimes \mathcal{O}_X(jH|_Y)) = 0$ for $i > 0$ and $1 \leq j \leq d = \dim Y$. By induction on the dimension, $\mathcal{F}|_Y \otimes \mathcal{O}_X(jH|_Y)$ is generated for any $j \geq \dim X$. Since $H^0(\mathcal{F}|_Y \otimes \mathcal{O}_X(jH|_Y)) \to H^0(\mathcal{F}|_Y \otimes \mathcal{O}_X(jH|_Y))$ is surjective, it follows that $\mathcal{F}$ is generated at x as required. □

We next investigate how multiplier ideals behave under restriction to a divisor. Suppose that $H \subset X$ is a smooth divisor on a smooth variety, then it may happen that for an effective $\mathbb{Q}$-divisor D we have $\mathcal{J}(H, D|_H) \subset \mathcal{J}(X, D) \cdot \mathcal{O}_H$. consider in fact $H = \{x = 0\} \subset X = \mathbb{C}^2$. If $D = \frac{1}{2}\{x - y^2 = 0\}$, then $\mathcal{J}(X, D) = \mathcal{O}_X$, however $\mathcal{J}(H, D|_H) = \mathfrak{m}_O$ where $O \in H$ is the origin. We think of this as saying that $(H, D|_H)$ is more singular than (X, D). The next result shows that $(H, D|_H)$ is at least as singular as (X, D).

Theorem 2.36. *Let $H \subset X$ be a smooth divisor on a smooth variety and $D \geq 0$ a $\mathbb{Q}$-divisor whose support does not contain H. Then*

$$\mathcal{J}(H, D|_H) \subset \mathcal{J}(X, D) \cdot \mathcal{O}_H$$

where $\mathcal{J}(X, D) \cdot \mathcal{O}_H := \operatorname{Im}(\mathcal{J}(X, D) \hookrightarrow \mathcal{O}_X \to \mathcal{O}_H) \subset \mathcal{O}_H$.

Moreover, if $0 < s < 1$, then for all $0 < t \ll 1$, we have

$$\mathcal{J}(X, D + (1-t)H) \cdot \mathcal{O}_H \subset \mathcal{J}(H, (1-s)D|_H).$$

Proof. Let $f : X' \to X$ be a log resolution of $(X, D + H)$ and write $f^*H = H' + \sum a_j E_j$ where $H' = f^{-1}_* H$, $a_j \geq 0$ and E_j are f-exceptional. By adjunction $K_{H'} = (K_{X'} + H')|_{H'}$ and $K_H = (K_X + H)|_H$. Thus $K_{H'/H} = (K_{X'/X} - \sum a_j E_j)|_{H'}$. Consider the short exact sequence

$$0 \to \mathcal{O}_{X'}(K_{X'/X} - \lfloor f^*D \rfloor - f^*H) \to \mathcal{O}_{X'}(K_{X'/X} - \lfloor f^*D \rfloor - \sum a_j E_j)$$
$$\to \mathcal{O}_{H'}(K_{H'/H} - \lfloor f^*D|_H \rfloor) \to 0.$$

Since $-\lfloor f^*D \rfloor - f^*H \sim_{\mathbb{Q},f} \{f^*D\}$ and $(X', \{f^*D\})$, it follows by Theorem 2.21 that $R^1 f_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor f^*D \rfloor - f^*H) = 0$ and so

$$f_*\mathcal{O}_{X'}(K_{X'/X} - \lfloor f^*D \rfloor - \sum a_j E_j) \to f_*\mathcal{O}_{H'}(K_{H'/H} + \lfloor f^*D|_H \rfloor) = \mathcal{J}(H, D|_H)$$

is surjective. The first assertion now follows since

$$\mathcal{J}(X, D) = f_*\mathcal{O}_{X'}(K_{X'/X} - \lfloor f^*D \rfloor) \supset f_*\mathcal{O}_{X'}(K_{X'/X} - \lfloor f^*D \rfloor - \sum a_j E_j).$$

Since

$$\mathcal{J}(X, D + (1-t)H) = f_*\mathcal{O}_{X'}(K_{X'/X} - \lfloor f^*(D + (1-t)H) \rfloor)$$

and

$$\mathcal{J}(H, (1-s)D|_H) = f_*\mathcal{O}_{H'}(K_{H'/H} - \lfloor (1-s) f^*D|_H \rfloor),$$

we must show that for any divisor E on X' such that $E \cap H' \neq \emptyset$, we have that for any irreducible component V of $E \cap H'$, if $\mathrm{mult}_V(K_{H'/H} - f^*((1-s)D|_H)) \leq -1$, then

$$\mathrm{mult}_E(K_{X'/X} - \lfloor f^*(D + (1-t)H) \rfloor) \leq \mathrm{mult}_V(K_{H'/H} - \lfloor (1-s) f^*D|_H \rfloor).$$

Let $k = \mathrm{mult}_E(K_{X'/X})$, $a = \mathrm{mult}_E(f^*H)$ and $d = \mathrm{mult}_E(f^*D)$, then we must show that

$$k - \lfloor (1-t)a + d \rfloor \leq k - a - \lfloor (1-s)d) \rfloor.$$

But for $0 < t \ll 1$ this equation is easily seen to hold. □

We have the following important consequence relating singularities on X and H.

Corollary 2.37. *Let X, H, D be as above. Then*

1. *(Inversion of adjunction) If $\mathcal{J}(H, D|_H) = \mathcal{O}_H$ near a point $x \in H$, then $\mathcal{J}(X, D) = \mathcal{O}_X$ near $x \in X$. In other words if $(H, D|_H)$ is klt near $x \in H$ then (X, D) is klt near $x \in X$.*
2. *(Inversion of adjunction II) If $\mathcal{J}(H, (1-s)D|_H) \subset \mathfrak{m}_x$ for some point $x \in H$ and $0 < s < 1$, then $\mathcal{J}(X, D + (1-t)H) \subset \mathfrak{m}_x$ for any $0 < t \ll 1$. In other words if $(H, (1-s)D|_H)$ is not klt near $x \in H$, then $(X, D + (1-t)H)$ is not klt near $x \in X$.*

Proof. Exercise. □

Corollary 2.38. *If X is smooth and D is an effective $\mathbb{Q}$-divisor such that* $\mathrm{mult}_x(D) < 1$, *then $\mathcal{J}(X, D) = \mathcal{O}_X$ near $x \in X$.*

Proof. By induction on $d = \dim X$. The case $d = 1$ is clear since then $\mathcal{J}(X, D) = \mathcal{O}_X(-\lfloor D \rfloor) = \mathcal{O}_X$. If $d > 1$, then pick $x \in H \subset X$ a general very ample divisor so that H is smooth and not contained in the support of D. We have $\mathrm{mult}_x(D|_H) = \mathrm{mult}_x(D) < 1$ and so by induction $\mathcal{J}(H, D|_H) = \mathcal{O}_H$ near x. By Corollary 2.38, we have $\mathcal{J}(X, D) = \mathcal{O}_X$ near $x \in X$. □

Remark 2.39. *A more general version of inversion of adjunction is the following. Let $(X, S + B)$ be a pair such that S is a prime divisor not contained in the support of B, let $\nu : S' \to S$ be the normalization of S and $(S', B_{S'})$ be the log pair defined by the adjunction formula $\nu^*(K_X + S + B) = K_{S'} + B_{S'}$. Then*

(1) *$(X, S + B)$ is purely log terminal near S if and only if $(S', B_{S'})$ is kawamata log terminal.*
(2) *$(X, S + B)$ is log canonical near S if and only if $(S', B_{S'})$ is log canonical.*

Proposition 2.40. *If X is smooth, $Z \subset X$ an irreducible d-dimensional subvariety and D is an effective $\mathbb{Q}$-divisor such that*

(1) *Z is a non-klt center of (X, D),*
(2) *(X, D) is lc along the general point of Z,*
(3) $\operatorname{Supp}(B)$ *does not contain Z, and*
(4) $\operatorname{mult}_z(B|_Z) > d$ *at a smooth point $z \in Z$.*

Then, for any $0 < \epsilon \ll 1$ we have

$$\mathcal{J}(X, (1 - \epsilon)D + B) \subset \mathfrak{m}_z.$$

Proof. Consider $f : X' \to X$ a log resolution of (X, D), then there is a divisor E on X' with center Z such that $a_E(X, D) = -1$. If $k = \operatorname{mult}_E(K_{X'/X})$, then $\operatorname{mult}_E(f^*D) = k + 1$. Since $z \in Z$ is general, we may assume that $f|_E$ is smooth over z. We have

$$\mathcal{J}(X, (1 - \epsilon)D + B) = f_*\mathcal{J}(X', f^*((1 - \epsilon)D + B) - K_{X'/X}).$$

Since $\operatorname{mult}_E(B|_Z) > d$, it follows that

$$\operatorname{mult}_{E_z}(f^*((1 - \epsilon)D + B) - K_{X'/X}) \geq d + 1 = \operatorname{codim}_Y E_z,$$

where E_z is the fiber of $E \to Z$ over $z \in Z$. Thus $\mathcal{J}(X', f^*((1 - \epsilon)D + B) - K_{X'/X}) \subset \mathcal{I}_{E_z}$ and the proposition follows. □

Exercise 2.41. *Show that (X, B) is klt iff $\mathcal{J}(X, B) = \mathcal{O}_X$.*

Exercise 2.42. *Show that if $(X, S + B)$ is purely log terminal (resp. log canonical) near S, then $(S', B_{S'})$ is kawamata log terminal (resp. log canonical).*

Exercise 2.43. *Use the Connectedness lemma to show that if $(S', B_{S'})$ is kawamata log terminal, then $(X, S + B)$ is purely log terminal near S.*

2.4. **Adjoint ideals.**

Definition 2.44. *Let (X, S) be a log smooth pair where S is a reduced irreducible divisor and D be an effective $\mathbb{R}$-divisor whose support does not contain S. Let $f : Y \to X$ be a log resolution of $(X, S+D)$. We define the adjoint ideal*

$$\mathcal{J}_{S,D} = f_*\mathcal{O}_Y(K_{Y/X} + f_*^{-1}S - \lfloor f^*(D+S) \rfloor).$$

Exercise 2.45. *Show that $\mathcal{J}_{S,D}$ is independent of the chosen log resolution.*

Exercise 2.46. *Show that $\mathcal{J}(X, S+D) \subset \mathcal{J}_{S,D} \subset \mathcal{J}(X, D)$.*

Proposition 2.47. *Let (X, S) be a log smooth pair where S is a reduced irreducible divisor and D be an effective $\mathbb{R}$-divisor whose support does not contain S. There is a short exact sequence*

$$0 \to \mathcal{J}(X, S+D) \to \mathcal{J}_{S,D} \to \mathcal{J}(S, D|_S) \to 0.$$

Proof. Let $f : Y \to X$ be a log resolution of $(X, S+D)$ and consider the short exact sequence

$$0 \to \mathcal{O}_Y(K_{Y/X} - \lfloor f^*(D+S) \rfloor) \to \mathcal{O}_Y(K_{Y/X} + S' - \lfloor f^*(D+S) \rfloor) \to \mathcal{O}_{S'}(K_{S'/S} - \lfloor f|_{S'}{}^*(D|_S) \rfloor) \to 0,$$

where $S' = f_*^{-1}S$, $K_S = (K_X + S)|_S$, and $K_{S'} = (K_Y + S')|_{S'}$. By Theorem 2.21, we have $R^1 f_*\mathcal{O}_Y(K_{Y/X} - \lfloor f^*(D+S) \rfloor) = 0$ and hence the proposition follows simply by pushing forward via f the above short exact sequence on Y. $\square$

Corollary 2.48. *Let (X, S) be a log smooth pair where S is a reduced irreducible divisor and D be an effective $\mathbb{R}$-divisor whose support does not contain S. If L is a Cartier divisor such that $L - (K_X + D + S)$ is nef and big, then*

$$H^0(L|_S \otimes \mathcal{J}(S, D|_S)) \subset \operatorname{Im}\left(H^0(L) \to H^0(L|_S)\right).$$

Proof. By Theorem 2.20, $H^1(L \otimes \mathcal{J}(X, S+D)) = 0$ and so by Proposition 2.47, $H^0(L \otimes \mathcal{J}_{S,D}) \to H^0(L|_S \otimes \mathcal{J}(S, D|_S))$ is surjective and the corollary follows. $\square$

Lemma 2.49. *Let X be a smooth variety, D, D' an effective $\mathbb{R}$-divisor and Σ be an effective Cartier divisor such that $D \leq \Sigma + D'$ and $\mathcal{J}(X, D') = \mathcal{O}_X$, then $\mathcal{O}_X(-\Sigma) \subset \mathcal{J}(D)$.*

Proof. Let $f : Y \to X$ be a log resolution of $(X, D + D' + \Sigma)$. Since $-f^*D \geq -f^*(\Sigma + D')$ and $f_*\mathcal{O}_Y(K_{Y/X} - \lfloor f^*D' \rfloor) = \mathcal{J}(X, D') = \mathcal{O}_X$, by the projection formula we have

$$\mathcal{J}(X, D) = f_*\mathcal{O}_Y(K_{Y/X} - \lfloor f^*D \rfloor) \supset f_*\mathcal{O}_Y(K_{Y/X} - \lfloor f^*(\Sigma + D') \rfloor) =$$

$$f_*\mathcal{O}_Y(K_{Y/X} - \lfloor f^*D' \rfloor) \otimes \mathcal{O}_X(-\Sigma) = \mathcal{O}_X(-\Sigma).$$

□

2.5. Deformation invariance of plurigenera.

Theorem 2.50. *Let $f : X \to T$ be a smooth morphism of smooth quasi-projective varieties such that $K_{X/T}$ is pseudo-effective and H an ample divisor. Then $H^0(X_t, \mathcal{O}_{X_t}(mK_{X_t}+H_t))$ is deformation invariant for all $t \in T$ and $m \geq 0$.*

If moreover K_{X_t} is big for some $t \in T$, then $H^0(X_t, \mathcal{O}_{X_t}(mK_{X_t}))$ is deformation invariant for all $t \in T$ and $m \geq 1$.

Proof. The question is local over T so we may assume that T is affine. Cutting by general hyperplanes in T we may assume that $\dim T = 1$. Pick $t \in T$, we must show that $f_*\mathcal{O}_X(mK_X + H) \to H^0(\mathcal{O}_{X_t}(mK_{X_t} + H_t))$ is surjective. Here $X_t = f^{-1}(t)$ is the fiber over t and $H_t = X|_{X_t}$. Pick $\Sigma \in |mK_{X_t} + H_t|$ and A such that $|rK_{X_t} + A_t|$ is free and

$$|rK_{X_t} + H_t + A_t| = |rK_X + H + A|_t \qquad \forall\ 0 \leq r \leq m.$$

Here $A_t = A|_{X_t}$ and $|rK_X+H+A|_t$ denotes the image of the restriction map $|rK_X + A| \to |rK_{X_t} + A_t|$.

We will first show that

$$k\Sigma + |rK_{X_t} + A_t| \subset |k(mK_X + H) + rK_X + A|_t \qquad \forall k \geq 0,\ 1 \leq r \leq m.$$

We proceed by induction on $l = mk + r$. By assumption, the cases $0 \leq l \leq m$ hold. Assume that the above inclusion holds for all integers $< l = mk+r$ where $1 \leq r \leq m$, in particular for a very general element $U \in |(r-1)K_{X_t}+A_t|$ there exists a divisor $S \in |(mk+r-1)K_X+kH+A|$ such that $S|_{X_t} = k\Sigma+U$. Since $K_{X/T}$ is pseudo-effective over T, for any $\delta > 0$ there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} K_{X/T} + \delta A$. Consider now the divisor $G = (1-\epsilon)S + \epsilon(mk + r - 1)D$, then

$$(mk + r - 1)K_X + kH + A - G \sim_{\mathbb{Q}} \epsilon(kH - (1 - (mk + r - 1)\delta)A)$$

is ample (for $0 < \delta \ll 1$) and so by Corollary 2.48 $H^0(\mathcal{O}_{X_t}(((mk + r)K_{X_t}+kH_t+A_t)\otimes\mathcal{J}(G|_{X_t})))$ is contained in the image of the restriction map

$$H^0(\mathcal{O}_X((mk+r)K_X+kH+A)) \to H^0(\mathcal{O}_{X_t}((mk+r)K_{X_t}+kH_t+A_t)).$$

Since $\mathcal{J}(X_t, \epsilon(mk + r - 1)D_t) = \mathcal{O}_{X_t}$ for $0 < \epsilon \ll 1$ and since U is a general smooth divisor intersecting D_t transversely, we have

$$\mathcal{J}(X_t, (1 - \epsilon)U + \epsilon(mk + r - 1)D_t) = \mathcal{O}_{X_t}.$$

Since $G_t \leq k\Sigma + (1 - \epsilon)U + \epsilon(mk + r - 1)D_t$, then by Lemma 2.49,

$$\mathcal{J}(G|_{X_t}) = \mathcal{J}((1 - \epsilon)(k\Sigma + U) + \epsilon(mk + r - 1)D_{X_t}) \supset \mathcal{O}_{X_t}(-k\Sigma)$$

and so for any $U' \in |rK_{X_t}+A|$, we have that $k\Sigma+U' \in |(mk+r)K_X+kH+A|_t$ as required. The induction is complete.

Consider now an element $D \in |k(mK_X+H)+A|$ such that $D_t = k\Sigma+U$ where $U \in |A_t|$ is general. Since

$$(m-1)K_X + H - \frac{m-1}{mk}D \sim_{\mathbb{Q}} \frac{1}{m}(H - \frac{m-1}{k}A)$$

is ample for $k \gg 0$, by Corollary 2.48 we have that $H^0(\mathcal{O}_{X_t}(mK_{X_t}+H_t) \otimes \mathcal{J}(\frac{m-1}{mk}D|_{X_t}))$ is contained in the image of the homomorphism

$$H^0(\mathcal{O}_X(mK_X+H)) \to H^0(\mathcal{O}_{X_t}(mK_{X_t}+H_t)).$$

Since $\frac{m-1}{mk}D|_{X_t} \leq \Sigma + \frac{m-1}{mk}U$ and $\mathcal{J}(X_t, \frac{m-1}{mk}U) = \mathcal{O}_{X_t}$ for $k \gg 0$, by Lemma 2.49 we have $\mathcal{J}(\frac{m-1}{mk}D_t) \subset \mathcal{O}_{X_t}(-\Sigma)$, it follows that $\Sigma \in |mK_X+H|_t$.

Suppose now that K_{X_t} is big for some $t \in T$. Since $H^0(X_t, \mathcal{O}_{X_t}(mK_{X_t}+H_t))$ is deformation invariant for all $t \in T$ and $m \geq 0$, it follows easily that K_{X_t} is big for all $t \in T$ and K_X is big over T. Let $\Sigma \in |mK_{X_t}|$ and consider an element $D \in |kmK_X+A|$ such that $D_t = k\Sigma+U$ where $U \in |A_t|$ is general. Since K_X is big over T, we may write $(m-1)K_X \sim_{\mathbb{Q}} A'+E$ where A' is ample and $E \geq 0$. Let $G = \frac{(1-\epsilon)(m-1)}{mk}D + \epsilon E$, then

$$(m-1)K_X - G \sim_{\mathbb{Q}} \epsilon A' - \frac{(1-\epsilon)(m-1)}{mk}A$$

is ample for $k \gg 0$. By corollary 2.48, $H^0(\mathcal{O}_{X_t}(mK_{X_t}) \otimes \mathcal{J}(G|_{X_t}))$ is contained in the image of $H^0(mK_X) \to H^0(mK_{X_t})$. Since $G|_{X_t} \leq \Sigma + \frac{(1-\epsilon)(m-1)}{mk}U + \epsilon E_t$ where $(X, \frac{(1-\epsilon)(m-1)}{mk}U + \epsilon E_t)$ is klt, it follows by Lemma 2.49 that $\Sigma \in |mK_X|_t$ as required. □

Remark 2.51. *Y.T. Siu has shown that if $f : X \to T$ is a smooth morphism of smooth quasi-projective varieties, then $h^0(mK_{X_t})$ is independent of $t \in T$. The proof is analytic and there is no known algebraic proof of this fact.*

2.6. **Fujita's Conjecture and the Theorem of Anhern and Siu.**

Conjecture 2.52. *[Fujita's Conjecture] Let X be a smooth projective variety and A an ample divisor on X, then K_X+tA is generated for any integer $t > \dim X$.*

The above result is known in dimension ≤ 5 by results of Kawamata [Kawamata97] and Ye-Zhu [YZ15]. In what follows we will prove several closely related statements.

Theorem 2.53. *Fujita's Conjecture holds if A is ample and globally generated.*

Proof. For simplicity let $t = n+1$ where $n = \dim X$. Since A is generated, it defines a finite (onto its image) morphism $f : X \to \mathbb{P}^n$ such that $f^*\mathcal{O}_{\mathbb{P}^n}(1) \cong A$. Fix $x \in X$ we wish to show that $K_X + (n+1)A$ is generated at x. Pick general hyperplanes $A_1, A_2, A_3, \ldots$ passing through $f(x)$. For $k \gg 0$ consider $D_k = \frac{n}{k}(A_1 + \ldots + A_k)$. It is easy to see that $\mathrm{mult}_x(D_k) \geq \dim X$ and $\mathrm{mult}_y(D_k) < 1$ for $y \neq x$ in a neighborhood of $x \in X$. But then by Remark 2.32, we have that the support of $\mathcal{O}_X/\mathcal{J}(D_k)$ has a component supported at x so that $\mathfrak{m}_x^l \subset \mathcal{O}_X/\mathcal{J}(D_k) \cong \mathcal{I}_Z \subset \mathfrak{m}_x$ near $x \in X$ for some $l > 0$. Since $(n+1)A - D_k$ is ample, by Theorem 2.33 we have $H^1(\mathcal{O}_X(K_X + (n+1)A) \otimes \mathcal{J}(D_k)) = 0$. From the short exact sequence

$$0 \to \mathcal{O}_X(K_X + (n+1)A) \otimes \mathcal{J}(D_k) \to \mathcal{O}_X(K_X + (n+1)A) \\ \to \mathcal{O}_X(K_X + (n+1)A) \otimes \mathcal{O}_X/\mathcal{J}(D_k) \to 0$$

we see that

$$H^0(\mathcal{O}_X(K_X + (n+1)A)) \to H^0(\mathcal{O}_X(K_X + (n+1)A) \otimes \mathcal{O}_X/\mathcal{J}(D_k))$$

is surjective. Since $H^0(\mathcal{O}_X(K_X + (n+1)A) \otimes \mathcal{O}_X/\mathcal{I}_Z)$ is a summand of $H^0(\mathcal{O}_X(K_X + (n+1)A) \otimes \mathcal{O}_X/\mathcal{J}(D_k))$ and since the short exact sequence

$$0 \to \mathcal{O}_X(K_X + (n+1)A) \otimes \mathfrak{m}_x/\mathcal{I}_Z \to \mathcal{O}_X(K_X + (n+1)A) \otimes \mathcal{O}_X/\mathcal{I}_Z \\ \to \mathcal{O}_X(K_X + (n+1)A) \otimes \mathcal{O}_X/\mathfrak{m}_x \to 0$$

induces a short exact sequence of global sections, it follows that

$$H^0(\mathcal{O}_X(K_X + (n+1)A)) \to H^0(\mathcal{O}_X(K_X + (n+1)A) \otimes \mathcal{O}_X/\mathfrak{m}_x)$$

is surjective and hence $K_X + (n+1)A$ is generated at $x \in X$. $\square$

Exercise 2.54. *Show that* $\mathrm{mult}_y(D_k) < 1$ *for* $y \neq x$ *in a neighborhood of* $x \in X$.

Theorem 2.55. *Fujita's Conjecture holds in dimension* 2.

Proof. Since A is ample, by Serre vanisghing and Riemann Roch, for $k \gg 0$

$$h^0(kA) = \chi(kA) = \frac{kA(kA - K_X)}{2} + \chi(\mathcal{O}_X) = k^2\frac{A^2}{2} + O(k).$$

Vanishing to order m at a smooth point $x \in X$ of a surface is imposed by $\leq \binom{m+1}{2}$ conditions. Thus, for any $\epsilon > 0$ and for $k \gg 0$, there is divisor $D_k \in |kA|$ such that $\mathrm{mult}_x(D_k) > k(1-\epsilon)$. Let

$$\lambda = \mathrm{lct}_x(X, \frac{D_k}{k}) = \sup\{t > 0 | (X, t\frac{D_k}{k}) \text{ is lc at } x \in X\}.$$

Notice that by Remark 2.32, we have $\lambda < 2/(1-\epsilon)$. By Nadel vanishing, since $3 > \lambda$, it follows that if $D = \frac{\lambda}{k} D_k$, then

$$H^0(\mathcal{O}_X(K_X + 3A)) \to H^0(\mathcal{O}_X(K_X + 3A) \otimes \mathcal{O}_X/\mathcal{J}(X, D))$$

is surjective and hence that $K_X + 3A$ is generated at $x \in X$ (since $\mathcal{O}_X(K_X+3A)\otimes\mathcal{O}_X/\mathfrak{m}_x$ is a summand of $\mathcal{O}_X(K_X+3A)\otimes\mathcal{O}_X/\mathcal{J}(X, D)$).

Suppose now that x is not a minimal non-klt center of (X, D), then the minimal non-klt center is a curve $x \in C \subset X$ which is normal and hence smooth at $x \in X$. This means that $D = C + D'$ where C is not contained in the support of D' and $\mathrm{mult}_x(D') < 1$. Let $f : X' \to X$ be a log resolution and $C' = f^{-1}_* C$. Consider the short exact sequence

$$0 \to \mathcal{O}_{X'}(K_{X'} + 3f^*A - \lfloor f^*D \rfloor) \to \mathcal{O}_{X'}(K_{X'} + C' + 3f^*A - \lfloor f^*D \rfloor)$$
$$\to \mathcal{O}_{C'}(K_{C'} + 3f^*A - \lfloor f^*D \rfloor) \to 0.$$

By Kawamata-Viehweg vanishing we obtain a short exact sequence

$$0 \to H^0(\mathcal{O}_X(K_X+3A)\otimes\mathcal{J}(X, D)) \to H^0(f_*\mathcal{O}_{X'}(K_{X'}+C'+3f^*A-\lfloor f^*D \rfloor))$$
$$\to H^0(f_*\mathcal{O}_{C'}(K_{C'} + 3f^*A - \lfloor f^*D \rfloor)) \to 0.$$

We claim that $\deg(L_{C'}) \geq 2$ where $L_{C'} = 3f^*A - \lfloor f^*D \rfloor$ and hence $K_{C'} + L_{C'}$ is generated. Grant this for the time being, then since $C' \to C$ is an isomorphism near $x \in C$, we have that $f_*\mathcal{O}_{C'}(K_{C'} + 3f^*A - \lfloor f^*D \rfloor)$ is locally free and generated near $x \in C$ and so the same holds for $f_*\mathcal{O}_{X'}(K_{X'} + C' + 3f^*A - \lfloor f^*D \rfloor)$. But then $K_X + 3A$ is generated near $x \in X$.

We have $3f^*A - \lfloor f^*D \rfloor \sim_{\mathbb{Q}} (3 - \lambda)f^*A + \{\lambda f^*D\}$ and hence $L_{C'} = (3f^*A-\lfloor f^*D \rfloor)|_{C'}$ is a Cartier divisor of positive degree and it suffices to show that $\deg((3-\lambda)f^*A+\{f^*D\})|_{C'} > 1$. Since $\mathrm{mult}_x(D_k/k) > (1-\epsilon)$, it follows that $\mathrm{mult}_x(\{\lambda f^*D_k\}) \geq \lambda(1 - \epsilon) - 1$ (since $\lfloor f^*D \rfloor = C'$ near $x \in C'$). But then $\deg(L_{C'}) \geq \lambda(1-\epsilon)-1+3-\lambda > 1$ for $0 < \epsilon \ll 1$. $\square$

The next result implies that Fujita's conjecture holds for $t > \binom{n+1}{2}$.

Theorem 2.56 (Anhern-Siu)**.** *Let X be a smooth projective n-dimensional variety and A be an ample line bundle such that*

$$A^{\dim Z} \cdot Z > \binom{n+1}{2}^{\dim Z}$$

for any subvariety $x \in Z \subset X$, then $K_X + A$ is generated at $x \in X$.

Proof. As we have seen above, it suffices to show that there exists a $\mathbb{Q}$-divisor $D > 0$ such that

(1) x is an isolated point of the support of $\mathcal{O}_X/\mathcal{J}(D)$ and
(2) $D \sim_{\mathbb{Q}} cA$ for some $c < 1$.

The first step is to produce a divisor D with $\mathcal{J}(D) \subset \mathfrak{m}_x$. This can be achieved by the following easy argument.

Lemma 2.57. *Let $x \in V$ be a smooth point on an irreducible $d-$dimensional projective variety and A an ample divisor such that $A^d > a^d$ for some number $a > 0$. Then for any $k \gg 0$ there exists a divisor $A_k \in |kA|$ such that $\operatorname{mult}_x(A_k) > ka$.*

Proof of Lemma 2.57. By Riemann-Roch, we have

$$h^0(\mathcal{O}_V(kA)) = \frac{k^d A^d}{d!} + O(k^{d-1}).$$

Vanishing to order m at the smooth point $x \in V$ imposes at most

$$\binom{d+m-1}{d} = \frac{m^d}{d!} + O(m^{d-1})$$

independent conditions. The lemma follows easily. $\square$

Since by assumption $A^n > \binom{n+1}{2}$, it follows that there exists a divisor $D_1 \sim_{\mathbb{Q}} c_1 A$ with $\operatorname{mult}_x(D_1) \geq n$ and $c_1 < n/\binom{n+1}{2}$. Replacing D_1 by λD_1 where

$$\lambda = \operatorname{lct}_x(X, 0; D_1) = \sup\{t > 0 | (X, tD_1) \text{ is lc at } x \in X\} \leq 1,$$

we may assume that (X, D_1) is lc but not klt at $x \in X$. Perturbing D_1, we may assume that in fact (X, D_1) has a unique non-klt center through $x \in X$. Thus $\mathcal{J}(X, D_1) = \mathcal{I}_{Z_1} \subset \mathfrak{m}_x$ locally near $x \in X$. We will now show by induction that for any $k > 0$ there exists an effective $\mathbb{Q}$-divisor $D_k \sim c_k A$ such that

(1) (X, D_k) is lc but not klt at $x \in X$,
(2) (X, D_k) has a unique non-klt center $x \in Z_k \subset X$ of dimension $\dim Z_k \leq n-k$, and
(3) $c_k < \left(\sum_{i=1}^k (n-i+1)\right)/\binom{n+1}{n}$.

Note that the case $k = 1$ was established above. Assume now that the claim holds for all integers $\leq k$. Consider the normalization of a general curve in Z_k passing through $x \in Z_k$ say $g : T \to X$ with $g(t_0) = x$. For general $t \in T$, $g(t)$ is a smooth point on Z_k and hence there is a divisor $G_t \sim g_t A|_{Z_k}$ such that $\operatorname{mult}_{g(t)}(G_t) > \dim Z_k$ and $g_t < k/\binom{n+1}{n}$. By Serre vanishing

$$H^0(\mathcal{O}_X(tA)) \to H^0(\mathcal{O}_{Z_k}(tA))$$

is surjective for $t \gg 0$ and so there is a $\mathbb{Q}$-divisor $\tilde{G}_t$ on X such that $\tilde{G}_t \sim g_t A$ and $\tilde{G}_t|_{Z_k} = G_t$. By Proposition 2.40, it follows that $\mathcal{J}((1-\delta)D_k + G_t) \subset \mathfrak{m}_{g(t)}$ and the cosupport of $\mathcal{J}((1-\delta)D_k + G_t)$ at $g(t)$ is strictly contained in Z_k. It then follows that

Claim 2.58. *There exists a divisor $G_{t_0} \sim gA|_{Z_k}$ such that $\mathcal{J}((1-\delta)D_k + G_{t_0}) \subset \mathfrak{m}_x$ and the cosupport of $\mathcal{J}((1-\delta)D_k + G_{t_0})$ at x is strictly contained in Z_k.*

The proof of this claim is a lengthy exercise. Granting the claim then, after multiplying $(1-\delta)D_k + G_{t_0}$ by its log canonical threshold at $x \in X$ and peturbing it so that there is a unique non-klt center at $x \in X$, we obtain the required D_{k+1}.

□

Exercise 2.59. *Let X be a smooth projective variety and $B \geq 0$ a $\mathbb{Q}$-divisor. Suppose that $\mathcal{O}_X/\mathcal{J}(X,B)$ is supported on $Z \subset X$ and that $D \in V$ is a general element of a linear series whose base locus is contained in Z. Then for any $t < 1$ whe have that $\mathcal{O}_X/\mathcal{J}(X, B+tD)$ is supported on $Z \subset X$.*

Exercise 2.60. *Let T be a normal curve, $g : T \to X$ a a morphism to a smooth projective variety, $B \geq 0$ an effective $\mathbb{Q}$-divisor on X and G a $\mathbb{Q}$-divisor on X. Suppose that for any $t \in T \setminus \{t_0\}$ there is a divisor $G_t \sim_{\mathbb{Q}} G$ such that $\mathfrak{m}_{g(t)} \subset \mathcal{J}(X, B+G_t)$, then $\mathfrak{m}_{g(0)} \subset \mathcal{J}(X, B+G_0)$ for some $G_0 \sim_{\mathbb{Q}} G$.*

3. Boundedness of varieties of general type

The goal of this section is to prove the following result.

Theorem 3.1. *Fix $d \in \mathbb{N}$. Then there exists an integer $m = m_d$ depending only on d such that if X is a smooth d-dimensional projective variety, then $|kK_X|$ is birational for all $k \geq m$.*

We begin with a few preparations.

3.1. Kawamata sub-adjunction.

Theorem 3.2 (Kawamata subadjunction)**.** *Let (X, B_0) be a klt pair and V be a non-klt center of a pair (X,B) which is minimal on a neighborhood of its generic point $\eta_V \subset U \subset X$. Assume that (X,B) is lc at η_V, then for any ample $\mathbb{Q}$-divisor H, we have*

$$(K_X + B + H)|_{V^\nu} = K_{V^\nu} + B_{V^\nu}$$

where $V^\nu \to V$ is the normalization and the pair $(V^\nu|_U, B_{V^\nu}|_U)$ is klt

Proof. See [Kawamata98]. □

We illustrate the above result with the following basic example. Let $(X, S+B)$ be a plt surface pair so that $\lfloor S+B \rfloor = S$. Then S is a minimal log canonical center and in particular it is a smooth curve. Let

$f : X' \to X$ be a log resolution of $(X, S+B)$ and write $K_{X'}+S'+B' = f^*(K_X+S+B)$ where $S' = f^{-1}_*S$. Let $K_{S'}+B_{S'} = (K_{X'}+S'+B')|_{S'} = K_{S'} + B_{S'}$ and $K_S + B_S = (f|_{S'})_*(K_{S'} + B_{S'}$ (note that $f|_{S'}$ is an isomorphism). We have

$$K_S + B_S = (K_X + S + B)|_S,$$

where (S, B_S) is klt.

From the classification of PLT surface sinularities, one can check that the coefficients of B_S are of the form $1 - \frac{1-\sum k_i b_i}{n} \in [0, 1)$ where b_i are coefficients of B and $k_i \in \mathbb{N}$. It is easy to see that if the coefficients of B lie in a DCC set, then so do the coefficients of B_S.

3.2. **Easy addition.**

Theorem 3.3. *Let $f : X \to T$ be a morphism of smooth complex projective varieties, then $\kappa(X) \leq \kappa(X_t) + \dim T$ where X_t is a general fiber.*

Proof. Let $g : X \to Z$ be the Iitaka fibration. Replacing X by an appropriate birational model, we may assume that g is a morphism. By definition $\dim Z = \kappa(X)$ and $K_X \sim_{\mathbb{Q}} g^*A + E$ where E is effective and A is ample. It is easy to see that

$$\kappa(X_t) \geq \dim(g(X_t)) \geq \dim X_t - \dim(X/Z) = \dim Z - \dim T = \kappa(X) - \dim T.$$

$\square$

Corollary 3.4. *Let $f : Z \to T$ be a projective morphism and $g : Z \to X$ be a dominant morphism to a variety of general type, then X_t is a variety of general type for general $t \in T$.*

Proof. Replacing X, Z, T by appropriate birational models, we may assume that X, Z, T are smooth (recall that by definition X is of general type if so is any of its resolutions). Cutting by general hyperplanes on T, we may assume that g is generically finite (i.e. that $\dim Z = \dim X$). Since g is generically finite, $K_Z = g^*K_X + R$ where $R \geq 0$ is the ramification divisor and so Z is of general type. By Theorem 3.3

$$\dim Z = \kappa(Z) \leq \kappa(Z_t) + \dim T$$

and so $\dim Z_t \leq \kappa(Z_t)$ i.e. Z_t is of general type. $\square$

3.3. **Finite generation of the canonical ring.** We recall the following result from [BCHM10]

Theorem 3.5. *Let $f : X \to T$ be a projective morphism and (X, B) a klt pair such that $B \geq 0$ is a $\mathbb{Q}$-divisor. Then the pluricanonical ring*

$$R(K_X + B/Z) = \oplus_{m \geq 0} f_*\mathcal{O}_X(m(K_X + B))$$

is a finitely generated $\mathcal{O}_T$ module.

Remark 3.6. *If X is projective and $K_X + B$ is big, then $X_{\mathrm{can}} :=$ $\mathrm{Proj}R(K_X+B)$ is a projective variety and $\phi : X \dashrightarrow X_{\mathrm{can}}$ is a birational morphism which we may think of as follows. Let $m > 0$ be sufficiently divisible so that $R(K_X + B)$ is generated in degree m, then ϕ is defined by the linear series $|mK_X|$ so that $X_{\mathrm{can}} \subset \mathbb{P}^N = |mK_X|$. If $p : X' \to X$ is a log resolution of $|mK_X|$ so that $|p^*m(K_X + B)| = M + F$ where F is the fixed divisor and M is base point free, then M defines the morphism $q : X' \to X_{\mathrm{can}}$ and we have*

$$M = q^*\mathcal{O}_{X_{\mathrm{can}}}(1) = q^*\mathcal{O}_{X_{\mathrm{can}}}(m(K_{X_{\mathrm{can}}} + B_{X_{\mathrm{can}}}))$$

where $B_{X_{\mathrm{can}}} = f_*B$. In particular $K_{X_{\mathrm{can}}} + B_{X_{\mathrm{can}}}$ is an ample $\mathbb{Q}$-Cartier divisor and

$$p^*(K_X + B) = q^*(K_{X_{\mathrm{can}}} + B_{X_{\mathrm{can}}}) + \frac{F}{m}.$$

It follows easily that $(X_{\mathrm{can}}, B_{X_{\mathrm{can}}})$ has klt singularities. It is expected that pluricanonical rings of lc pairs are also finitely generated.

Definition 3.7. *Let $f : X \to T$ be a projective morphism and $(X, B \geq 0)$ be a lc pair such that $K_X + B$ is ample over T then we say that (X, B) is a log canonical model over T. If $K_X + B$ is nef over T then we say that (X, B) is a weak log canonical model over T.*

Exercise 3.8. *Show that if X is projective and $K_X + B$ is big, then X_{can} is birational to X.*

Exercise 3.9. *Show that if (X, B) is klt then so is $(X_{\mathrm{can}}, B_{X_{\mathrm{can}}})$.*

Exercise 3.10. *Show that if X is projective and K_X is big and canonical, then X_{can} has canonical singularities (this is of course not true if X does not have canonical singularities).*

3.4. **Proof of Theorem 3.1.** Tsuji's observation is that in order to prove Theorem 3.1, it suffices to prove the apparently weaker result

Theorem 3.11. *Fix $d \in \mathbb{N}$. Then there exists constants $A, B > 0$ depending only on d such that if X is a smooth d-dimensional projective variety, then $|kK_X|$ is birational for all*

$$k \geq \frac{A}{(\mathrm{vol}(K_X))^{1/d}} + B.$$

Claim 3.12. *Theorem 3.11 implies Theorem 3.1*

Proof. If $\mathrm{vol}(K_X) > 1$, then we may pick $m = \lceil A + B \rceil$. Therefore, we may assume that $\mathrm{vol}(K_X) \leq 1$. Let $r = \lceil \frac{A}{v^{1/d}} + B \rceil$ where $v = \mathrm{vol}(K_X)$, then $Z = \overline{\phi_r(X)} \subset |rK_X|$ is a variety of degree $r^d v \leq (A + B + 1)^d$. By a Hilbert scheme argument, there exists a projective morphism of quasi-projective varieties $\mathcal{Z} \to T$ such that if X is a smooth projective variety of dimension d and volume $\mathrm{vol}(K_X) \leq 1$, then X is birational to a fiber $\mathcal{Z}_t$ for some $t \in T$. We replace T by the closure of the points $t \in T$ such that $\mathcal{Z}_t$ is birational to a smooth projective variety of dimension d and volume $\mathrm{vol}(K_X) \leq 1$. Replacing T by a union of locally closed subsets and passing to a log resolution, we may assume that the fibers $\mathcal{Z}_t$ are smooth. By Siu's theorem on the deformation invariance of plurigenera, we may assume that $\mathrm{vol}(\mathcal{Z}_t)$ is constant on all connected components of T. It follows that there is a minimum value for $\mathrm{vol}(\mathcal{Z}_t)$ say v_d. But then we may pick $m = \lceil \frac{A}{v_d^{1/d}} + B \rceil$. $\square$

By a similar argument, we also have

Claim 3.13. *Theorem 3.1 implies Theorem 1.19.*

Proof. We begin by showing that $\mathcal{V}_d$ is discrete. It suffices to show that $\mathcal{V}_d \cap [0, L]$ is discrete for any $L > 0$. Arguing as above, we may assume that there is a smooth projective morphism $f : \mathcal{Z} \to T$ such that the fibers $\mathcal{Z}_t$ parametrizes all smooth d-dimensional projective varieties of general type with $\mathrm{vol}(K_X) \leq L$. Since $\mathrm{vol}(\mathcal{Z}_t)$ is constant on all connected components of T, the claim follows.

(1) now follows since $\mathcal{V}_d$ has a minimal positive element say $v_d > 0$ and as observed above we may pick $m = \lceil \frac{A}{v_d^{1/d}} + B \rceil$.

Consider now $\mathcal{X} = \mathrm{Proj}_T(R(K_{\mathcal{Z}}))$. Since for any $t \in T$ we have $f_*\mathcal{O}_{\mathcal{Z}}(mK_{\mathcal{Z}}) \to H^0(mK_{\mathcal{Z}_t})$, is surjective, it follows that $\mathcal{X}_t \cong \mathrm{Proj}(R(K_{\mathcal{X}_t}))$. This is (3). $\square$

Proof of Theorem 3.1. We claim that it suffices to show that for any two distinct very general points $x, y \in X$, there exists an effective $\mathbb{Q}$-divisor $D_{x,y}$ such that:

(1) $D_{x,y} \sim_{\mathbb{Q}} \lambda K_X$ where $\lambda < \frac{A}{\mathrm{vol}(K_X)^{1/d}} + B - 1$;
(2) x is an isolated point of the co-support of $\mathcal{J}(D_{x,y})$ and y is contained in the co-support of $\mathcal{J}(D_{x,y})$.

To see the claim, note that K_X is big and so we may pick $m > 0$ such that $mK_X \sim G + H$ where H is ample and $G \geq 0$. We may assume that x, y are not contained in the support of G and we let $D'_{x,y} = D_{x,y} + \frac{r-1-\lambda}{m} G$. It follows that $(r-1)K_X - D'_{x,y} \sim_{\mathbb{Q}} \frac{r-1-\lambda}{m} H$ is ample, so that by Theorem 2.33 $H^1(X, \omega_X^{\otimes m} \otimes \mathcal{J}(D'_{x,y})) = 0$. Consider the short exact sequence

$$0 \to \omega_X^{\otimes m} \otimes \mathcal{J}(D'_{x,y}) \to \omega_X^{\otimes r} \to \mathcal{Q} \to 0,$$

which induces a surjection

$$H^0(X, \omega_X^{\otimes r}) \to H^0(X, \mathcal{Q}).$$

Since x is an isolated point of the co-support of $\mathcal{J}(D'_{x,y})$, $\mathcal{Q}$ has a summand say $\mathcal{Q}_x$ supported at x and hence admitting a surjection $\mathcal{Q}_x \to \mathbb{C}(x)$. Since y is also contained in the co-support of $\mathcal{Q}$, one sees that there is a section of $\omega_X^{\otimes r}$ vanishing at y but not at x. Thus $|rK_X|$ defines a birational map and the claim now follows.

In what follows, we will show that there is a divisor $D_x \sim_{\mathbb{Q}} \lambda K_X$ where $\lambda < \frac{A}{\mathrm{vol}(K_X)^{1/d}} + B - 1$ and x is an isolated point of the co-support of $\mathcal{J}(D_x)$. The argument for producing the divisor $D_{x,y}$ is similar and we refer the reader to [Tsuji07] or [Takayama06] for the details. For simplicity, we will also assume that K_X is ample. This can be achieved replacing X by its canonical model. Of course, X will no longer be smooth but this only introduces minor technical challanges.

We proceed by induction on the dimension and so we may assume that Theorem 1.19 holds in dimension $\leq d-1$. Since

$$h^0(\mathcal{O}_X(mK_X)) = \frac{\mathrm{vol}(\omega_X)}{d!} m^d + O(m^{d-1})$$

and since vanishing to order k at a smooth point $x \in X$ imposes at most $k^d/d! + O(k^{d-1})$ conditions, by an easy computation, it follows that for any smooth point $x \in X$ and $m \gg 0$, there exists a $\mathbb{Q}$-divisor $D_x^m \sim mK_X$ such that $\mathrm{mult}_x(D_x^m) > \frac{m}{2}\mathrm{vol}(K_X)^{1//d}$. (Note that by assuming that $x \in X$ is very general, we may assume that m is independent of $x \in X$.) Let

$$\tau := \sup\{t \geq 0 | (X, tD_x^m) \text{ is lc at } x \in X\}.$$

By Remark 2.32, we have $\tau < \frac{2d}{m \cdot \mathrm{vol}(K_X)^{1/d}}$. Let $D_x = \tau D_x^m$, then $\mathfrak{m}_x \subset \mathcal{J}(X, D_x)$ and $D_x \sim_{\mathbb{Q}} \lambda K_X$ where $\lambda \leq \frac{2d}{\cdot \mathrm{vol}(K_X)^{1/d}}$. Perturbing D_x, we may assume that on a neighborhood of $x \in X$, we have $\mathcal{J}(D_x) = \mathcal{I}_{V_x}$ where $x \in V_x \subset X$ is an irreducible subvariety.

We now plan to follow the ideas in the proof of the Theorem of Anhern-Siu to cut down the dimension of the non-klt center V_x until we arrive to the case where $\dim V_x = 0$. We will therefore assume that $\dim X > \dim V_x > 0$. Following the proof of Theorem 2.56, it suffices to

(1) produce a divisor $E_{x'}$ on V_x such that $\mathrm{mult}_{x'}(E_{x'}) > \dim V_x$ and

(2) lift E_x to a divisor on X say $F_{x'} \sim_{\mathbb{Q}} \lambda' K_X$ whose support does not contain V_x and such that $F_{x'}|_{V_x} \geq E_{x'}$ where $\lambda' = O(\mathrm{vol}(K_X)^{-1/n}$ (i.e. $\lambda' \leq \frac{A'}{\mathrm{vol}(K_X)^{1/n}} + B'$ for appropriate constants A', B').

In what follows, we will assume that K_X s ample (this can be achieved by replacing X by its canonical model) and for simplicity that X is smooth (in practice X has canonical singularities, but this only adds minor technical difficulties). Notice that by Kawamata subadjunction, we have

$$(K_X + D_x + \epsilon K_X)|_{V^\nu} = K_{V^\nu} + B_\epsilon$$

where $V^\nu \to V$ is the normalization abd $B_\epsilon > 0$. By Theorem 3.4, V^ν is of general type and so $\mathrm{vol}(K_{V'}) > v'_n$ where $V' \to V^\nu$ is any resolution, $\dim V' = n' < n$ and $v_{n'} > 0$ is a constant. By induction on the dimension, we may assume that there is an effective $\mathbb{Q}$-divisor $E' \sim_{\mathbb{Q}} \gamma K_{V'}$ on V' with $\mathrm{mult}_{x'}(E') > n' := \dim V'$ and $0 < \gamma < n/v_{n'}$ (i.e. γ is bounded above by a constant). Pushing forward to V^ν and adding B_ϵ, we obtain a $\mathbb{Q}$ divisor on V^ν

$$E^\nu + B_\epsilon \sim_{\mathbb{Q}} (K_X + D_x + \epsilon K_X)|_{V^\nu} = (1 + \lambda + \epsilon) K_X|_{V^\nu}.$$

Assume for simplicity that V_x is normal (this is true on a neighborhood of $x \in X$) Since K_X is ample, it follows (by Serre vanishing) that

$$H^0(X, \mathcal{O}_X(mK_X)) \to H^0(X, \mathcal{O}_{V^\nu}(mK_X|_V))$$

is surjective for all $m \gg 0$. It then follows that

$$E^\nu + B_\epsilon \sim_{\mathbb{Q}} F_{x'}|_{V_x} = F_{x'}|V_x \qquad \text{where} \qquad F_{x'} \sim_{\mathbb{Q}} (1 + \lambda + \epsilon) K_X.$$

Set $\lambda' = 1 + \lambda + \epsilon$.

□

4. Varieties of log general type

In this section we will discuss results related to the boundedness of varieties of log general type. The first question that we encounter is why should we consider log pairs of general type and what generality should we consider.

4.1. **Automorphisms of varieties of general type.** Let X be a smooth projective variety of general type, then it is known that the automorphism group of X is finite. It is natural to ask how big can this automorphism group be. The well known answer in dimension 1 is the following.

Theorem 4.1. *Let C be a curve of general type and G its automorphism group. Then $|G| \leq 42(2g-2)$.*

Proof. Let $f : C \to B = C/G$ be the induced morphism, then we

$$K_C \sim_{\mathbb{Q}} f^*(K_B + D) \qquad \text{where} \qquad D = \sum_{i=1}^{k}(1 - \frac{1}{n_i})P_i.$$

here P_i denote the branch points of f and n_i the ramification indices of f over P_i. To prove the above formula, assume that $P \in B$ is a point of ramification index n so that $f^*P = nQ$. In local coordinates we have $y = f(x) = x^n$, then $dy = nx^{n-1}dx$ or equivalently $dy/y = ndx/x$ and so $f^*(K_B + P) = K_C + Q \sim_{\mathbb{Q}} K_C + f^*P$.

It then follows that

$$2g - 2 = \deg(K_C) = |G| \cdot \deg(K_B + D),$$

and so it suffices to show that $\deg(K_B + D) \geq \frac{1}{42}$. This can be done by a tedious case by case analisys. Since $1 - \frac{1}{n} \geq 1/2$, we may assume that there are $k \leq 4$ points and that $g = -1$ so that $\deg(K_B + D) = -2 + \sum_{i=1}^{4}(1 - \frac{1}{n_i})$. Assume that $n_1 \geq n_2 \geq n_3 \geq n_4$. Since $\deg(K_B + D) > 0$, $n_1 \geq 3$ and hence $\deg(K_B + D) \geq \frac{1}{6}$. Thus $k = 3$ (the case $k \leq 2$, $g = -1$ gives $\deg(K_B + D) < 0$). If $n_3 \geq 4$, then $\deg(K_B + D) \geq \frac{1}{4}$. If $n_3 = 3$, then $n_1 \geq 4$ and $\deg(K_B + D) \geq \frac{1}{12}$. Thus $n_3 = 2$ and then $n_2 \geq 3$. If $n_2 \geq 5$, then $\deg(K_B + D) \geq \frac{1}{10}$. If $n_2 = 4$, then $n_1 \geq 5$ and so $\deg(K_B + D) \geq \frac{1}{20}$. If $n_2 = 3$, then $n_1 \geq 7$ and so $\deg(K_B + D) \geq \frac{1}{42}$. □

By the same argument, if X is a variety of general type with automorphism group G, $f : X \to Y = X/G$ and D is a $\mathbb{Q}$-divisor such that $K_X = f^*(K_Y + D)$, then

$$|G| = \frac{\mathrm{vol}(K_X)}{\mathrm{vol}(K_Y + D)}.$$

Proposition 4.2. (Y, D) *is klt and the coefficients of* D *belong to the set* $\{1 - \frac{1}{n} | n \in \mathbb{N}\}$.

Proof. We refer the reader to [KM98, Proposition 5.20]. The main steps of the proof are as follows.

0) It is easy to see that Y is normal (if not, let $\nu : Y^\nu \to Y$ be the normalization, f factors through Y^ν....).

1) It is easy to see that the statement holds for curves and hence it holds in codimension 1.

2) We must check that $K_Y + D$ is $\mathbb{Q}$-Cartier. let $g = |G|$ and consider gK_X which is a G invariant Cartier divisor. Locally this is given by G invariant principal divisors (h) on X.....

3) Since $K_Y + D$ is $\mathbb{Q}$-Cartier and $f^*(K_Y + D) = K_X$ (it suffices to check this in codimension 1).

4) We check that (Y, D) is klt. Let E be an exceptional divisor for $h : Y' \to Y$ and let $g : X' \to X$ be obtained by fiber product. If E' is a divisor mapping to E via h, such that the ramification index of h along E is r, then near E' we have

$$K_{X'} = h^*K_X + a(E', X, 0)E' =$$
$$h^*f^*(K_Y + D) + a(E', X, 0)E' = f'^*g^*(K_Y + D) + a(E', X, 0)E'.$$
$$K_{X'} = f'^*K_{Y'} + (r-1)E' = f'^*(g^*(K_Y) + a(E, Y, 0)E) + (r-1)E'$$
$$= f'^*K_{Y'} + (r-1)E' = f'^*(g^*(K_Y + D) + a(E, Y, D)E) + (r-1)E'$$
$$= f'^*g^*(K_Y + D) + (ra(E, Y, D) + (r-1))E'.$$

It follows that $a(E', X, 0)+1 = r(a(E, Y, D)+1)$ and so if $a(E', X, 0) > -1$ then $a(E, Y, D) > -1$. $\square$

To generalize Theorem 4.1, it "suffices" to show the following.

Conjecture 4.3. *Let $\mathcal{P}_d$ be the set of klt pairs (Y, D) be a such that $\dim Y = d$ and the coefficients of D lie in the set $\{1 - \frac{1}{n} | n \in \mathbb{N}\}$ (or more generally in a fixed DCC set $\mathcal{C}$), then the set of volumes*

$$\mathcal{V}_d = \{\mathrm{vol}(K_Y + D) | (Y, D) \in \mathcal{P}_d\}$$

has a minimum (or even $\mathcal{V}_d$ is a DCC set).

Remark 4.4. *Note that if $(Y, D) \in \mathcal{P}_d$, then one can consider a log resolution $f : Y' \to Y$ and the divisor D' given by the strict transform of D plus the exceptional divisors with coefficient 1. Note then that $K_{Y'} + D' - f^*(K_Y + D) \geq 0$ is exceptional and so* $\mathrm{vol}(K_Y + D) = \mathrm{vol}(K_{Y'} + D')$ *and so in the above conjectures, it suffices to consider log smooth pairs.*

Remark 4.5. *It is conjectured that* $\min(\mathcal{V}_2) = \frac{1}{42^2}$.

Proposition 4.6. *The set of volumes* $\mathrm{vol}(K_X + B)$ *where (X, B) is log smooth and $B = \lfloor B \rfloor$ is not discrete.*

Proof. For example let $X_0 = \mathbb{P}^2$ and $B_0 = L + H_1 + H_2 + H_3$ be the union of 4 general lines. We define $\nu_1 : X_1 \to X$ by blowing up the point $x = L \cap H_1$. Let E_1 be the corresponding exceptional divisor and blow up the intersection point $E_1 \cap L_1$ where L_1 is the strict transform of L to obtain $X_2 \to X_1$ with exeptional divisor E_2. Inductively blowing up the intersection of the strict transform of L with the exceptional divisor E_n for $X_n \to X_{n-1}$, we obtain morphisms $\nu_n : X_n \to X_0$. Let $B_n = (\nu_n^{-1})_* B_0 + \mathrm{Ex}(\nu_n)$, then $K_{X_n} + B_n = \nu_n^*(K_{X_0} + B_0)$. If $B'_n = B_n - E_n$, then we claim that $\mathrm{vol}(K_{X_n} + B'_n) = 1 - \frac{1}{n}$. To see this we compute the corresponding minimal model. Note that $K_{X_n}+B'_n=f^*L-E_n$ is not nef. One sees that the minimal model $\pi : X_n\to \bar{X}$ contracts the curves $E_1,\ldots,E_{n-1}$ and the induced morphism $\eta:\bar{X}\to X_0$ contracts the curve $\bar{E}=\pi_*E_n$. Since $E_i^2=-2$ for $1\leq i\leq n-1$ and $E_n^2=-1$ and $E_i\cdot E_j=1$ for $|i-j|=1$ and $E_i\cdot E_j=0$ for $|i-j|>1$, it follows that

$$(E_1+2E_2+\ldots+nE_n)\cdot E_i=0 \qquad \text{for } 1\leq i\leq n-1.$$

But then $\pi^*(n\bar{E})=E_1+2E_2+\ldots+nE_n$ and so

$$\bar{E}^2=\frac{1}{n^2}(E_1+2E_2+\ldots+nE_n)^2=$$

$$\frac{1}{n^2}(E_1+2E_2+\ldots+nE_n)nE_n=\frac{n(n-1)-n^2}{n^2}=-\frac{1}{n}.$$

Finally

$$\mathrm{vol}(K_{X_n}+B'_n)=(K_{\bar{X}})^2=(\eta^*(K_{X_0}+B_0)-\bar{E})^2=(K_{X_0}+B_0)^2+\bar{E}^2=1-\frac{1}{n}.$$

□

Remark 4.7. *Many interesting examples (with empty boundary) can be constructed by taking hypersurfaces of the form* $x_1^{a_1}x_2+x_2^{a_2}x_3+x_3^{a_3}x_4+x_4^{a_4}x_1=0$ *in weighted projective space* $\mathbb{P}(w_1,w_2,w_3,w_4)$ *(known as Kollár surfaces).*

Exercise 4.8. *Let* (X,B) *be a log canonical surface, then* $\mathrm{vol}(K_X+B)\geq (K_X+B)^2$ *and equality holds iff* K_X+B *is nef. (Hint: consider* $\nu : X\to X'$ *the canonical model, then* $K_X+B=\nu^*(K_{X'}+B')+E$ *where* $E\geq 0$ *(*$E=0$ *iff* K_X+B *is nef. But then* $(K_X+B)^2=(K_{X'}+B')^2+E^2=(K_{X'}+B')^2-e$ *where* $e\geq 0$ *(and* $e=0$ *iff* K_X+B *is nef).*

4.2. **Open varieties.** Let U be a smooth quasi-projective variety and X its closure (in the corresponding projective space). By Hironaka's Theorem, we may assume that X is also smooth and $D:=X\setminus U$ is a simple normal crossings divisor.

Lemma 4.9. *The vector spaces* $H^0(m(K_X+D))$ *depend only on* U *(and not the choice of* X *and* D*).*

Proof. Suppose that $\nu : X'\to X$ is a morphism of smooth projective varieties which is an isomorphism over U such that $D'=\nu^{-1}(D)$ also has has simple normal crossings. Since (X,D) is log canonical and $D'\geq \mathrm{Ex}(\nu)$ it follows thaty $K_{X'}+D'=\nu^*(K_X+D)+E$ where $E\geq 0$ is ν-exceptional. But then

$$H^0(m(K_{X'}+D'))=H^0(m\nu^*(K_X+D)+mE)=H^0(m(K_X+D)).$$

The proof now follows easily. □

Note that the vector spaces $H^0(\Omega^i_X(log D))$ are also uniquely determined by U.

4.3. **Iitaka fibration.** Let (X,B) be a proper klt pair and $R(K_X+B)=\oplus_{m\geq 0}H^0(m(K_X+B))$, then $R(K_X+B)$ is finitely generated and so we may consider the corresponding projective variety $Z:=\mathrm{Proj}R(K_X+B)$. By a Theorem of Mori and Fujino there exists a klt pair (Z,B_Z) and integers l,m such that $R(K_X+B)^{(l)}=R(K_Z+B_Z)^{(m)}$.

The construction is somewhat involved. Replacing X by a higher model, we may assume that $f:X\to Z$ is a morphism. Replacing X,Z by appropriate birational models, we may assume that f is smooth other the complement of snc divisor. Assume for simplicity that $K_X\sim_{\mathbb{Q},Z}=0$ so that $K_X\sim_{\mathbb{Q}} f^*L$ where L is a $\mathbb{Q}$-Cartier divisor on Z. (conjecturally this can be accomplished by running a relative minimal model so that K_X is semiample over Z and replacing Z by $\mathrm{Proj}R(K_X+B/Z)$). We define $B_Z=\sum(1-t_P)P$ where for any prime divisor $P\subset Z$, $t_P=\sup\{t\geq 0|(X,B+tf^*P)$ is lc over $\eta P\}$. We let $M_Z=L-(K_Z+B_Z)$. The divisor B_Z is known as the *boundary part* and the divisor M_Z is the moduli part. It is known that M_Z is a nef $\mathbb{Q}$-divisor and conjectured that it is semiample. If f is an elliptic fibration, then $M_Z=j^*\mathcal{O}_{\mathbb{P}^1}(1/12)$. It is easy to see that the coefficients of B_Z are <1 and hence (Z,B_Z) is klt

Since $K_Z+B_Z+M_Z$ is big, it is $\mathbb{Q}$-linearly equivalent to $A+E$ where A is ample and $E\geq 0$. But then

4.4. **Proper moduli spaces.** Another reason to consider log pairs is to be able to construct proper moduli spaces. In order to obtain a proper moduli space (functor), we must consider semilog canonical pairs (X,B).

Definition 4.10. *A SLC (semi-log-canonical) pair (X,B) is given by an S_2 quasiprojective variety X with SNC singularities in codimension 1 and a $\mathbb{R}$-divisor B whose support has no component contained in* $\mathrm{Sing}(X)$*, such that K_X+B is $\mathbb{R}$-Cartier and if $\nu:X^\nu\to X$ is the normalization and $K_{X^\nu}+B^\nu=\nu^*(K_X+B)$, then (X^ν,B^ν) is a disjoint union of lc pairs.*

For example if X is a smooth variety and S is a simple normal crossings divisor and $(X,S+B)$ is a lc pair, then (S,B_S) is a slc pair where $(K_X+S+B)|_S=K_S+B_S$. More generally we have:

Lemma 4.11. *Let $(X, S+B)$ be a dlt pair, where $S = \lfloor S+B \rfloor$ and write $K_S + B_S = (K_X + S + B)|_S$, then (S, B_S) is slc. Note that if $D \subset S^\nu$ is the double locus, then $B_S \geq D$.*

Proof. See [???]. $\square$

Suppose now that $(\mathcal{X}^0, \mathcal{B}^0)$ is a lc pair and $f : \mathcal{X}^0 \to T^0 = T \setminus O$ is a projective morphism to a smooth curve (or to the generic point of a DVR). In what follows we will often replace T by an appropriate neighborhood of $O \in T$. Assume that for exery $t \in T^0$ the pairs $(\mathcal{X}^0_t, \mathcal{B}^0_t)$ are log canonical models i.e. pairs with lc singularities such that $K_{\mathcal{X}^0_t} + \mathcal{B}^0_t$ is ample. We let $\mathcal{X} \to T$ be a compactification and $\mathcal{X}' \to \mathcal{X}$ be a log resolution of $(\mathcal{X}, \mathcal{B} + \mathcal{X}_0)$. Replacing T by a cover ramified only over O, we may assume that $\mathcal{X}' \to T$ is semistable so that $\mathcal{X}'_t$ is smooth for $t \in T \setminus O$ and $\mathcal{X}'_O$ is a snc divisor. Let $\mathcal{B}'$ be the divisor given by the strict transform of $\mathcal{B}$ plus all exceptional divisors dominating T plus $\mathcal{X}_0$. By [HX13], the relative canonical ring $R(K_{\mathcal{X}'} + \mathcal{B}')$ is finitely generated over T and so we may consider

$$\bar{\mathcal{X}} := \mathrm{Proj}_T R(K_{\mathcal{X}'} + \mathcal{B}').$$

Let $\bar{\mathcal{B}}$ be the strict transform of $\mathcal{B}'$, then $(\bar{\mathcal{X}}, \bar{\mathcal{B}}) \times_T (T \setminus O)$ is isomorphic to $(\mathcal{X}^0, \mathcal{B}^0)$. Moreover, since $\bar{\mathcal{B}} \geq \bar{\mathcal{X}}_O$, the pair $(\bar{\mathcal{X}}_O, \bar{\mathcal{B}}_O)$ is a slc model where $K_{\bar{\mathcal{X}}_O} + \bar{\mathcal{B}}_O = (K_{\bar{\mathcal{X}}} + \bar{\mathcal{B}})|_{\bar{\mathcal{X}}_O}$. We also have the following.

Lemma 4.12. *Suppose that $(\bar{\mathcal{X}}', \bar{\mathcal{B}}')$ is another lc model over T isomorphic to $(\bar{\mathcal{X}}, \bar{\mathcal{B}})$ over $T \setminus O$ and such that $\bar{\mathcal{B}}' \geq \bar{\mathcal{X}}'_O$, then $(\bar{\mathcal{X}}', \bar{\mathcal{B}}')$ is isomorphic to $(\bar{\mathcal{X}}, \bar{\mathcal{B}})$ over T.*

Proof. Let $p : W \to \bar{\mathcal{X}}$ and $q : W \to \bar{\mathcal{X}}'$ be a common resolution and write $p^*(K_{\bar{\mathcal{X}}} + \bar{\mathcal{B}}) = q^*(K_{\bar{\mathcal{X}}'} + \bar{\mathcal{B}}') + E$. Since $\bar{\mathcal{B}} \geq \bar{\mathcal{X}}'_O$, it is easy to see that $p_*E \geq 0$ and since $q^*(K_{\bar{\mathcal{X}}'} + \bar{\mathcal{B}}')$ is p-nef, then by the Negativity Lemma, $E \geq 0$. Similarly $q_*E \leq 0$ and $p^*(K_{\bar{\mathcal{X}}} + \bar{\mathcal{B}})$ is q-nef so that $E \leq 0$. Thus $E = 0$ and

$$\bar{\mathcal{X}} = \mathrm{Proj} R(K_{\bar{\mathcal{X}}} + \bar{\mathcal{B}}/T) = \mathrm{Proj} R(K_{\bar{\mathcal{X}}'} + \bar{\mathcal{B}}'/T) = \bar{\mathcal{X}}'.$$

$\square$

5. The MMP

5.1. **Non-vanishing, base point free and cone theorems.** Recall that for a normal projective variety X, the set of $\mathbb{R}$-Cartier divisors modulo numerical equivalence is denoted by $N^1(X)$ and $N_1(X)$ denotes the dual space of $\mathbb{R}$ linear combinations of curves up to numerical equivalence. We let the *effective cone* $NE(X) \subset N_1(X)$ be the cone generated by effective curves. For any cone $C \subset N_1(X)$, and any

$L \in N^1(X)$, $C_{L\geq 0} \subset N_1(X)$ denotes the set of curves $\Sigma \in C$ such that $C \cdot L \geq 0$ (and similarly for $C_{L<0}$, $C_{L=0}$ etc.). If $C_{L\geq 0} = \mathbb{R}_{\geq 0}[\Sigma]$ for some curve class $0 \neq [\Sigma] \in N_1(X)$, then we say that $\mathbb{R}_{\geq 0}[\Sigma]$ (or simply $[\Sigma]$) is an *extremal ray* of C. Similarly if $C_{L\geq 0} = C_{L=0}$, then $F = C_{L=0}$ is an *extremal face*. Next we recall the following fundamental results.

Theorem 5.1 (Non-vanishing theorem)**.** *Let (X, B) be a projective sub-klt pair and D a nef Cartier divisor such that $aD - (K_X + B)$ is nef and big for some $a > 0$, then $H^0(\mathcal{O}_X(mD - \lfloor B \rfloor)) \neq 0$ for all $m \gg 0$.*

Proof. **Step 0.** We may assume that X is smooth and $aD-(K_X+B)$ is ample.

Let $f : X' \to X$ be a resolution $D' = f^*D$ and $K_{X'} + B' = f^*(K_X + B)$. Then $aD' - (K_{X'} + B') = f^*(aD - (K_X + B))$ is nef and big and so $aD' - (K_{X'} + B') \sim_{\mathbb{Q}} A + F$ where A is ample and $F \geq 0$. For $0 < \epsilon \ll 1$, $(X', B' + \epsilon F)$ is sub-klt and

$$aD' - (K_{X'} + B' + \epsilon F) \sim_{\mathbb{Q}} (1 - \epsilon)(aD' - (K_{X'} + B')) + \epsilon A$$

is ample. Since $f_*(B' + \epsilon F) \geq B$, we have

$$h^0(\mathcal{O}_{X'}(mD' - \lfloor B' + \epsilon F \rfloor)) \leq H^0(\mathcal{O}_X(mD - \lfloor B \rfloor))$$

and so we may replace D and (X, B) by D' and $(X', B' + \epsilon F')$.

Step 1. We may assume that D is not numerically equivalent to 0.

Suppose that in fact $D \equiv 0$, then for any $k, t \in \mathbb{Z}$, we write

$$kD - \lfloor B \rfloor \equiv K_X + \{B\} + tD - (K_X + B).$$

Note that $tD - (K_X + B)$ is ample for all $t \in \mathbb{Z}$ and so by Kawamata Viehweg vanishing we have

$$h^0(\mathcal{O}_X(mD - \lfloor B \rfloor)) = \chi(\mathcal{O}_X(mD - \lfloor B \rfloor)) =$$

$$\chi(\mathcal{O}_X(-\lfloor B \rfloor)) = h^0(\mathcal{O}_X(-\lfloor B \rfloor)) \neq 0.$$

Step 2. For any $x \in (X \setminus \mathrm{Supp}(B))$ there is an integer q_0 such that for any integer $q \geq q_0$ there is a $\mathbb{Q}$-divisor

$$M(q) \equiv qD - (K_X + B)$$

with $\mathrm{mult}_x M(q) > 2 \dim X$.

Let $d = \dim X$ and A be an ample divisor. Since D is nef, $D^e \cdot A^{d-e} \geq 0$ for $1 \leq e \leq d$ and since D is not numerically trivial, $D \cdot A^{d-1} > 0$. But then

$$(qD-K_X-B)^d = ((q-a)D+aD-K_X-B)^d \geq d(q-a)D\cdot(aD-K_X-B) > 0.$$

Since the RHS goes to infinity as q goes to infinity, by Serre vanishing and Riemann Roch, we have

$$h^0(\mathcal{O}_X(e(qD-K_X-B))) \geq \frac{e^d}{d!}(2d)^d + O(e^{d-1}).$$

Vanishing at x with multiplicity $> 2de$ imposes at most

$$\frac{(2de)^d}{d!} + O(e^{d-1})$$

conditions and so there is a divisor

$$M(q,e) \in |e(qD-K_X-B)| \qquad \text{with} \qquad \mathrm{mult}_x M(q,e) > 2de.$$

let $M(q) = M(q,e)/e$.

Step 3. Let t be the log canonical threshold of (X,B) with respect to $M = M(q)$, then $t < 1/2$. Since $A := qD - K_X - B$ is ample, perturbing, we may assume that $(X, B+tM)$ has a unique non-klt center V. By Theorem 2.29, V is normal and by Theorem 3.2, $(K_X + B + tM + \epsilon A)|_V \sim_{\mathbb{Q}} K_V + B_{V,\epsilon}$ where $(V, B_{V,\epsilon})$ is sub-klt. Consider the short exact sequences

$$0 \to \mathcal{O}_X(mD) \otimes \mathcal{J}(X, B+tM) \to \mathcal{O}_X(mD) \to \mathcal{O}_V(mD|_V) \to 0.$$

Since

$$mD - (K_X + B + tM) \sim_{\mathbb{R}} (m - tq)D - (1+t)(K_X + B)$$

is ample for all $m \gg 0$, by Nadel vanishing $H^1(\mathcal{O}_X(mD) \otimes \mathcal{J}(X, B + tM)) = 0$ and so it suffices to show that $H^0(\mathcal{O}_V(mD)) \neq 0$. Since $mD|_V$ is nef and

$$mD|_V - (K_V + B_{V,\epsilon}) \sim_{\mathbb{R}} (mD - K_X - B - tM - \epsilon A)|_V \sim_{\mathbb{R}}$$
$$(m - (t+\epsilon)q)D - (1 + t + \epsilon)(K_X + B)$$

is ample, by induction on the dimension $H^0(\mathcal{O}_V(mD)) \neq 0$.

□

Exercise 5.2. *Show that if D is nef and not numerically equivalent to 0, then $D \cdot A^{d-1} > 0$ for any ample divisor A.*

Theorem 5.3. *[Base point free theorem] Let (X,B) be a projective klt pair and D be a nef Cartier divisor such that $aD - K_X - B$ is nef and big for some $a > 0$, then $|bD|$ is base point free for all $b \gg 0$.*

Proof. As in the proof of the non-vanishing theorem, we may assume that $aD - K_X - B$ is ample. Pick $m > 0$ such that $\mathrm{Bs}(mD) = \mathbf{B}(mD)$. (Recall that the *stable base locus* is defined as $\mathbf{B}(D) = \cap_{m>0}\mathrm{Bs}(mD)$. It is easy to see that $\mathrm{Bs}(mD) = \mathbf{B}(mD)$ for any $m > 0$ sufficiently divisible.) If $D \equiv 0$, then $h^0(mD) \neq 0$ implies that $mD \sim 0$ and so $\mathrm{Bs}(mD) = \emptyset$ as required.

Otherwise, assume for simplicity that X is smooth. Pick general sections $D_1, D_2, \ldots, D_{d+1} \in |mD|$. Then $c := \mathrm{lct}(X, B; \sum D_i) < 1$ and the non-klt locus of $(X, B + c\sum B_i)$ is contained in $\mathrm{Bs}(mD)$. Perturbing, we may assume that there is a unique such non-klt center say V. Arguing as in the proof of the non-vanishing theorem, we may assume that $H^0(\mathcal{O}_X(mD)) \to H^0(\mathcal{O}_V(mD))$ is surjective and $H^0(\mathcal{O}_V(mD)) \neq 0$. But then $V \not\subset \mathrm{Bs}(mD)$ contradicting the assumption that $V \subset \mathbf{B}(D)$. $\square$

Exercise 5.4. *Show that* $\mathrm{Bs}(mD) \subset \mathrm{Bs}(mpD)$ *for any* $m, p \in \mathbb{N}$, *but it can happen that* $\mathrm{Bs}(mD) \not\subset \mathrm{Bs}((m+1)D)$. *Deduce that* $\mathbf{B}(D) = \mathrm{Bs}(mD)$ *for all* $m > 0$ *sufficiently big and divisible.*

Theorem 5.5 (Cone Theorem)**.** *Let* (X, B) *be a projective klt pair. Then*

(1) *There are countably many rational curves* $C_j \subset X$ *such that* $0 < -(K_X + B) \cdot C_j \leq 2 \dim X$, *and*
$$\overline{NE} = \overline{NE}_{(K_X+B) \geq 0} + \sum \mathbb{R}_{\geq 0}[C_j].$$
(2) *For any* $\epsilon > 0$, *there are only finitely many rays*
$$[C_j] \in \overline{NE}(X)_{(K_X+B+\epsilon H)<0}.$$
(3) *If* $F \subset \overline{NE}(X)$ *is a* K_X+B *negative extremal face, then there is a unique morphism* $\mathrm{cont}_F : X \to Z$ *such that* $(\mathrm{cont}_F)_*\mathcal{O}_X = \mathcal{O}_Z$ *and an irreducible curve* $C \subset X$ *is contracted to a point if and only if* $[C] \in F$.
(4) *Let* L *be a line bundle on* X *such that* $L \cdot C = 0$ *for all curves* $[C] \in F$, *then there is a line bundle* L_Z *on* Z *such that* $(\mathrm{cont}_F)^* L_Z = L$.

Proof. For (1) and (2) please see [KM98, §3.3]. Next we sketch the proof of (3) and (4). Let $F \subset \overline{NE}(X)$ be a $K_X + B$ negative extremal face $F = F_D$ for some $\mathbb{Q}$-Cartier divisor D. For any $m \gg 0$, the $\mathbb{Q}$-Cartier divisor $mD - (K_X + B)$ is strictly positive on $\overline{NE}(X) \setminus \{0\}$. Therefore $mD - (K_X + B)$ is ample and mD is nef and by Theorem 5.3, mD is base point free for all $m \gg 0$. Let $g_m : X \to Z = Z_m$ be the Stein factorization of $X \to |mD|$ so that Z_m is normal and $g_{m,*}\mathcal{O}_X = \mathcal{O}_Z$. Let M_{Z_m} be the pull-back of the hyperplane line bundle on $|mD|$ to Z_m so that $g_m^* M_{Z_m} = mD$. This proves (3).

A curve $C \subset X$ is contracted by g_m iff $C \cdot D = 0$ and so $g = g_m : X \to Z$ is independent of $m \gg 0$. But then $D = (m+1)D - mD =$

$g^*M_{m+1} - g^*M_m$ is Cartier. Suppose now that $L \cdot C = 0$ for all $[C] \in F$, then $L + mD$ also supports F for $m \gg 0$ and so it defines g. By the arguments above $L + mD = g^*N_Z$ for some Cartier divisor N_Z on Z so that $L = g^*(N_Z - M_{Z,m})$. $\square$

5.2. **Minimal model preliminaries.** In this section we recall some of the results from the minimal model program.

Let (X, B) and (X', B') be lc pairs and $f : X \to T$, $f' : X' \to T'$ be projective morphisms, then a birational map $\phi : X \dashrightarrow X'$ is a *birational contraction* if X' contains no divisors exceptional over X. Given a birational contraction $\phi : X \dashrightarrow X'$ such that $B' = \phi_* B$, then ϕ is $K_X + B$ *non-positive* (resp. $K_X + B$ *negative*) if for any common resolution $p : W \to X$ and $q : W \to Y$ we have $p^*(K_X + B) - q^*(K_{X'} + B')$ is effective (resp. effective and its support contains all ϕ exceptional divisors.

Exercise 5.6. *Show that if $\phi : X \dashrightarrow X'$ is $K_X + B$ non-positive, then $R(K_X + B/T) \cong R(K_{X'} + B'/T)$.*

Exercise 5.7. *That a composition of two $K_X + B$ non-positive birational contraction is again a $K_X + B$ non-positive birational contraction.*

Exercise 5.8. *Show that a contraction of a -1curve on a smooth surface X (resp. a -2 curve on a surface with canonical singularities) is K_X negative (resp. K_X non positive). Conclude that the map to the minimal model $X \to X_{\min}$ (resp. to the canonical model $X \to X_{\text{can}}$) is K_X negative (resp. K_X non-positive).*

Exercise 5.9. *Let $\phi : X \to X'$ be a $K_X + B$ non positive birational contraction. Show that for any divisor E over X we have $a_E(X, B) \leq a_E(X', B')$.*

Exercise 5.10. *Let $\phi : X \to X'$ be a $K_X + B$ negative birational contraction. Show that for any divisor E over X we have $a_E(X, B) \leq a_E(X', B')$ and strict inequality holds if the center of E is contained in the exceptional set of ϕ.*

Definition 5.11. *If (X, B) is klt and ϕ is a $K_X + B$ negative birational contraction such that X' is $\mathbb{Q}$-factorial and $K_{X'} + B'$ is nef over T, then we say that X' is a log terminal model of (X, B) over T.*

Definition 5.12. *If (X, B) is lc and ϕ is a $K_X + B$ non-positive birational contraction such that $K_{X'} + B'$ is ample, then we say that (X', B') is a log canonical model of (X, B).*

Exercise 5.13. *Show that log canonical models are unique and given by $X \dashrightarrow \mathrm{Proj} R(K_X + B)$.*

Exercise 5.14. *Show that minimal models are not smooth in dimension $d \geq 3$. (Hint consider the quotient of an abelian threefold via an involution.)*

Exercise 5.15. *Show that minimal models are not unique in dimension $d \geq 3$.*

Definition 5.16. *Let (X, B) be a $\mathbb{Q}$-factorial klt pair and $f : X \to Z$ be a small (so that $\dim(\mathrm{Ex}(f)) = \dim X - 1$) projective birational morphism of normal varieties with $f_*\mathcal{O}_X = \mathcal{O}_Z$ and relative picard number $\rho(X/Z) = 1$ such that $-(K_X + B)$ is ample over Z, then f is a $K_X + B$ flipping contraction. We say that $\mathrm{Ex}(f)$ is the flipping locus.*

Definition 5.17. *If $f : X \to Z$ is a flipping contraction, then the flip $f^+ : X^+ \to Z$ (if it exists) is given by $X^+ = \mathrm{Proj}_Z(R(K_X + B))$. We say that $\mathrm{Ex}(f^+)$ is the flipped locus.*

Remark 5.18. *The existence of flips in the klt case follows from* [BCHM10], *and in the log canonical case from results of Birkar and of Hacon-Xu.*

Lemma 5.19. *Let $f : X \to Z$ be a $K_X + B$ flipping contraction and $f^+ : X^+ \to Z$ be the corresponding flip, then*

(1) *$\phi : X \dashrightarrow X^+$ is small,*
(2) *X^+ is $\mathbb{Q}$-factorial,*
(3) *$K_{X^+} + B^+$ is f^+-ample,*
(4) *$\phi : X \dashrightarrow X^+$ is $K_X + B$ non-positive (in particular if (X, B) is klt/lc, then so is (X^+, B^+)).*
(5) *If E is a divisor with center contained in the flipping or flipped locus, then $a_E(X, B) < a_E(X^+, B^+)$.*

Proof. (4) and (5) are an easy consequence of the negativity lemma. Let $p : W \to X$ and $q : W \to Y$ be a common resolution, then $p^*(K_X + B) - q^*(K_{X^+} + B^+) = E$ where $-E$ is exceptional and nef over X and so $E \geq 0$ which proves (4). Moreover, the support of E dominates both the flipping and flipped locus (by the ampleness over Z condition on $K_{X^+} + B^+$ and $-(K_X + B)$). But then the fibers of $W \to Z$ are contained in E and (5) follows.

By assumption $K_{X^+} + B^+$ is $\mathbb{Q}$-Cartier and f^+ ample and (3) holds.

Suppose that f^+ is not small and let E be an f^+-exceptional divisor. Since $K_{X^+} + B^+$ is ample over Z, we may assume that Z is affine and $\mathcal{O}_{X^+}(m(K_{X^+} + B^+)) = \mathcal{O}_{X^+}(1)$ is very ample. For any integer $t \gg 0$

$$\mathcal{O}_Z(tm(K_Z + f_*B)) = f^+_*\mathcal{O}_{X^+}(t) \subsetneq f^+_*\mathcal{O}_{X^+}(t)(E),$$

where the last inclusion is strict for $t \gg 0$ as $\mathcal{O}_{X^+}(1)$ is ample. Since $\mathcal{O}_Z(tm(K_Z+f_*B))$ is reflexive, there is a natural inclusion $f^+_*\mathcal{O}_{X^+}(t)(E) \subset \mathcal{O}_Z(tm(K_Z+f_*B))$ which is impossible. Thus f^+ is small and (1) holds.

Let G^+ be a $\mathbb{Q}$ divisor on X^+ and set $G = \phi_*^{-1}G^+$. Since X is $\mathbb{Q}$-factorial, G is $\mathbb{Q}$-Cartier. Since $\rho(X/Z) = 1$, any $\mathbb{Q}$-Cartier divisor G on X is of the form $G \sim_{\mathbb{Q}} aD + b(K_X + B)$ where $D = f^*D_Z$ for a $\mathbb{Q}$-Cartier divisor D_Z on Z. But then $G^+ = \phi_*G = a{f^+}^*D_Z + b(K_{X^+} + B^+)$ is $\mathbb{Q}$-Cartier, and hence (2) follows.

□

Exercise 5.20. *Let $X \dashrightarrow X^+$ be a $K_X + B$ flip. If (X, B) is dlt (resp. plt), show that (X^+, B^+) is dlt (resp. plt).*

Exercise 5.21. *Let $f : X \to Z$ be a flipping contraction, show that $K_Z + f_*B$ is not $\mathbb{Q}$-Cartier and hence (Z, f_*B) is not lc.*

Exercise 5.22. *Show that the properties of Lemma 5.19 determine the flip uniquely.*

Definition 5.23. *Let (X, B) be a $\mathbb{Q}$-factorial klt pair and $f : X \to Z$ be a projective morphism of normal varieties such that $\dim \mathrm{Ex}(f) = \dim X - 1$, $f_*\mathcal{O}_X = \mathcal{O}_Z$, $-(K_X + B)$ is ample over Z and $\rho(X/Z) = 1$, then f is a divisorial contraction.*

Lemma 5.24. *Let $f : X \to Z$ be a divisorial contraction, then*

(1) *$E = \mathrm{Ex}(f)$ is a prime divisor,*
(2) *$-E$ is ample over Z,*
(3) *f is $K_X + B$ negative and hence (Z, f_*B) is klt, and*
(4) *X is $\mathbb{Q}$-factorial.*

Proof. Let E be an exceptional prime divisor. Since the fibers of f are connected, if $E \neq \mathrm{Ex}(f)$, then there is a curve C intersecting E but not contained in E. It follows that $C \cdot E > 0$ and so E is relatively ample (as $\rho(X/Z) = 1$). By the negativity lemma we obtain an immediate contradiction and so $E = \mathrm{Ex}(f)$ and (1) follows.

It also follows from the negativity lemma that $-E$ is not relatively trivial and hence either ample or anti-ample (as $\rho(X/Z) = 1$, but then $-E$ is ample over Z and (2) follows.

We may write $K_X + B - aE \sim_{\mathbb{Q},Z} 0$ for some $a > 0$ and so $K_X + B = f^*(K_Z + f_*B) + aE$ and hence (3) holds.

Let G be a divisor on Z. Pick e so that $f_*^{-1}G + eE \sim_{\mathbb{Q},Z} 0$, then $G + eE \sim_{\mathbb{Q}} f^*G$ and so G is $\mathbb{Q}$-Cartier.

□

Exercise 5.25. *Let ϕ be a birational contraction. Show that if $D \sim_{\mathbb{Q}} E$ on X, then $\phi_*D \sim_{\mathbb{Q}} \phi_*G$. In particular, if D is big or pseudo-effective, then so is ϕ_*D.*

Exercise 5.26. *Let $f : X \to Z$ be a birational morphism and G a $\mathbb{Q}$-Cartier divisor on Z, then f^*G is $\mathbb{Q}$-Cartier on X.*

Exercise 5.27. *Let $f : X \dashrightarrow Y$ be a small birational morphism, then $|G|_{\mathbb{Q}} \cong |f_*G|_{\mathbb{Q}}$ for any $\mathbb{Q}$-divisor on X.*

Definition 5.28. *Let (X, B) be a $\mathbb{Q}$-factorial klt pair and $f : X \to Z$ be a morphism of normal varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Z$, $\rho(X/Z) = 1$, $-(K_X + B)$ is ample over Z and $\dim X > \dim Z$, then f is a Mori fiber space.*

Exercise 5.29. *Show by example that there exist Mori fiber spaces such that the general fiber X_z has Picard number $\rho(X_z) > 1$.*

5.3. **Running the minimal model program.**

Theorem 5.30. *Let (X, B) be a klt pair and $f : X \to T$ be a projective morphism such that $K_X + B$ is big over T (resp. $K_X + B$ is not pseudo-effective over T), then (X, B) has a minimal model over T (resp. a Mori fiber space) which can be obtained by a finite sequence of flips and divisorial contractions over T.*

In order to construct the minimal model (when $K_X + B$ is pseudo-effective) or a Mori fiber space (when $K_X + B$ is not pseudo-effective), we run a minimal model program. The traditional strategy is as follows:

(1) Start with a projective $\mathbb{Q}$-factorial klt pair (X, B).
(2) If $K_X + B$ is nef, stop: this is a minimal model.
(3) If $K_X + B$ is not nef, pick a $K_X + B$ negative extremal ray $\Sigma \subset \overline{NE}(X)$.
(4) If Σ defines a Mori fiber space, stop.
(5) If Σ defines a divisorial contraction (resp. a flipping contraction) $f : X \to Z$, then replace (X, B) by (Z, f_*B) (resp. by the flip (X^+, B^+)) and return to (2).

Note that if if we have a flip $X^+ \dashrightarrow X^+$, then $\rho(X^+) = \rho(X)$ but if we have a divisorial contraction, then $\rho(Z) = \rho(X) - 1$. Since $\rho(X) \in \mathbb{N}$, it follows easily that any sequence of steps of the minimal model program has at most $\rho(X)$ divisorial contractions and hence to show that the above strategy ends after finitely many steps, we must show that any sequence of flips is finite.

Conjecture 5.31 (Termination of flips)**.** *Let (X, B) be a $\mathbb{Q}$-factorial projective klt pair. There is no infinite sequence of $K_X + B$ flips $X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \ldots$.*

Remark 5.32. *To be more precise $X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \ldots$ is a sequence of $K_X + B$ flips if there are $K_{X_i} + B_i$ flipping contractions $X_i \to Z_i$ such that $X_{i+1} \to Z_i$ is the corresponding $K_{X_i} + B_i$ flip where B_i is the strict transform of B on X_i.*

Remark 5.33. *The termination of flips conjecture is known in dimension 3 and many cases of dimension 4. In higher dimension it appears to be a very difficult conjecture. For this reason an alternative approach is used in* [BCHM10] *that reduces the question of termination of flips to a question of termination of special sequences of flips with nice properties.*

Definition 5.34 (Minimal model program with scaling)**.** *In this version of the MMP we start with a projective $\mathbb{Q}$-factorial pair $(X, B+H)$ such that $K_X + B + H$ is nef and B is big. (Given (X, B) one may for example pick H a general $\mathbb{Q}$ divisor $\mathbb{Q}$-linearly equivalent to a sufficiently ample divisor.) Consider now the nef threshold*

$$\lambda = \inf\{t \geq 0 | K_X + B + tH \text{ is nef}\}.$$

(1) *If $\lambda = 0$, then $K_X + B$ is nef and (X, B) is a minimal model.*
(2) *If $\lambda > 0$, pick a $(K_X + B)$-negative extremal ray R such that $(K_X + B + \lambda H) \cdot R = 0$. Let $f : X \to Z$ be the corresponding contraction.*
(3) *If $\dim Z < \dim X$, then we have a $K_X + B$ Mori fiber space.*
(4) *If $\dim Z = \dim X$, then f is either a $K_X + B$ flipping or divisorial contraction and we replace $(X, B + \lambda H)$ by the corresponding $K_X + B$ flip $(X^+, B^+ + \lambda H^+)$ or divisorial contraction $(Z, f_*B + \lambda f_*H)$. Note that by the Base Point Free Theorem 5.3, $K_{X^+} + B^+ + \lambda H^+$ or $K_Z + f_*B + \lambda f_*H$ is also nef and klt and so we may replace (X, B, H) by $(X, B, \lambda H)$ or $(Z, f_*, \lambda f_*H)$ and repeat the process.*

Remark 5.35. *Therefore, running the MMP with scaling we produce a sequence of rational numbers $1 \geq \ldots t_i \geq t_{i+1} \ldots \geq 0$ and flips or divisorial contractions $X_i \dashrightarrow X_{i+1}$ such that*

(1) *$K_{X_i} + B_i + tH_i$ is nef for $t_i \geq t \geq t_{i+1}$,*
(2) *$X \dashrightarrow X_i$ is a sequence of steps of the $K_X + B + tH$ mmp for any $0 \leq t \leq t_i$,*
(3) *$(X_i, B_i + tH_i)$ is klt for $0 \leq t \leq t_i$,*
(4) *$K_X + B + tH$ is pseudo-effective for any $t \geq t_i$.*

We then have the following.

Theorem 5.36. *[Termination of the minimal model program with scaling] Let $(X, B + H)$ be a projective $\mathbb{Q}$-factorial klt pair such that $K_X + B + H$ is nef and B is big (or $K_X + B$ is big or $K_X + B$ is not pseudo-effective), then the $K_X + B$ mmp with scaling of H terminates after finitely many steps.*

Proof. For a detailed proof we refer the reader to [BCHM10]. □

Remark 5.37. *The case when $K_X + B$ is big is easily reduced to the case when B is big. Consider in fact a rational number $0 < e \ll 1$ and write $K_X + B \sim_{\mathbb{Q}} A + E$ where A is ample and $E \geq 0$ (which is possible if $K_X + B$ is big). We then have that $(X, B' := B + e(A + E))$ is klt, B' is big, and $K_X + B' \sim_{\mathbb{Q}} (1 + e)(K_X + B)$. It follows easily that every step of the $K_X + B'$ mmp is a step of the $K_X + B$ mmp and so the termination of the $K_X + B'$ mmp gives the termination of the $K_X + B$ mmp.*

Remark 5.38. *The case when $K_X + B$ is not pseudo-effective is easily reduced to the case when B is big. Consider in fact a rational number $0 < e \ll 1$, then $K_X + B + eH$ is not pseudo-effective and in particular $t_i > e$ for any $i \geq 0$. Let $B' = B + eH$, then every step of the $K_X + B$ mmp with scaling of H is a step of the $K_X + B'$ mmp with scaling of H (or more precisely of $(1 - e)H$). Termination of the $K_X + B'$ mmp gives the termination of the $K_X + B$ mmp.*

As a consequence of Theorem 5.36, we obtain the following.

Theorem 5.39 (Existence of minimal models)**.** *Let (X, B) be a $\mathbb{Q}$-factorial projective klt pair such that B (or $K_X + B$) is big. If $K_X + B$ is pseudo-effective, then (X, B) has a good minimal model. If $K_X + B$ is not pseudo-effective, then (X, B) has a Mori fiber space.*

Proof. Let H be a general sufficiently ample divisor, then $(X, B + H)$ is klt and $K_X + B + H$ is ample and in particular nef. By Theorem 5.36, the $K_X + B$ mmp with scaling of H terminates $\phi : X \dashrightarrow X'$. It is easy to see that ϕ is $K_X + B$ negative birational contraction. If $K_{X'} + B'$ is nef, then this is a minimal model for $K_X + B$. Note that by Theorem 5.3, $K_{X'} + B'$ is semiample (assume for simplicity that B is a $\mathbb{Q}$-divisor) and so this is a good minimal model. In particular $\kappa(K_X + B) = \kappa(K_{X'} + B') \geq 0$ and so $K_X + B$ is pseudo-effective. If $K_{X'} + B'$ is not nef, then there is a Mori fiber space $g : X' \to Z$ such that $-(K_{X'} + B')$ is g ample. It follows easily that $K_X + B$ is not pseudo-effective. $\square$

Theorem 5.40. *[Finite generation] Let (X, B) be a projective klt pair such that $B \geq 0$ has rational coefficients, then $R(K_X + B)$ is finitely generated.*

Proof. If B is big, then by Theorem 5.39, consider the minimal model $f : X \dashrightarrow X'$. Then $R(K_X + B) \cong R(K_{X'} + B')$. Note that B' is big, and by Theorem 5.3, $K_{X'} + B'$ is semiample so that there exists a morphism $g : X' \to Z$ such that $K_{X'} + B' \sim_{\mathbb{Q}} g^*A$ where A is an ample $\mathbb{Q}$-divisor on Z and $f_*\mathcal{O}_X = \mathcal{O}_Z$. It is well known that $R(K_{X'} + B')$ is finitely generated if and only if so is the truncation

$$R(K_{X'}+B')^{(m)}=\oplus_{k\in\mathbb{N}}H^0(\mathcal{O}_{X'}(km(K_{X'}+B')))$$

for any $m>0$. Suppose that $m(K_{X'}+B')\sim g^*mA$ where mA is Cartier, then by the projection formula

$$\oplus_{k\in\mathbb{N}}H^0(\mathcal{O}_{X'}(km(K_{X'}+B')))=\oplus_{k\in\mathbb{N}}H^0(\mathcal{O}_Z(kmA))$$

which is easily seen to be finitely generated.

We now prove the general case. If $|m(K_X+B)|=\emptyset$ for all $m>0$, then the claim is obvious. If this is not the case, then consider the rational map $X\to Z$ defined by $|m(K_X+B)|$ for $m\gg 0$ sufficiently divisible. After replacing X and Y by appropriate birational models, we may assume that Y is smooth and $f:X\to Y$ is a morphism. By a result of Mori and Fujino, we have that $R(K_X+B)\cong R(K_Z+B_Z+M_Z)$ where (Z,B_Z) is a klt log smooth pair and M_Z is a nef divisor. By construction $K_Z+B_Z+M_Z$ is big and hence $\mathbb{Q}$ linearly equivalent to A_Z+E_Z where A_Z is ample and $E_Z\geq 0$. Fix $0<e\ll 1$, then M_Z+eA_Z is ample. Pick a general element $G_Z\in|M_Z+eA_Z|$, then $(Z,B'_Z:=B_Z+eE_Z+G_Z)$ is klt and B'_Z is big. Fix $m>0$ sufficiently divisible, then $R(K_Z+B_Z+M_Z)^{(m')}\cong R(K_Z+B'_Z)^{(m)}$ where $m'=m(1+e)$. It follows that $R(K_Z+B_Z+M_Z)$ is finitely generated if and only if so is $R(K_Z+B'_Z)$. The claim follows by what we proved above. □

Exercise 5.41. *Let Z be a normal variety and A an ample divisor. Show that $R(A)=\oplus_{k\in\mathbb{N}}H^0(\mathcal{O}_Z(kA))$ is finitely generated.*

To gain some intuition on why Theorem 5.36 holds, we show the following.

Proposition 5.42. *Assume that Theorem 5.39 holds, then Theorem 5.36 also holds.*

Proof. The idea of the proof is quite simple (if we assume Theorem 5.39). Since each step of the MMP with scaling of H produces a minimal model for K_X+B+tH for some $0\leq t\leq 1$, it suffices to show that there are only finitely many such minimal models. Assume by way of contradiction that there is an infinite sequence of flips $X_i\dashrightarrow X_{i+1}$ such that $K_{X_i}+B_i+tH_i$ is nef and klt for $t\in[s_{i+1},s_i]$ where $1\geq s_1\geq\ldots\geq s_i\geq s_{i+1}\geq\ldots>0$. Let $\sigma=\lim s_i$. By Theorem 5.39, there exists a minimal model $f:X\dashrightarrow X'$ for $K_X+B+\sigma H$. Note that we may that f is $K_X+B+\sigma H$ negative and $K_{X'}+B'+\sigma H'$ is semiample (by Theorem 5.3) so that there is a morphism $g:X'\to Y$ and an ample $\mathbb{R}$-divisor A on Y

such that $g^*A \sim_{\mathbb{R}} K_{X'} + B' + \sigma H'$. It is easy to see that the map f is $K_X + B + sH$ negative for $s \in [\sigma, \sigma + \epsilon]$ and $0 < \epsilon \ll 1$ (in the sense that if E is a divisor on X which is f-exceptional, then $a_E(X, B + sH) < a_E(X', B' + sH')$). Let $h : X' \dashrightarrow X''$ be a minimal model for $K_{X'} + B' + (\sigma + \epsilon)H'$ over Y, then in fact h is a minimal model over Y for $K_{X'} + B' + sH'$ for any $s \in (\sigma, \sigma + -\epsilon]$. The reason being that in this case $s = t(\sigma + \epsilon) + (1 - t)\sigma$ for some $1 \geq t > 0$ and so

$$K_{X'}+B'+sH' = t(K_{X'}+B'+(\sigma+\epsilon)H')+(1-t)(K_{X'}+B'+\sigma H') \sim_{\mathbb{R},Y} t(K_{X'} + B' + (\sigma + \epsilon)H').$$

But then it is clear that the $K_{X'} + B' + (\sigma + \epsilon)H'$ mmp over Y is automatically a $K_{X'} + B' + sH'$ mmp over Y. It is easy to see that if $0 < t \ll 1$, then X'' is in fact a minimal model for $K_{X'} + B' + sH'$ (not just over Y). Otherwise, let Σ be a curve with $(K_{X'} + B' + sH') \cdot \Sigma < 0$, then $g_*\Sigma \neq 0$ and so $\Sigma \cdot (K_{X'} + B' + \sigma H') \geq \delta > 0$ (eg. if $r(K_{X'} + B' + \sigma H')$ is Cartier, then let $\delta = 1/r$). We may assume that Σ is a $K_{X'} + B' + (\sigma + \epsilon)H'$ negative extremal ray and so $(K_{X'} + B' + (\sigma + \epsilon)H') \cdot \Sigma \geq -2 \dim X$. But then

$$0 > (K_{X'} + B' + sH') \cdot \Sigma =$$
$$t(K_{X'} + B' + (\sigma + \epsilon)H') \cdot \Sigma + (1 - t)(K_{X'} + B' + \sigma H') \cdot \Sigma \geq$$
$$2t \dim X - (1 - t)\delta > 0 \qquad \text{for } 0 < t \ll 1.$$

This is a contradiction and so (replacing ϵ by a smaller number) we may assume that X'' is a $K_{X'}+B'+(\sigma+\epsilon)H'$ minimal model. Let $X'' \to W$ be the corresponding log canonical model, then for any $s_i \in (\sigma, \sigma + \epsilon]$, $X_i \to W$ is the log canonical model for $K_{X_i} + B_i + s_iH_i$. Let η_i be the corresponding morphism so that $K_{X_i} + B_i + s_iH_i = \eta_i^*(K_{X_W} + B_W + s_iH_W)$. Let $X_i \to Z_i$ be the flipping contraction with flip $X_{i+1} \to Z_i$, then $K_{X_i} + B_i + s_{i+1}H_i$ is trivial over Z_i and hence there is a morphism $Z_i \to W$. let C_i be a flipping curve, then $(K_{X_i} + B_i + s_iH_i) \cdot C_i < 0$, but this is impossible as $K_{X_i} + B_i + s_iH_i$ is pulled back from W and $\eta_{i,*}C_i = 0$. □

5.4. **Useful consequences.**

Theorem 5.43. *Let (X, B) be a projective klt pair and $E_1, \ldots, E_n$ a finite collection of divisors over X such that $a_{E_i}(X, B) \leq 0$, then there exists a proper birational morphism $f : X' \to X$ such that the set of f-exceptional divisors is in bijection with $E_1, \ldots, E_n$. In particular:*

(1) *If we pick $n = 0$, then $X' \to X$ is a small birational morphism such that X' is $\mathbb{Q}$-factorial and $K_{X'} + B' = f^*(K_X + B)$ where $B' = f_*^{-1}B$ and (X', B') is klt.*

(2) *If we pick $E_1, \dots, E_n$ to be the set of all divisors exceptional over X such that $a_E(X, B) \leq 0$, then (X', B') is terminal where $K_{X'} + B' = f^*(K_X + B)$.*

Proof. Let $g : X'' \to X$ be a log resolution such that $E_1, \dots, E_n$ are divisors on X''. Write $K_{X''} + B'' = g^*(K_X + B) + E''$ where B'' and E'' are effective $\mathbb{Q}$-divisors with no common components. Let F be the sum of all exceptional divisors distinct from $E_1, \dots, E_n$, then $(X'', B'' + \epsilon F)$ is klt for any $0 < \epsilon \ll 1$ and $K_{X''} + B'' + \epsilon F$ is clearly big over X and so there exists a minimal model $X'' \dashrightarrow X'$ for $K_{X''} + B'' + \epsilon F$ over X. But then $E'' + \epsilon F$ is nef over X and exceptional over X. By the negativity lemma $E'' + \epsilon F \leq 0$ by the negativity lemma and so $E'' + \epsilon F = 0$. But then the set of $X' \to X$ exceptional divisors is in bijection with $E_1, \dots, E_n$.

(1) is immediate, but (2) is more subtle. First of all, one must show that the set of divisors exceptional over X such that $a_E(X, B) < 0$ is finite. This can be done by (an easy but tedious argument) first replacing X by a log resolution and then blowing up along strata of the support of B. Since $K_{X'} + B' = f^*(K_X + B)$, for any divisor E exceptional over X', we have $a_E(X', B') = a_E(X, B) > 0$ (since otherwise E is a divisor on X'). □

We will also need the following stronger result which is proven by a similar method.

Theorem 5.44. *Let (X, B) be a log pair where $B \in [0, 1]$. Then there exists a projective birational morphism $f : Y \to X$ such that*

(1) *Y is $\mathbb{Q}$-factorial,*
(2) *f only extracts divisors of discrepancy $a_E(X, B) \leq -1$,*
(3) *If $E = \sum E_i$ is the sum of the f-exceptional divisors and $B_Y = f^{-1}_* B$, then*

$$K_Y + B_Y + E = f^*(K_X + B) + \sum_{a_E(X,B) < -1} (a_E(X, B) + 1)E.$$

(4) *If in addition (X, B) is log canonical and $B \in [0, 1)$, then we may chose f so that there is a divisor with support equal to E which is nef over X. In particular the inverse image of the non-klt locus of (X, B) is equal to the support of E.*

Proof. See [HMX14a, Proposition 3.3.1]. □

Lemma 5.45. *Let $f : X \to X'$ be a birational morphism between log canonical pairs (X, B) and (X', B') (where B, B' are effective $\mathbb{R}$-divisors). Suppose that $K_X + B$ is big and (X, B) has a log canonical model $g : X \dashrightarrow X^c$.*

If $f_*B \leq B'$ and $\operatorname{vol}(K_X + B) = \operatorname{vol}(K_{X'} + B')$ *then the induced birational map $X' \dashrightarrow X^c$ is also the log canonical model of (X', B').*

Proof. Let $\pi : W \to X$ be a log resolution of $(X, f_*^{-1}B' + \operatorname{Ex}(f))$ and of g. We write

$$K_W + \Theta = \pi^*(K_X + B) + E$$

where Θ and E are effective $\mathbb{R}$-divisors with no common components, then the log canonical model of (W, Θ) is the same as the log canonical model of (X, B). Replacing (X, B) by (W, Θ) we may assume that $(X, f_*^{-1}B' + \operatorname{Ex}(f))$ is log smooth and $g : X \dashrightarrow X^c$ is a morphism. Replacing (X', B') by $(X, f_*^{-1}B' + \operatorname{Ex}(f))$, we may assume that $X = X'$.

Let $A = g_*(K_X + B)$, then A is ample and $K_X + B - g^*A \geq 0$. Let $L = B' - B \geq 0$ and S a component of L with coefficient $a > 0$. Let

$$v(t) = \operatorname{vol}(g^*A + tS),$$

then $v(t)$ is a non-decreasing function of t and

$$v(0) = \operatorname{vol}(g^*A) = \operatorname{vol}(K_X + B) = \operatorname{vol}(K_X + B') \geq \operatorname{vol}(g^*A + L) \geq \operatorname{vol}(g^*A + aS) = v(a).$$

Therefore $v(t)$ is constant for $t \in [0, a]$. By [LM09, 4.25 (iii)] we have

$$\frac{1}{n}\frac{dv}{dt}|_{t=0} = \operatorname{vol}_{X|S}(g^*A) \geq S \cdot g^*A^{n-1} = g_*S \cdot A^{n-1}$$

where $n = \dim X$. But then $g_*S = 0$. It follows that every component of L is exceptional for g and so g is the log canonical model of (X, B'). $\square$

Remark 5.46. *If H is ample, then $H + tS$ is ample for $0 < t \ll 1$ and so*

$$\frac{1}{n}\frac{dv}{dt}|_{t=0} = \lim_{t \to 0} \frac{(H + tS)^n - H^n}{nt} = H^{n-1} \cdot S = \operatorname{vol}_{X|S}(H).$$

5.5. **Adjunction.** Let $(X, S + B)$ be a pair where S is a prime divisor not contained in the support of B, and $S^\nu \to S$ the normalization, then we define the different as follows.

$$K_{S^\nu} + \operatorname{Diff}_{S^\nu}(B) = (K_X + S + B)|_S.$$

In practice, to compute $\operatorname{Diff}_{S^\nu}(B)$, we consider a log resolution $f : X' \to X$ of $(X, S + B)$ and we write $K_{X'} + S' + B' = f^*(K_X + S + B)$ where $S' = f_*^{-1}S$. We define $\operatorname{Diff}_S(B) = B|_S$ and we have

$$K_{S^\nu} + \operatorname{Diff}_{S^\nu}(B) = (f|_S)_*(K_{S'} + \operatorname{Diff}_S(B)).$$

For example if X is the cone over a rational curve of degree n and S is a line on S, then the blow up of the vertex P gives a log resolution $f : X' \to X$ with exceptional divisor E. We have $K_{X'} + S' + (1 - 1/n)E = f^*(K_X + S)$. (To check this observe that $E^2 = -n$ and

$-2 = (K_{X'} + E) \cdot E = (f^*(K_X + S) + 1/nE - S') \cdot E = -1 - 1$ as required). But then

$$(f|_{S'})_*((K_{X'}+S'+(1-1/n)E)|_{S'}) = (f|_{S'})_*(K_{S'}+(1-1/n)P') = K_S+(1-1/n)P$$

and hence $\mathrm{Diff}\, S(0) = (1-1/n)P$. In general, computing the different can be reduced to a computation about surfaces. When X is a log canonical surface, one can perform these computations explicitely. Shokurov shows the following.

Lemma 5.47 (Shokurovs Log Adjunction Formula)**.** *Let $(X, S + B)$ be a log canonical surface where $B = \sum b_i B_i$ and S is a prime divisor with normalization $S^\nu \to S$, then*

$$(K_X + S + B)|_{S^\nu} = K_{S^\nu} + \mathrm{Diff}_{S^\nu}(B) = K_{S^\nu} + \mathrm{Diff}_{S^\nu} + B|_{S^\nu}$$

where the coefficients of $\mathrm{Diff}_{S^\nu}(B)$ are 1 or of the form $(n - 1 + \sum a_i b_i)/n \in [0, 1]$ for some $a_i \in \mathbb{N}$. In particular if the coefficients of B are in the set I, then the coefficients of $\mathrm{Diff}_{S^\nu}(B)$ are in the set $D(I)$.

5.6. **Deformation invariance of plurigenera.**

Theorem 5.48. *If (X, B) is a snc pair and $X \to T$ is a morphism such that (X, B) is log smooth over T, then*

(1) *$h^0(X_t, \mathcal{O}_{X_t}(m(K_{X_t}+B_t)))$ is independent of $t \in T$. In particular*

$$f_*\mathcal{O}_X(m(K_X + B)) \to H^0(X_t, \mathcal{O}_{X_t}(m(K_{X_t} + B_t)))$$

is surjective for all $t \in T$.

(2) *If there is a point $t \in T$ such that (X_t, B_t) has a good minimal model, then (X, B) has a good minimal model over T and every fiber has a good minimal model. Furthermore, the relative log canonical model of (X, B) over T gives the relative log canonical model for each fiber.*

Proof. See [HMX14b, 1.2, 1.4]. □

6. Boundedness of pairs of log general type

Theorem 6.1. *Fix $d \in \mathbb{N}$ and $I \subset [0, 1]$ a DCC set whose only accumulation point is 1. Let $\mathcal{P}_{d,I}$ be the set of all projective log canonical models (X, B) such that $\dim X = d$ and $B \in I$.*

(1) *There exists an integer $m_{d,I}$ depending only on d, I such that if $(X, B) \in \mathcal{P}_{d,I}$, then $|m(K_X + B)|$ is birational for all $m \geq m_{d,I}$.*

(2) *The set $\mathcal{V}_{d,I} := \{\mathrm{vol}(K_X + B)|(X, B) \in \mathcal{P}_{d,I}\}$ is a DCC set.*

(3) *For any $v \in \mathcal{V}_{d,I}$, there exists a projective morphism of quasi-projective varieties $\mathcal{X} \to T$ and a pair $(\mathcal{X}, \mathcal{B})$ such that if $(X, B) \in \mathcal{P}_{d,I}$ and $\mathrm{vol}(K_X + B) = v$, then $(X, B) \cong (\mathcal{X}_t, \mathcal{B}_t)$ for some $t \in T$.*

Remark 6.2. *Let $I' = I \cup \{1\} \subset [0, 1]$, then the set $\mathcal{V}_{d,I} = \{\mathrm{vol}(K_X + B)\}$ where (X, B) are projective log smooth d-dimensional log pairs with $B \in I'$. This is because given a pair $(X, B) \in \mathcal{P}_{d,I}$ we can consider a log resolution $\nu : X' \to X$ and the divisor $B' = \nu_*^{-1}B + \mathrm{Ex}(\nu)$ so that $K_{X'} + B' - \nu^*(K_X + B) = E$ where E is effective and exceptional and hence $h^0(m(K_{X'} + B')) = h^0(m(K_X + B))$ and in particular $\mathrm{vol}(K_{X'} + B') = \mathrm{vol}(K_X + B)$.*

Conjecturally, if (X, B) is a log smooth pair with $\mathrm{vol}(K_X + B) > 0$, then it has a log canonical model $\bar{X} = \mathrm{Proj}(R(K_X + B))$ and $\mathrm{vol}(K_X + B) = \mathrm{vol}(K_{\bar{X}} + \bar{B})$.

Remark 6.3. *Suppose that (X, B) is a semi log canonical pair and let $\nu : \cup(X_i, B_i) \to (X, B)$ be the normalization. If $B \in I$, then $B_i \in I' = I \cup \{1\} \subset [0, 1]$ and $\mathrm{vol}(K_X + B) = \sum \mathrm{vol}(K_{X_i} + B_i)$ is a sum of elements in the DCC set $\mathcal{V}_{d,I'}$. It is easy to see that if $\mathrm{vol}(K_X + B) = v$ is fixed, then it can be written in finitely many ways as a sum of elements of $\mathcal{V}_{d,I'}$. This implies that (the normalization of) semi-log canonical models of fixed volume with coefficients in a fixed DCC set also form a bounded family.*

Corollary 6.4. *Fix d and $\mathcal{C}$ a DCC set, then there exists a number $1 > \tau > 0$ such that if (X, B) is a d-dimensional log canonical pair such that $K_X + B$ is big, then $K_X + \tau B$ is big.*

Proof. By Theorem 6.1, there exists an integer $m = m_{d,\mathcal{C}}$ such that $|m(K_X + B)|$ is birational. Replacing X by an appropriate birational model, we may assume that $|m(K_X + B)| = |H| + F$ where H is base point free and $F \geq 0$ is the fixed locus. Consider the birational map $f : X \to Z \subset \mathbb{P}^N = |H|$ so that $f^*\mathcal{O}_Z(1) = H$. Pick $x, x' \in X$ general points and $H_1, \ldots, H_{d+1}$ the pullbacks of general hyperplanes through $f(x)$, and $H'_1, \ldots, H'_{d+1}$ the pullbacks of general hyperplanes through $f(x')$. We then have that x, x' are isolated points in the cosupport of $\mathcal{J}(X, \frac{d}{d+1}\sum(H_i + H'_i))$ and so, since by Nadel vanishing

$$H^1(\mathcal{O}_X(K_X + (2d+1)H) \otimes \mathcal{J}(X, \frac{d}{d+1}\sum(H_i + H'_i)) = 0$$

it follows that $H^0(K_X+(2d+1)H)\to \mathbb{C}_{x,x'}$ is surjective and in particular $|K_X+(2d+1)m(K_X+B)|$ is birational and hence big. Since

$$K_X+(2d+1)m(K_X+B)=(1+(2d+1)m)(K_X+\frac{(2d+1)m}{1+(2d+1)m}B)$$

the assertion follows letting $\tau=(2d+1)m/(1+(2d+1)m)$. $\square$

Exercise 6.5. *Show that if X is a smooth projective variety of dimension d and M is a big Cartier divisor such that $|M|$ is base point free, then $|K_X+(2d+1)M|$ is birational and hence $K_X+(2d+1)M$ is big.*

Corollary 6.6 (ACC for Pseudo-effective thresholds)**.** *Fix $d\in\mathbb{N}$ and $\mathcal{C}\subset[0,1]$ a DCC set. Consider the set of all pseudo-effective thresholds*

$$\mathcal{T}_{d,\mathcal{C}}=\{\tau(X_i,B_i;C_i)\}$$

where (X_i,B_i) is a d-dimensional log canonical pair, $C_i\geq 0$ belongs to a base point free linear series, $B_i\in\mathcal{C}$ and $\tau(X_i,B_i;C_i)$ is the pseudo-effective threshold

$$\tau(X_i,B_i;C_i)=\inf\{t\geq 0|K_{X_i}+B_i+tC_i \text{ is big}\}.$$

Then $\mathcal{T}_{d,\mathcal{C}}$ is an ACC set.

Proof. Suppose to the contrary that we have an increasing set of pseudo-effective thresholds $t_i=\tau(X_i,B_i;C_i)$. Let $t=\lim t_i$, then $K_{X_i}+B_i+tC_i$ is big and the coefficients of B_i+tC_i belong to the DCC set $\mathcal{C}'=\mathcal{C}\cup\{t\}$. We may assume that the pairs (X_i,B_i+tC_i) are log canonical and so by Corollary 6.4 there exists a number $0<\tau<1$ such that $K_{X_i}+\tau(B_i+tC_i)$ is big and hence so is $K_{X_i}+B_i+t_iC_i$ for any $i\gg 0$ (as $\tau(B_i+tC_i)\leq B_i+t_iC_i$). This is a contradiction. $\square$

The above results are closely related to the following conjecture of Shokurov.

Theorem 6.7. *Fix $n\in\mathbb{N}$ and $\mathcal{C}\subset[0,1]$ a DCC set. let $LCT_n(\mathcal{C})=\{\mathrm{lct}(X,B;M)\}$ where (X,B) is log canonical, $B,M\in\mathcal{C}$. Then $LCT_n(\mathcal{C})$ satisfies the DCC.*

Proof. Suppose that there is a sequence of d-dimensional log canonical pairs (X_i,B_i) and divisors $0\neq M_i\geq 0$ such that $B_i,M_i\in\mathcal{C}$ and $t_i=\mathrm{lct}(X_i,B_i;M_i)$ is an increasing sequence. Replacing (X_i,B_i) by dlt models, we may assume that X_i is dlt and $\mathbb{Q}$-factorial. Since $\mathcal{C}$ is a DCC set, it has a positive minimum say $c>0$ and so $t_i\leq 1/c$. Let $t=\lim t_i$. Clearly $t>t_i$. For all $i>0$ let $\nu_i:Y_i\to X_i$ be the proper birational morphism extracting a unique divisor E_i of discrepancy -1 with center a minimal non klt center of $(X_i,B_i+t_iM_i)$. Cutting by hyperplanes, we may assume that this minimal non klt center is a closed point $x_i \in X_i$. We may assume that $\rho(Y_i/X_i) = 1$ and we define

$$K_{E_i} + \Delta_i = (K_{Y_i} + E_i + \nu_{i,*}^{-1}(B_i + t_i M_i))|_{E_i},$$

$$K_{E_i} + \Delta_i' = (K_{Y_i} + E_i + \nu_{i,*}^{-1}(B_i + tM_i))|_{E_i}.$$

Note that the coefficients of $B_i + t_i M_i$ and $B_i + tM_i$ are in the DCC set $\mathcal{C}' = \mathcal{C} \times \{1, t, t_i 1, t_2, \ldots\}$. and hence the coefficients of Δ_i and Δ_i' are in the DCC set $D(\mathcal{C}')$. Since $\lim t_i = t$, $K_{E_i} + \Delta_i'$ is log canonical by induction on the dimension (otherwise $\mathrm{lct}(E_i, 0; \Delta_i') < 1$ and $\lim \mathrm{lct}(E_i, 0; \Delta_i') = 1$ contradicting the ACC for LCT's). Since $t > t_i$ and $\rho(Y_i/X_i) = 1$, we have that

$$K_{E_i} + \Delta_i' = (K_{Y_i} + E_i + \nu_{i,*}^{-1}(B_i + tM_i))|_{E_i} = (t - t_i)\nu_{i,*}^{-1} M_i|_{E_i}$$

is ample and hence $K_{E_i} + \Delta_i'$ is ample. We claim that there exists a number $\tau < 1$ such that $K_{E_i} + \tau\Delta_i'$ is big (see Corollary 6.4) below (in fact $\tau =$). But then $\Delta_i \geq \tau\Delta_i'$ for $i \gg 0$ so that

$$K_{E_i} + \tau\Delta_i' \leq K_{E_i} + \tau\Delta_i \sim_{\mathbb{Q}} 0$$

which is an obvious contradiction.

□

The first step in proving Theorem 6.1 is to prove that the pairs $(X, B) \in \mathcal{P}_{d,I}$ with $\mathrm{vol}(K_X + B) = v$ are log birationally bounded.

Definition 6.8. *A set of log pairs $(X, B) \in I$ is log birationally bounded if there exists a log pair $(\mathcal{X}, \mathcal{B})$ and a projective morphism $\mathcal{X} \to T$ such that for any pair $(X, B) \in I$, there is a $t \in T$ and a birational map $f : X \dashrightarrow \mathcal{X}_s$ such that the support of the strict transform of B plus the $\mathcal{X}_s \dashrightarrow X$ exceptional divisors are contained in the support of $\mathcal{B}$.*

Theorem 6.9. *Fix $d \in \mathbb{N}$,$v > 0$, and I a DCC set. Let $\mathcal{Q}_{d,v,I}$ be the set of projective log canonical pairs (X, B) such that $\dim X = d$, $B \in I$ and $\mathrm{vol}(K_X + B) \leq v$, then $\mathcal{Q}_{d,v,I}$ is birationally bounded.*

Proof. As in the proof of Theorem 3.1, it suffices to show that there exists an integer $m_{d,v,I} = O(v^{-1/d})$ such that $m(K_X + B)$ is birational for any $m \geq m_{d,v,I}$ (this means that $m_{d,v,I} \leq Av^{-1/d} + B$ for some constants A, B depending on d, I but not on v). Suppose in fact that such an integer $m = m_{d,v,I}$ exists, then $|m(K_X+B)|$ induces a birational map whose image Z has bounded degree

$$\deg Z \leq \mathrm{vol}(m(K_X + B)) = m^d v \leq (Av^{-1/d} + B)^d v \leq (A + uB)^d$$

where $u = \max\{v^{1/d}, 1\}$. Fix $v > 0$, then there is a family $\mathcal{Z} \to T$ (depending on I, d, v) such that if $\mathrm{vol}(K_X + B) \leq v$, then $Z = \mathcal{Z}_t$ for some $t \in T$. This explains why X is birationally bounded, but why is the pair (X, B) log birationally bounded? Replacing (X, B) by an appropriate birational model, we may assume that

$$\phi = \phi_{K_X + m(K_X+B)} : X \to Z$$

is a birational morphism (note that we may apply Exercise 6.5 and replace m by $(2d+1)m$). Thus we may write

$$|K_X + m(K_X + B)| = |M| + E,$$

where M is a big and base point free Cartier divisor and E is an effective $\mathbb{R}$-divisor. Let $H = \mathcal{O}_Z(1)$ be the very ample divisor on Z such that $\phi^* H = M$. Note that

$$\mathrm{vol}(K_X + m(K_X + B)) \leq \mathrm{vol}((m+1)(K_X + B)) \leq 2^d v.$$

Fix $\Gamma \in |\lfloor K_X + m(K_X + B) \rfloor|$, δ be the smallest positive element in $\mathcal{C}$, and let

$$\alpha = \max\{\frac{1}{\delta}, 2(2d+1)\}.$$

Let D_0 be the sum of the components of B and Γ that are not ϕ exceptional, then

$$D_0 \leq \delta(B + \Gamma).$$

Note that there is a divisor $C \geq 0$ such that

$$\delta(B + \Gamma) + C \sim_{\mathbb{Q}} \delta(m+1)(K_X + B).$$

Since ϕ is a morphism and M is base point free, by Lemma 6.12, we have that

$$\begin{aligned}
\phi_*(B_{red}) \cdot H^{d-1} &\leq D_0 \cdot (2(2d+1)M)^{d-1} \leq 2^d \mathrm{vol}(K_X + D_0 + 2(2d+1)M) \\
&\leq 2^d \mathrm{vol}(K_X + \alpha(B + \Gamma) + 2(2d+1)(M + E + B)) \\
&\leq 2^d \mathrm{vol}(K_X + B + (\alpha + 2(2d+1))(m+1)(K_X + B)) \leq 2^d (1 + 2\alpha(m+1))^d \mathrm{vol}(K_X + B) \\
&\leq 2^{3d} \alpha^d \mathrm{vol}((m+1)(K_X + B)) \leq 2^{4d} \alpha^d v.
\end{aligned}$$

The rest of the proof also closely follows the proof of Theorem 3.1, but there is a key difficulty that appears when we try to bound the volume of $K_X + B$ along a non-klt center V. Assume that (X, B) is as above ($\dim X = d$, $B \in I$ and $\mathrm{vol}(K_X + B) = v$) and for simplicity $K_X + B$ is ample. Suppose that $D \sim_{\mathbb{Q}} \lambda(K_X + B)$ where $\mathcal{J}(X, D) = \mathcal{I}_V$ near a general point $x \in V \subset X$. We must find a log canonical pair (V, B_V) such that

(1) $K_V + B_V$ is log canonical, big and $B_V \in J$ for some DCC set J and
(2) $(K_X + B + D)|_V \geq K_V + B_V$.

It is clear that $(K_X+B+D)|_V$ is big, however (V, B_V) is typically not lc and we have no control over the coefficients of D and hence of B_V. Instead we proceed as follows. Let $V^\nu \to V$ be the normalization and $V' \to V$ a resolution,

$$D(I) := \{a \leq 1 | a = \frac{m-1-\sum i_t}{m},\ m \in \mathbb{N},\ i_t \in I\}$$

be the derived set of I.

Exercise 6.10. *Show that if I is a DCC set then so is $D(I)$.*

Claim 6.11. *There exists a divisor Θ on V^ν, with $\Theta \in \{1-t|t \in \mathrm{LCT}_{d-1}(D(I)) \cup 1\}$ such that*

1. *$(K_X+B+D)|_{V^\nu} - (K_{V^\nu}+\Theta)$ is PSEF and*
2. *if $x \in V \subset X$ is a general point, then $K_{V'}+\Theta' \geq (K_X+B)|_{V'}$ where Θ' is the strict transform of Θ plus the $V' \to V^\nu$ exceptional divisors.*

Granting the claim, since K_X+B is big, it follows that $K_{V'}+\Theta'$ is also big. By Theorem 6.7 in dimension $\leq d-1$, the set of log canonical thresholds

$$\{t \in \mathrm{LCT}_{d-1}(D(I)) \cup 1\}$$

is an ACC set (any non-decreasing sequence is eventually constant) and so, the coefficients of Θ' belong to a DCC set J. But then by Theorem 6.1 in dimension $\leq d-1$, the volume of $K_{V'}+\Theta'$ is bounded from below (by a constant), and so proceeding as in the proof of Theorem 3.1, after finitely many steps, we may assume that $\dim V = 0$ as required. (In more detail, if $\mathrm{vol}(K_{V'}+\Theta') > C > 0$, then there is a divisor $D_{V'} \sim \lambda'(K_{V'}+\Theta')$ such that

1. $\mathrm{mult}_{x'}(D_{V'}) > \dim V$, where $x' \in V'$ is a general point, and
2. $\lambda' < \dim V / C^{1/d}$.

Note that if $h : V' \to V$ is the induced morphism, then there is a divisor D' on X such that

1. $D'|_V \geq h_* D_{V'}$ (on an open subset $x' \in U \subset V$), and
2. $D' \sim_{\mathbb{Q}} \lambda'(K_X+B+D) \sim_{\mathbb{Q}} \lambda'(1+\lambda)(K_X+B)$.

by Proposition 2.40, the non klt locus of $(X, D+(1-\epsilon)B+B')$ contains x' but does not contain V. Replacing D by $D+(1-\epsilon)D'$ and repeating this process at most $d-1$ times, we may assume that x' is an isolated component of $\mathcal{J}(X, B+D)$. It follows that $|m(K_X+B)|$ defines a birational map for any $m \geq A\mathrm{vol}(K_X+B)^{-1/d}+B$ where $A, B > 0$ are constants.

To gain some intuition on why the above claim works, we begin by defining Θ. After perturbing D, we may assume that $(X, B+D)$ is log canonical at the generic point of V and has a unique non-klt center S dominating the generic point of V. By Theorem 5.44 there exists a birational morphism $\nu : Y \to X$ such that Y is $\mathbb{Q}$-factorial, if E is a ν-exceptional divisor, then $a_E(X, B+D) \leq -1$, and S is a ν exceptional divisor. We write

$$K_Y + S + \Gamma = \nu^*(K_X + B) + E, \qquad K_S + \Phi = (K_Y + S + \Gamma)|_S,$$

$$K_Y+S+\Gamma+\Gamma' = \nu^*(K_X+B+D)+E, \qquad K_S+\Phi' = (K_Y+S+\Gamma+\Gamma')|_S.$$

Assume for simplicity that S is normal. Notice that $\Gamma \in I$ and so $\Phi \in D(I)$. For any codimension 1 point $P \in V^\nu$ define $t_p = \mathrm{lct}_{\eta P}(S, \Phi; f|_S^* P)$ where the log canonical threshold is computed over η_P the generic point of P. We then let $\Theta = \sum(1-t_P)P$ and define Θ' similarly for (S, Φ'). By Kawamata subadjunction $(K_X+B+D)|_{V^\nu} - (K_{V^\nu} + \Theta')$ is pseudo-effective. Since $\Theta' \geq \Theta$, it follows that $(K_X + B + D)|_{V^\nu} - (K_{V^\nu} + \Theta)$ is pseudo-effective. (1) of the claim follows.

Part (2) of the claim is much more technical. However, suppose that $V = S$ is a divisor on X, then $\Theta = \Phi$ and so $K_V + \Theta = (K_X+S+\Gamma)|_S = (K_X+B)|_S + E|_S$, however note that $E = (1-b)S$ where $b = \mathrm{mult}_S(B)$. Since $x \in S \subset X$ is a very general point, we may assume that S varies in a family covering X and hence $S \sim S' \neq S$ and so $S|_S \sim S'|_S \geq 0$. $\square$

Lemma 6.12. *Let X be a normal projective variety of dimension d and M a base point free Cartier divisor such that the induced morphism ϕ_M is birational. Let $H = 2(2d+1)M$ and D a sum of distinct prime divisors, then*

$$D \cdot H^{d-1} \leq 2^d \mathrm{vol}(K_X + D + H).$$

Proof. We may discard all ϕ_M-exceptional components of D. Let $f : Y \to X$ be a log resolution of the pair (X, D) and G the strict transform of D on Y, then

$$D \cdot H^{d-1} = G \cdot (f^*H)^{d-1} \qquad \text{and}$$

$$\mathrm{vol}(K_Y + G + f^*H) \leq \mathrm{vol}(K_X + D + H).$$

Therefore, replacing X, D, M by Y, G, f^*M, we may assume that (X, D) is log smooth and that the components of D are disjoint.

We may write $M \sim_{\mathbb{Q}} A+B$ where A is ample and $B \geq 0$ has no components in common with D (as no component of D is Φ_M exceptional). But then $K_X + D + \delta B$ is dlt for any $1 \gg \delta > 0$ and so

$$H^i(\mathcal{O}_X(K_X + E + pM)) = 0 \qquad \forall p, i \in \mathbb{N}$$

and any integral Weil divisor $0 \leq E \leq D$.[1] Let

$$A_m = K_X + D + mH, \qquad \text{then} \qquad H^i(\mathcal{O}_D(A_m)) = 0,$$

for all $i, m > 0$. therefore, there is a degree $d-1$ polynomial

$$p(M) = H^0(\mathcal{O}_D(A_m)).$$

By Exercise 6.5, $A_1 = K_X + D + H$ is big and so $K_X + D + H$ has a log canonical model $X \dashrightarrow X_{lc}$ (Theorem 5.40). Thus there is a degree d polynomial

$$Q(m) := h^0(\mathcal{O}_X(2mA_1)) = h^0(\mathcal{O}_{X_{lc}}(2m(K_{X_{lc}} + D_{lc} + H_{lc}))),$$

for any sufficiently divisible integer m. The leading coefficients of $P(m)$ and $Q(m)$ are

$$\frac{D \cdot H^{d-1}}{(d-1)!} \qquad \text{and} \qquad \frac{2^d \operatorname{vol}(K_X + D + H)}{d!}.$$

If $D = \sum D_i$ where the D_i are prime and $M_i = (D - D_i + (2d+1)M)|_{D_i}$, then

$$H^0(\mathcal{O}_X(K_X + D + (2d+1)M)) \to H^0(\mathcal{O}_{D_i}(K_{D_i} + M_i)),$$

is surjective, and so the general section of $H^0(\mathcal{O}_X(K_X+D+(2d+1)M))$ does not vanish identically along any component of D. Let

$$s \in H^0(\mathcal{O}_X(K_X+D+(2d+1)M)), \qquad \text{and} \qquad l \in H^0(\mathcal{O}_X((2d+1)M))$$

be sections not vanishing identically along any component of D. If

$$t = s^{\otimes 2d-1} \otimes l \in H^0(\mathcal{O}_X(2dA_1 - A_m)),$$

then t induces an injection $H^0(\mathcal{O}_X(A_m)) \to H^0(\mathcal{O}_X(2mA_1))$. Since $H^1(\mathcal{O}_X(A_m - D)) = 0$, we have a surjection

$$H^0(\mathcal{O}_X(A_m)) \to H^0(\mathcal{O}_D(A_m))$$

and so

$$t|_D \cdot H^0(\mathcal{O}_D(A_m)) \subset \operatorname{Im}\left(H^0(\mathcal{O}_X(2mA_1)) \to H^0(\mathcal{O}_D(2mA_1))\right).$$

Thus

$$P(m) \leq h^0(\mathcal{O}_X(2mA_1)) - h^0(\mathcal{O}_X(2mA_1 - D)).$$

Since $h^0(\mathcal{O}_X(2K_X + D + 2H)) \neq 0$, we have

$$Q(m-1) = h^0(\mathcal{O}_X(2(m-1))A_1)) \leq h^0(\mathcal{O}_X(2mA_1 - D)).$$

Therefore $P(m) \leq Q(m) - Q(m-1)$ and the claim follows by comparing the leading coefficients of $P(m)$ and $Q(m)$. $\square$

[1] In fact $E + pM \sim_{\mathbb{Q}} (1-\epsilon)E + \delta B + \epsilon E + \delta A$ where, for $0 < \epsilon \ll \delta \ll 1$, $(X, (1-\epsilon)E + \delta B)$ is klt and $\epsilon E + \delta A$ is ample.

Proof that Theorem 6.9 implies Theorem 6.1. By the proof of Theorem 6.9 to prove (1), it suffices to show that the set $\mathcal{V}_{d,I}$ has a positive lower bound. In particular it suffices to show (2).

We will now prove (2). Let $f : \mathcal{Z} \to T$ be the family (depending on d, I, v) constructed in Theorem 6.9. Replacing T be a closed subset, we may assume that the pairs $(X, B) \in \mathcal{P}_{d,I}$ such that $\mathrm{vol}(K_X + B) \leq v$ are birational to fibers $(\mathcal{Z}_t, \mathcal{B}_t)$ for some $t \in T$. We may assume that $(\mathcal{Z}, \mathcal{B})$ is log smooth over T. Blowing up (X, B), we may assume that $f : X \to \mathcal{Z}_t$ is a morphism. Unluckily it is not clear that $\mathrm{vol}(K_X+B) = \mathrm{vol}(K_{\mathcal{Z}_t} + \mathcal{B}_t)$ (even though the inequality $\leq$ is clear). Notice however that

Claim 6.13. *We have*

$$\mathrm{vol}(K_X + B) = \mathrm{vol}(K_X + B \wedge \mathbf{L}_{f_*B,X})$$

where

$$L_{f_*B,X} = -\sum_{E \subset X | a_E(\mathcal{Z}_t, f_*B) \leq 0} a_E(\mathcal{Z}_t, f_*B)E.$$

Proof. Notice that

$$H^0(m(K_X + B)) \subset H^0(m(K_{\mathcal{Z}_t} + f_*B)) = H^0(m(K_X + \mathbf{L}_{f_*B,X}))$$

so that every section of $H^0(m(K_X+B))$ actually belongs to $H^0(m(K_X+B \wedge \mathbf{L}_{f_*B,X}))$. $\square$

Notice that as $(\mathcal{Z}_t, \mathcal{B}_t)$ has simple normal crossings and $f_*B \leq \mathcal{B}_t$, every divisor in $\mathbf{L}_{f_*B,X}$ may be obtained by a finite sequence of blow ups along strata of the strict transform of $\mathcal{B}_t$ and exceptional divisors. Let $X' \to \mathcal{Z}_t$ be a sequence of blow ups along strata of the strict transform of $\mathcal{B}_t$ and exceptional divisors such that each component of $\mathbf{L}_{f_*B,X}$ is a divisor on X' and let B' be the strict transform of B plus the $X' \dashrightarrow X$ exceptional divisors.

Claim 6.14. *We have* $\mathrm{vol}(K_X + B) = \mathrm{vol}(K_{X'} + B')$*.*

Proof. Let $g : W \to X$ be a resolution of the indeterminacies of $X \dashrightarrow X'$. Clearly $\mathrm{vol}(K_X + B) = \mathrm{vol}(K_W + \mathbf{M}_{B,W})$ where $\mathbf{M}_{B,W} = g_*^{-1}B + \mathrm{Ex}(g)$. It is easy to see that $\mathbf{M}_{B,W} \geq \mathbf{M}_{B',W} \wedge \mathbf{L}_{f_*B,W}$. We only need to check divisors E that are exceptional for $W \to X'$ but not for $W \to X$, but for these divisors we have $\mathrm{mult}_E(\mathbf{L}_{f_*B,W}) = 0$ and so the inequality follows. $\square$

Consider now the connected component of T containing t, $t_0 \in T$ a fixed point of this component and let $\mathcal{Z}' \to \mathcal{Z}$ be the sequence of blow ups of $\mathcal{Z}$ along strata of $\mathbf{M}_{\mathcal{B}}$ such that $\mathcal{Z}'_t \cong X'$. Let $\mathcal{B}'$ be the $\mathbb{Q}$-divisor such that $\mathcal{B}'_t = B'$. By Theorem 5.48, we have

$$\mathrm{vol}(K_X + B) = \mathrm{vol}(K_{X'} + B') = \mathrm{vol}(K_{\mathcal{Z}'_t} + \mathcal{B}'_t) = \mathrm{vol}(K_{\mathcal{Z}'_{t_0}} + \mathcal{B}'_{t_0}).$$

It follows that we may assume that (X, B) is obtained from a fixed pair (Z, B_Z) via a finite sequence of blow ups (here we assume that B_Z is reduced, the support of B is contained in $\mathbf{B}_{B_Z,X}$, and the coefficients of B belong to $I \cup \{1\}$).

Suppose now by way of contradiction that we have a sequence of pairs (X_i, B_i) and morphisms $f_i : X_i \to Z$ such that $f_{i,*}B_i \le B_Z$ and $\mathrm{vol}(K_{X_i} + B_i)$ is strictly decreasing. Since the coefficients of B_i are in a DCC set, we may assume that (after passing to a subsequence) the coefficients of $f_{i,*}B_i$ are non decreasing and in particular they admit a limit. Let $\Delta = \lim f_{i,*}B_i$ be this limit. Suppose that $\lfloor \Delta \rfloor = 0$ and let $Z' \to Z$ be a terminalization of (Z, Δ). We may assume that $\nu : Z' \to Z$ is given by a sequence of blow ups along strata of $\mathbf{M}_\Delta$ and writing $K_{Z'} + \Delta' = \nu^*(K_Z + \Delta)$, then $\Delta' \ge 0$ and (Z', Δ') is terminal. Replacing X_i by a higher model, we may assume that $f'_i : X_i \dashrightarrow Z'$ is a morphism. Passing to another subsequence, we may assume that the coefficients of $f'_{i,*}B_i$ are non-decreasing. Since (Z', Δ') is terminal and

$$f'_{i,*}B_i \wedge \mathbf{L}_{f_{i,*}B_i,Z'} \le \Delta',$$

we have that $(Z', f'_{i,*}B_i \wedge \mathbf{L}_{f_{i,*}B_i,Z'})$ is also terminal and hence

$$\mathrm{vol}(K_{X_i}+B_i) \le \mathrm{vol}(K_{Z'}+f'_{i,*}B_i) = \mathrm{vol}(K_{Z'}+f'_{i,*}B_i\wedge\mathbf{L}_{f_{i,*}B_i,Z'}) \le \mathrm{vol}(K_{X_i}+B_i).$$

(the second equality follows as in Claim 6.13 and the last inequality since $K_{Z'} + f'_{i,*}B_i \wedge \mathbf{L}_{f_{i,*}B_i,Z'}$ is terminal and $(f'_i)^{-1}_*(f'_{i,*}B_i \wedge \mathbf{L}_{f_{i,*}B_i,Z'}) \le B_i$.) Since $f'_{i,*}B_i \ge f'_{i+1,*}B_{i+1}$, it follows that

$$\mathrm{vol}(K_{X_i}+B_i) = \mathrm{vol}(K_{Z'}+f'_{i,*}B_i) \ge \mathrm{vol}(K_{Z'}+f'_{i+1,*}B_{i+1}) = \mathrm{vol}(K_{X_{i+1}}+B_{i+1}).$$

This contradicts the fact that $\mathrm{vol}(K_{X_i} + B_i)$ is decreasing.

Finally, we prove (3). Fix $v = \mathrm{vol}(K_{X_i^c} + B_i^c)$. Assume by way of contradiction that (X_i^c, B_i^c) (and any infinite subsequence thereof) is an unbounded sequence. Arguing as above (replacing X_i^c by an appropriate birational model $X_i \to X_i^c$) , we may assume that $f_i : X_i \to \mathcal{Z}_{t_i}$ is a birational morphism and $f_{i,*}B_i$ is supported on $\mathcal{B}_{t_i}$. passing to a subsequence, we assume that all t_i belong to the same component of T. We define Δ_i on $\mathcal{Z}$ as the divisor supported on $\mathcal{B}$ such that $\Delta_i|_{\mathcal{Z}_{t_i}} = f_{i,*}B_i$. Passing to a subsequence, we may assume that $\Delta_i \le \Delta_{i+1}$. let $\Delta = \lim \Delta_i$. Assume that $(\mathcal{Z}, \Delta)$ is terminal (see [HMX14b] for details of the log canonical case). Let $\nu : \mathcal{Z}' \to \mathcal{Z}$ be a terminalization so that $K_{\mathcal{Z}'} + \Delta' = \nu^*(K_{\mathcal{Z}} + \Delta)$ where $(\mathcal{Z}', \Delta' \ge 0)$ is terminal. We may assume that $f_i' : X_i \dashrightarrow \mathcal{Z}'_{t_i}$ is also morphism. As observed above,

$$v = \mathrm{vol}(K_{X_i} + B_i) = \mathrm{vol}(K_{\mathcal{Z}'_{t_i}} + (\Delta_i)_{t_i}) = \mathrm{vol}(K_{\mathcal{Z}'_{t_0}} + (\Delta_i)_{t_0})$$

and so

$$\mathrm{vol}(K_{\mathcal{Z}'_{t_i}} + \Delta_{t_i}) = \mathrm{vol}(K_{\mathcal{Z}'_{t_0}} + \Delta_{t_0}) = \lim \mathrm{vol}(K_{\mathcal{Z}'_{t_i}} + (\Delta_i)_{t_0}) = v.$$

By Lemma 5.45, X_i^c coincides with the log canonical model of $(\mathcal{Z}_{t_i}, \Delta_{t_i})$. Consider now the log canonical model $g : \mathcal{Z} \dashrightarrow \mathcal{Z}^c$ for $(\mathcal{Z}, \Delta)$ over T. This log canonical model exists by Theorem 5.48 and $(X_i^c, B_i^c) = (\mathcal{Z}^c, g_*\Delta_{t_i})$. □

7. Rational curves on varieties with negative K_X

We follow the treatment of [KM98, §1]

Theorem 7.1. *Let X be a smooth variety and $f : Y \to X$ a projective birational morphism. For any $x \in X$, the fiber $f^{-1}(x)$ is either a point or is covered by rational curves.*

Proof. Suppose that $\dim X = 2$. Consider the rational map $f^{-1} : X \dashrightarrow Y$ and resolve its indeterminacies by a finite sequence of blow ups (at smooth points), $\nu : X' \to X$ so that $g : X' \to Y$ is a morphism. Then $\nu^{-1}(x)$ is a union of rational curves and so $f^{-1}(x) = g(\nu^{-1}(x))$ is also a union of (possibly singular) rational curves. The general case was shown by Abhyankar (in 1956). □

Corollary 7.2. *Let $g : X \dashrightarrow Y$ be a rational map and $Z \subset X \times Y$ be the closure of the graph of g, and p, q the projections. If $x \in X$ is a smooth point such that g is not a morphism at x, then $q(p^{-1}(x))$ is covered by rational curves.*

Proof. By Theorem 7.1, $p^{-1}(x)$ is either a point or is covered by rational curves. If $p^{-1}(x)$ is a point, then it is easy to see that X is isomorphic to Z on a neighborhood of x and hence that g is a morphism at $x \in X$. Therefore $q(p^{-1}(x))$ is covered by rational curves. □

Corollary 7.3. *Let $g : X \dashrightarrow Y$ be a rational map which is not everywhere defined. If X is smooth and Y is projective, then Y contains a rational curve.*

Proof. Immediate from Corollary 7.2. □

Lemma 7.4. *Let $f : Y \to Z$ be a projective morphism such that Y is irreducible, f is surjective and every fiber is connected of dimension n. If $g : Y \to X$ is a morphism such that $g(f^{-1}(z_0))$ is a point for some $z_0 \in Z$, then $g(f^{-1}(z))$ is a point for every $z \in Z$.*

Proof. Let W be the image of $f \times g : Y \to Z \times X$ and $f = p \circ h : Y \to W \to Z$ be the induced morphism, then $\dim p^{-1}(z_0) = 0$ and so by semicontinuity of the fiber dimension, there is a neighborhood $z_0 \in U \subset Z$ such that $\dim p^{-1}(z) = 0$. But then the fibers of $Y \to W$ over points of $p^{-1}(U)$ are n-dimensional and thus every fiber of $Y \to W$ has dimension $\geq n$. Since $h^{-1}(w) \subset f^{-1}(p(w))$ while $\dim h^{-1}(w) \geq n$ and $\dim f^{-1}(p(w)) = n$, it follows that $h^{-1}(w)$ is a union of components of $f^{-1}(p(w))$. But then since $h^{-1}(p^{-1}(p(w))) = f^{-1}(p(w))$ it follows that each component of $f^{-1}(p(w))$ maps to a point in $(p^{-1}(p(w))$ and so $(p^{-1}(p(w))$ is finite and hence a single point (since f has connected fibers).

□

Corollary 7.5. *[Bend and Break] let X be a projective variety, $p \in C$ a point on a projective curve and $g_0 : C \to X$ be a non-constant morphism. Suppose that there is a smooth pointed curve $0 \in D$ and a morphism $G : C \times D \to X$ such that*

(1) $G|_{C\times 0} = g_0$,
(2) $G(\{p\} \times D) = g_0(p)$, *and*
(3) $G|_{C\times\{t\}} \neq g_0$ *for general* $t \in D$,

then there is a morphism $g_1 : C \to X$, and a positive linear combination of rational curves $Z = \sum a_i Z_i$ such that

(1) $(g_0)_*(C) \equiv (g_1)_*(C) + Z$, *and*
(2) $g_0(p) \in \cup_i Z_i$.

Therefore X contains a rational curve through $g_0(p)$.

Proof. We may assume that $D \subset \bar{D}$ is projective (after compactifying) and we have a rational map $\bar{G} : C \times \bar{D} \dashrightarrow X$. We claim that $\bar{G}$ is not defined at some point of $p \times \bar{D}$. If this were not the case, then there is a neighborhood $p \in U \subset C$, such that $\bar{G}$ is defined on $U \times \bar{D}$. Since $\bar{G}(p \times \bar{D})$ is a point, by Lemma 7.4, $\bar{G}(p' \times \bar{D})$ is a point for any $p' \in U$. But then $\bar{G}$ is constant on $U \times \bar{D}$ which is impossible.

Let S be the normalization of the closure of the graph of $\bar{G}$, $\pi : S \to C \times \bar{D}$, $G_S : S \to X$ and $h : S \to C \times \bar{D} \to \bar{D}$. By what we have seen above, there is a point $(p, d) \in C \times \bar{D}$ such that π is not an isomorphism over (p, d). Let $h^{-1}(d) = C' + E$ where $C' = \pi^{-1}_* C \times \{d\}$ and E is π-exceptional and hence a union of rational curves by Theorem 7.1. we let $g_1 = G_S|_{C'}$ and $Z = G_S(E)$. Then $g_{0,*}(C)$ is algebraically equivalent to $g_{1,*}(C) + Z$. □

Note that if C is rational, Corollary 7.5 gives no new information.

Lemma 7.6. *[Bend and Break for rational curves] let X be a projective variery and $g : \mathbb{P}^1 \to X$ a nonconstant morphism. Suppose that there is a smooth pointed curve $0 \in D$ and a morphism $G : \mathbb{P}^1 \times D \to X$ such that*

1. *$G|_{\mathbb{P}^1 \times 0} = g_0$,*
2. *$G(\{0\} \times D) = g_0(0)$, $G(\{\infty\} \times D) = g_0(\infty)$, and*
3. *$G|_{\mathbb{P}^1 \times D} \neq g_0$ is a surface.*

Then $g_{0,*}(\mathbb{P}^1)$ is algebraically equivalent to a reducible curve or a multiple curve.

Proof. We may assume that $D \subset \bar{D}$ is projective (after compactifying) and $q : S \to \bar{D}$ is a $\mathbb{P}^1$ bundle containing $\mathbb{P}^1 \times D$ extending the projection $\mathbb{P}^1 \times D \to D$. Let $\bar{G} : S \dashrightarrow X$ be the induced rational map and $\tilde{r} : \tilde{S} \to S$ a sequence of blow ups such that the induced map $\tilde{G} : \tilde{S} \to X$ is a morphism. We proceed by induction on the number k of blow ups in $\tilde{r}$.

If $k = 0$, then $\bar{G}$ is a morphism. Let H be ample on X and $C_0, C_\infty \subset S$ be the closures of $\{0\} \times D$ and $\{\infty\} \times D$. We have

$$(\bar{G}^*H)^2 > 0, \qquad (C_0 \cdot \bar{G}^*H) = 0 = (C_\infty \cdot \bar{G}^*H).$$

But then $(C_0)^2 < 0$ and $(C_\infty)^2 < 0$ (by the Hodge Index Theorem) and so $\bar{G}^*H, C_0, C_\infty$ are linearly independent in $NS(X)$. However, S is a $\mathbb{P}^1$ bundle and so this group is 2 dimensional (generated by a fiber and a section). This is the required contradiction.

Therefore $k > 0$ and we let $r : S' \to S$ be the first blow up (say at $P \in q^{-1}(y)$) and $r' : \tilde{S} \to S'$ the induced morphism. We may assume that $\tilde{G}_*((q \circ \tilde{r})^*(y))$ is irreducible and reduced (otherwise the statement holds). Let F_1 be the exceptional curve of r and F_2 the strict transform of q^*y so that $(q \circ r)^*(y) = F_1 + F_2$ is the union of two -1 curves meeting transversely at $Q = F_1 \cap F_2$.

We claim that $\bar{G}' : S' \to X$ is a morphism along F_2. Otherwise $\bar{G}'$ is undefined at $P' \in F_2$. If $P' \neq Q$, then

$$\tilde{G}_*((q \circ \tilde{r})^*(y)) = \tilde{G}_*\mathrm{red}(\tilde{r}^{-1}(P)) + \tilde{G}_*\mathrm{red}(\tilde{r}^{-1}(P') + (\text{effective cycle}),$$

which is a contradiction. Thus $P' = Q$, and r' extracts divisors over Q. However each one of these divisors must appear with multiplicity ≥ 2 in $\tilde{r}^*q^*(y)$ and hence must be contracted by $\tilde{G}$. Thus $\bar{G}' : S' \to X$ is a morphism.

Let $S' \to S''$ be the contraction of the -1 curve F_2, then the indeterminacies of $S'' \dashrightarrow X$ can be resolved via $k - 1$ blow ups and so we are done by induction. $\square$

Theorem 7.7. [Mori82] *let X be a smooth projective variety such that $-K_X$ is ample, then through every point $x \in X$ there is a rational curve D such that*

$$0 < -(D \cdot K_X) \leq \dim X + 1.$$

Proof. Let $0 \in C$ be a pointed curve and $f : C \to X$ be a non constant morphism such that $f(0) = x$. The morphism f admits a deformation space of dimension

$$\geq h^0(C, f^*T_X) - h^1(C, f^*T_X) = \chi(C, f^*T_X) =$$

$$-(f_*C \cdot K_X) + (1 - g(C)) \cdot \dim X.$$

Suppose that this dimension is $\geq m$, then there exists an m-dimensional pointed affine variety $0 \in Z$ and a morphism $F : C \times Z \to X$ such that $F|_{C\times\{0\}} = f$ and $F|_{C\times\{z\}} \neq F|_{C\times\{0\}}$ for $0 \neq z \in Z$. Fixing the image of the base point $0 \in C$ is at most $\dim X$ conditions, the deformation space of morphisms $f : (C, 0) \to (X, x)$ has dimension

$$\geq h^0(C, f^*T_X) - h^1(C, f^*T_X) - \dim X = -(f_*C \cdot K_X) - g(C) \cdot \dim X.$$

This means that if $-K_X \cdot f_*C > g(C) \cdot \dim X$, then there is a non-trivial one parameter family of deformations of C fixing $f(0) = x$. This inequality is automatically satisfied by rational curves $C \cong \mathbb{P}^1$. If C is an elliptic curve, then let $n : C \to C$ be the morphism induced by multiplication by an integer $n \in \mathbb{N}$. We have

$$-((f \circ n)_*C \cdot K_X) = n^2(f_*C \cdot K_X) > \dim X$$

as soon as $n > \sqrt{\dim X}$. Thus a multiple of C moves with a fixed point. If $g(C) \geq 2$ then there are no endomorphisms of degree > 1 so a similar approach does not work. However, in charactersitic $p > 0$ we can use the Frobenius morphism.

Suppose for simplicity that X and C are defined by equations

$$h_1, \ldots, h_r \in \mathbb{Z}[x_0, \ldots, x_n], \qquad c_1, \ldots, c_s \in \mathbb{Z}[y_0, \ldots, y_m].$$

In this case we have morphisms $X \to \operatorname{Spec}(\mathbb{Z})$ and $C \to \operatorname{Spec}(\mathbb{Z})$. Note that these morphisms are generically flat. Reducing modulo p (i.e. taking the fiber over $(p) \in \operatorname{Spec}(\mathbb{Z})$ we get equations in $\mathbb{F}_p[x_0, \ldots, x_n]$ and $\mathbb{F}_p[y_0, \ldots, y_m]$ which define projective varieties

$$X_p \subset \mathbb{P}^n_{\bar{\mathbb{F}}_p}, \qquad C_p \subset \mathbb{P}^m_{\bar{\mathbb{F}}_p}.$$

Note that for all but finitely many $p \in \operatorname{Spec}(\mathbb{Z})$, X_p and C_p are smooth. The Frobenius morphism $F = F_{\mathbb{P}^n} : \mathbb{P}^m \to \mathbb{P}^m$ is defined by $F(y_0, \ldots, y_m) = (y_0^p, \ldots, y_m^p)$ and similarly for $F : \mathbb{P}^n \to \mathbb{P}^n$. Notice that $h_i(y_0, \ldots, y_m)^p = h_i(y_0^p, \ldots, y_m^p)$ and so we get a morphism

$F_{C_p} : C_p \to C_p$. Note that F_{C_p} induces a morphism of degree p which is an isomorphism of topological spaces. The values of

$$(f_p)_* C_p \cdot K_{X_p}, \qquad g(C_p), \qquad \chi(T_X|_{C_p})$$

are constant for almost all p. Consider the morphism

$$f_p \circ F^m_{C_p} : C_p \to C_p \to X_p,$$

then

$$(f_p \circ F^m_{C_p})_*(C_p) \cdot K_{X_p} = C_p \cdot (F^m_{C_p})^* f_p^* K_{X_p} = p^m C_p \cdot f_p^* K_{X_p}.$$

Note that $C_p \cdot f_p^* K_{X_p}$ is a fixed negative number for almost all p. But then the space of deformations of $f_p \circ F^m_{C_p}$ with a fixed point has dimension at least

$$-p^m C_p \cdot f_p^* K_{X_p} - g(C_p) \dim X > 0, \qquad \text{for } m \gg 1.$$

Fix $x \in X$, then arguing as above, we obtain a rational curve $x \in \Sigma_p \subset X_p$. If $\Sigma_p \cdot (-K_{X_p}) > \dim X + 1$, we have that the deformation space of the corresponding morphism from the normalization $\Sigma'_p \to X_p$ that fixes two points p, p' has dimension at least 2. Since automorphisms of $\mathbb{P}^1$ fixing two points deform in a 1 parameter family, the image Σ_p must move in X_p and so Σ_p is algebraically equivalent to a sum of rational curves of lower degree. Therefore we may assume that $\Sigma_p \cdot (-K_{X_p}) \leq \dim X + 1$. It follows by standard arguments that there exists a rational curve $\Sigma \subset X$ such that $\Sigma \cdot -K_X \leq \dim X + 1$.

To see the last claim, note that a morphism $\mathbb{P}^1_{\mathbb{F}_p} \to X_p$. Since $-K_X$ is ample, we may assume that a fixed multiple is very ample and hence so is the corresponding multiple of $-K_{X_p}$. Since the degree of Σ_p is bounded (by $\dim X + 1$), we may assume that $\Sigma_p \to X_p \subset \mathbb{P}^n_{\mathbb{Z}_p}$ is defined by homogeneous forms $(g_{p,0}, \ldots, g_{p,n})$ such that

$$h_i(g_{p,0}, \ldots, g_{p,n}) \qquad 1 \leq i \leq r.$$

We view this as a system of polynomial equations in the coefficients of the polynomials g_{p_j}. Since these polynomials have a common solution for infinitely many primes p, they have a solution over $\bar{\mathbb{Q}}$ (or any algebraically closed field).[2] $\square$

Theorem 7.8. *Let X be a smooth Fano variety of dimension d and Picard number $\rho(X) = 1$, then there is an open subset $U \subset X$ such that if $x, y \in U$, then x, y can be joined by an irreducible rational curve of anticanonical degree at most $d(d+1)$.*

[2] These equations define a closed subset of $\mathbb{P}^N_{\mathrm{Spec}\mathbb{Z}}$. The projection to $\mathrm{Spec}\mathbb{Z}$ is proper and the image is dense and hence must contain the generic point.

Proof. See [Kollár96, V Proposition 2.6]. The idea is to show that we can connect x, y by a chain of rational curves of degree $\leq d(d+1)$ and then show that this chain can be smoothed. $\square$

Theorem 7.9. *Fix $d \in \mathbb{N}$ then there exists a constant $c = c(d)$ such that if X be a smooth Fano variety of dimension d, then there is an open subset $U \subset X$ such that if $x, y \in U$, then x, y can be joined by a chain of rational curves of anticanonical degree at most c.*

Proof. See [KMM92]. $\square$

7.1. **Boundedness of smooth Fano varieties.**

Theorem 7.10. *Let X be a smooth complex projective variety of dimension d such that $-K_X$ is ample, then $(-K_X)^d \leq c(d)$ where $c(d)$ is the constant from Theorem 7.9. If moreover $\rho(X) = 1$, then $(-K_X)^d \leq (d(d+1))^d$.*

Proof. Let $x \in X$ be a general point. Assume that $\rho(X) = 1$. If $(-K_X)^d > (d(d+1))^d$, then we may find a divisor $D \sim -kK_X$ such that $\operatorname{mult}_x(D) \geq kd(d+1) + 1$. (In order to assume that k does not depend on $x \in X$, we use the fact that $\mathbb{C}$ is uncountable.) By Theorem 7.8, there is a rational curve $C \subset X$ of anticanonical degree $\leq d(d+1)$ passing through x and not contained in the support of D. Since $\operatorname{mult}_x(D) > d(d+1)$, then

$$C \cdot (-K_X) = \frac{1}{k} C \cdot D \geq \frac{kd(d+1)+1}{k} > d+1.$$

This is the required contradiction. $\square$

Theorem 7.11. *The set of smooth complex projective Fano varieties of dimension d is bounded.*

Proof. By the result of Anhern-Siu (Theorem 2.56), we know that $B := K_X - (1 + \binom{d+1}{2})K_X$ is generated by global sections. Let $n = 1 - d\binom{d+1}{2}$, then by Kodaira vanishing nK_X is 0-regular i.e. $H^i(nK_X - iB) = 0$ for $i = 1, 2, \ldots, d = \dim X$. By Lemma 7.12 below, it follows that $nK_X + B = -((d+1)\binom{d+1}{2} - 1)K_X$ is very ample. By Theorem 7.11, $|nK_X + B|$ embeds X as a subvariety $\mathbb{P}^N$ of bounded degree. The claim now follows from a Hilbert scheme type argument. $\square$

Lemma 7.12. *Let B be an ample and generated line bundle and A a line bundle such that $H^i(A - iB) = 0$ for all $i > 0$, then $A + B$ is very ample.*

Proof. We must show that $A+B$ is generated and for any point $x \in X$, $\mathcal{O}_X(A+B) \otimes \mathfrak{m}_x$ is also generated. The first statement follows from the arguments of Lemma 2.35 and so we focus on the second part. Let $V \subset H^0(B)$ be a general d dimensional subset of sections vanishing at $x \in X$. We may assume that if $s_1, \ldots, s_d$ is a basis of V, then $Z = V(s_1, \ldots, s_d)$ is a zero dimensional scheme containing x. Consider the corresponding Koszul complex $E^\bullet$

$$0 \to \mathcal{O}_X(A-(d-1)B) \otimes \wedge^d V \to \mathcal{O}_X(A-(d-2)B) \otimes \wedge^{d-1} V \to \ldots \to \mathcal{O}_X(A) \otimes V$$

which is exact with cokernel $\mathcal{O}_X(A+B) \otimes I_Z$. It suffices to show that $\mathcal{O}_X(A+B) \otimes I_Z$ is 0-regular. Notice in fact that since $\mathfrak{m}_x/I_Z$ is zero dimensional and so the short exact sequence

$$0 \to \mathcal{O}_X(A+B) \otimes I_Z \to \mathcal{O}_X(A+B) \otimes \mathfrak{m}_x \to \mathfrak{m}_x/I_Z \to 0$$

easily implies that $\mathcal{O}_X(A+B) \otimes \mathfrak{m}_x$ is also 0-regular.

Since $H^i(A-iB) = 0$ for $i > 0$, it follows easily that $H^i(A-jB) = 0$ for $i > 0$ and $j \leq i$. Notice that $H^i(\mathcal{O}_X(A-(i-1)B) \otimes I_Z) = \mathbb{H}^i(E^\bullet \otimes \mathcal{O}_X(-iB))$. We have $H^k(E^{-j} \otimes \mathcal{O}_X(-iB)) = H^k(\mathcal{O}_X(A-(i+j)B)) \otimes \wedge^j V = 0$ for $k \geq i+j$. An easy spectral sequence argument shows that $\mathbb{H}^i(E^\bullet \otimes \mathcal{O}_X(-iB))$ for all $i > 0$. $\square$

8. The Pseudo-effective cone

In this section we recall the proof of the following fundamental result of Boucksom-Demailly-Paun-Peternell.

Theorem 8.1. *Let X be a normal irreducible complex projective variety of dimension d, then the cones $\overline{\mathrm{Mov}}(X)$ and $\overline{\mathrm{Eff}}(X)$ are dual.*

Recall that the pseudo-effective cone $\overline{\mathrm{Eff}}(X) \subset N^1(X)_{\mathbb{R}}$ is the closure of the big cone. A $\mathbb{Q}$ divisor G is big iff $\mathrm{limsup} h^0(mG)/m^d > 0$. It is known that if G and G' are numerically equivalent then G is big iff G' is big. It is also known that if G is big, then the limit $\lim h^0(mG)/m^d$ exists. We set $\mathrm{vol}(G) = \lim d! h^0(mG)/m^d$. The volume function can be extended to a continuous function on the big cone and to a semi-continuous function $\mathrm{vol} : N^1(X)_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$. We refer the reader to [Lazarsfeld04] for a detailed discussion.

The cone of movable (or mobile) curve $\overline{\mathrm{Mov}}(X) \subset N_1(X)_{\mathbb{R}}$ is the closure of the convex cone spanned by all curves of the for

$$f_*(A_1 \cdots A_{d-1})$$

where $f : Y \to X$ is a projective birational morphism and the A_i are ample divisors on Y.

Corollary 8.2. *Let X be a smooth projective variety, then X is uniruled if and only if K_X is not pseudo-effective.*

Proof. By the arguments in the proof of Theorem 3.3, it is easy to see that if X is uniruled, then K_X is not pseudo-effective. Suppose in fact that X is uniruled, then there exists a projective morphism $f : Z \to T$ whose generic fiber is a rational curve and a dominant morphism $g : Z \to X$. Since K_X is pseudo-effective and $K_Z = g^*K_X + R$ where $R \geq 0$, it follows that K_Y is also pseudo-effective. But then $K_Z|_{Z_t}$ is also pseudo-effective. However this is impossible since $K_Z|_{Z_t} \cong K_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$ is not pseudo-effective.

Suppose now that K_X is not pseudo-effective. By Theorem 8.1, there is a curve $\gamma \in \overline{\text{Mov}}(X)$ such that $K_X \cdot \gamma < 0$ and hence we may find a covering family of curves C_t such that $K_X \cdot C_t < 0$. By an argument similar to the proof of Theorem 7.7, it follows that X is covered by rational curves. $\square$

Before proving the above theorem, we will need to recall several results of independent interest. First of all we recall the notation of asymptotic multiplier ideal sheaves. Let L be a line bundle on a smooth projective variety and $c > 0$. The multiplier ideal shef $\mathcal{J}(c \cdot |L|)$ is computed as follows. let $f : Y \to X$ be a resolution of $|L|$ so that $f^*|L| = M + E$ where M is base point free and E plus the exceptional divisor has simple normal crossings support. Then

$$\mathcal{J}(c \cdot |L|) = f_*\mathcal{O}_Y(K_{Y/X} - \lfloor c \cdot F \rfloor).$$

For any $k \in \mathbb{N}$, we may assume that f is also a log resolution of $|kL|$. Writing $f^*|kL| = M_k + F_k$ it is easy to see that $F_k \leq k \cdot F_k$ and so

$$\mathcal{J}(c \cdot |L|) = f_*\mathcal{O}_Y(K_{Y/X} - \lfloor c \cdot F \rfloor) \subset f_*\mathcal{O}_Y(K_{Y/X} - \lfloor \frac{c}{k} \cdot F_k \rfloor) = \mathcal{J}(\frac{c}{k} \cdot |kL|).$$

Since X is Noetherian, the inclusions

$$\mathcal{J}(c \cdot |L|) \subset \mathcal{J}(\frac{c}{2} \cdot |2L|) \subset \mathcal{J}(\frac{c}{2!} \cdot |2!L|) \subset \mathcal{J}(\frac{c}{3!} \cdot |3!L|) \subset \ldots$$

eventually stabilizes and we let

$$\mathcal{J}(c \cdot ||L||) = \mathcal{J}(\frac{c}{k!} \cdot |k!L|).$$

It is easy to see that $\mathcal{J}(c \cdot ||L||) = \mathcal{J}(\frac{c}{k} \cdot |kL|)$ for any k sufficiently divisible. These asymptotic multiplier ideals satisfy many useful properties.

Lemma 8.3. $\mathcal{J}(|L|) \subset \mathcal{J}(||L||)$ *and in particular*

$$H^0(L \otimes \mathcal{J}(||L||)) = H^0(L).$$

Proof. The inclusion $\mathcal{J}(|L|) \subset \mathcal{J}(||L||)$ was established above. It suffices to check that $H^0(L \otimes \mathcal{J}(|L|)) = H^0(L)$. The inclusion $\subset$ is obvious. Conversely, $\mathcal{J}(|L|) = f_*\mathcal{O}_Y(K_{Y/X} - F) \subset f_*\mathcal{O}_Y(-F) = \mathfrak{b}(L)$ where $\mathfrak{b}(L)$ is the base locus ideal of $|L|$ given by the image of $H^0(L) \otimes \mathcal{O}_X \to L \otimes \mathfrak{b}(L) \subset L$. $\square$

Proposition 8.4. *If $M - K_X - cL$ is nef and big, then $H^i(M \otimes \mathcal{J}(c \cdot ||L||)) = 0$ for $i > 0$ and if $M - K_X - cL - (d+1)B$ is nef for an ample and generated divisor B, then $M \otimes \mathcal{J}(c \cdot ||L||)$ is generated.*

Proof. We have $\mathcal{J}(c \cdot ||L||) = \mathcal{J}(\frac{c}{k} \cdot |kL|)$ for some k sufficiently divisible. The vanishing $H^i(M \otimes \mathcal{J}(c \cdot ||L||)) = 0$ for $i > 0$ then follows immediately from Nadel vanishing and the global generation of $M \otimes \mathcal{J}(c \cdot ||L||)$ follows from Castelnuovo Mumford regularity. $\square$

Theorem 8.5. *[Subadditivity of multiplier ideal sheaves] Let L be a divisor on X with $\kappa(L) \geq 0$. For $l, m \in \mathbb{N}$ and $c > 0$, we have*

$$\mathcal{J}(c \cdot ||(m+l)L||) \subset \mathcal{J}(c \cdot ||mL||) \cdot \mathcal{J}(c \cdot ||lL||)$$

and in particular $\mathcal{J}(c \cdot ||mL||) \subset \mathcal{J}(c \cdot ||L||)^m$.

Proof. We refer the reader to [Lazarsfeld04]. $\square$

Theorem 8.6. *[Fujita's approximation Theorem] Let L be a big $\mathbb{Q}$-divisor on X an irreducible projective variety of dimension d. For every $\epsilon > 0$ there exists a projective birational morphism $f : Y \to X$, an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor B such that $f^*L \sim_{\mathbb{Q}} A + B$ and*

$$\mathrm{vol}(A) \geq \mathrm{vol}(L) - \epsilon.$$

Proof. We follow [Lazarsfeld04]. After resolving the singularities of X, we may assume that X is smooth and L is nef and big. It suffices to show that there exists a nef $\mathbb{Q}$-divisor A with the above properties. If this is the case, then A is nef and big so that $A \sim_{\mathbb{Q}} H + F$ where H is ample and $F \geq 0$. but then $A - \delta F \sim_{\mathbb{Q}} (1-\delta)A + \delta H$ is ample and $\lim \mathrm{vol}(A - \delta F) = \mathrm{vol}(A)$.

Let B be a very ample divisor on X such that $K_X + (d+1)B$ is very ample. For any $p \geq 0$ we set $M_p = pL - K_X - (d+1)B$ and $\mathcal{J}_p = \mathcal{J}(||M_p||)$. Note that M_p is big for $p \gg 0$. In fact we have

$$\frac{M_p}{p} = L - \frac{K_X + (d+1)B}{p} \qquad \text{so that} \qquad \mathrm{limvol}(\frac{M_p}{p}) = \mathrm{vol}(L).$$

In particular

$$\mathrm{vol}(M_p) \geq p^d(\mathrm{vol}(L) - \epsilon) \qquad \text{for } p \gg 0.$$

$L_p \otimes \mathcal{J}_p$ is generated by global sections (by Proposition 8.4).By Theorem 8.5 $\mathcal{J}(||lM_p||) \subset \mathcal{J}(||M_p||)^l$. It follows that

$$H^0(\mathcal{O}_X(lM_p)) = H^0(\mathcal{O}_X(lM_p) \otimes \mathcal{J}(||M_p||)^l) \subset H^0(\mathcal{O}_X(lpL) \otimes \mathcal{J}(||M_p||)^l)$$

and so

$$H^0(\mathcal{O}_X(lM_p)) \subset H^0(\mathcal{O}_X(lpL \otimes \mathcal{J}_p^l)) \qquad \forall l > 0.$$

Let $f : Y \to X$ be a log resolution of $\mathcal{J}_p$ so that $\mathcal{J}_p \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E_p)$ where $E_p \geq 0$. Then $f^*(pL) - E_p$ is generated by global sections and hence nef. Notice that

$$H^0(\mathcal{O}_Y(l(f^*(pL) - E_p))) \supset H^0(\mathcal{O}_X(plL) \otimes \mathcal{J}_p^l) \supset H^0(\mathcal{O}_X(lM_p)),$$

and so

$$(f^*(pL) - E_p)^d = \mathrm{vol}(f^*(pL) - E_p) \geq \mathrm{vol}(M_p) \geq p^d(\mathrm{vol}(L) - \epsilon).$$

But then the result follows letting $A = (1/p)(f^*(pL) - E_p)$ and $B = (1/p)E_p$. □

The final result that we will need is the following.

Theorem 8.7. *Let L be a big $\mathbb{Q}$-divisor on a normal irreducible projective variety of dimension d. let $f : Y \to X$ be a projective birational morphism, A an ample $\mathbb{Q}$-divisor and B an effective $\mathbb{Q}$-divisor such that $f^*L = A + B$. If H is a $\mathbb{Q}$-divisor such that $H \pm L$ is ample, then*

$$(A^{d-1} \cdot B)^2 \leq C \cdot H^d \cdot (\mathrm{vol}(L) - \mathrm{vol}(A)).$$

Proof. See [Lazarsfeld04, §11]. □

Proof of Theorem 8.1. It is clear that if $D \geq 0$ is an effective $\mathbb{R}$-divisor, then $D \cdot f_*(A_1 \cdots A_{d-1}) \geq 0$ and so

$$\overline{\mathrm{Mov}}(X) \subset \overline{\mathrm{Eff}}(X)^\vee.$$

We will now prove the reverse inclusion. Suppose by way of contradiction thet there is a class ξ on the boundary of $\overline{\mathrm{Eff}}(X)$ which is in the interior of $\overline{\mathrm{Mov}}(X)^\vee$. Note that $\mathrm{vol}(\xi) = 0$. Pick h an ample class such that $h \pm \xi$ are ample and note that $\xi - \epsilon h \in \overline{\mathrm{Mov}}(X)^\vee$ for $0 < \epsilon \ll 1$ and so

$$\frac{\xi \cdot \gamma}{h \cdot \gamma} \geq \epsilon$$

for any mobile class γ. Since $\xi + \delta h$ is big, for any $1 \gg \delta > 0$, by Theorem 8.6, there is

$$f_\delta : Y_\delta \to X, \qquad f_\delta^*(\xi + \delta h) = A_\delta + B_\delta$$

where A_δ is ample, $B_\delta \geq 0$, and

$$(\star) \qquad \mathrm{vol}(A_\delta) \geq \mathrm{vol}(\xi + \delta h) - \delta^{2d} \geq \frac{1}{2}\mathrm{vol}(\xi + \delta h) \geq \frac{\delta^d}{2} \cdot h^d.$$

The class $\gamma_\delta = (f_\delta)_*(A_\delta^{d-1})$ is movable and we have

$$(\sharp) \qquad h \cdot \gamma_\delta = f_\delta^* h \cdot A_\delta^{d-1} \geq (h^d)^{1/d} \cdot (A_\delta)^{(d-1)/d}$$

by the generalized Hodge inequalities. One sees that

$$\xi \cdot \gamma_\delta \leq (\xi + \delta h) \cdot \gamma_\delta = f_\delta^*(\xi + \delta h) \cdot A_\delta^{d-1} = A_\delta^d + B_\delta \cdot A_\delta^{d-1}.$$

By Theorem 8.7 and equation $(\star)$ above, we have that

$$B_\delta \cdot A_\delta^{d-1} \leq (C_1 \cdot h^d \cdot (\mathrm{vol}(\xi + \delta h) - \mathrm{vol}(A_\delta)))^{1/2} \leq C_2 \cdot \delta^d$$

where C_1, C_2 are constants independent of δ. Inequalities $(\star)$ and $(\sharp)$ imply that

$$(\flat) \qquad \frac{\xi \cdot \gamma_\delta}{h \cdot \gamma_\delta} \leq \frac{A_\delta^d + C_2 \cdot \delta^d}{(h^d)^{1/d} \cdot (A_\delta)^{(d-1)/d}} \leq C_3 \cdot (A_\delta^d)^{1/d} + C_4 \cdot \delta$$

where C_3, C_4 are constants independent of δ. Since $\mathrm{vol}(\xi) = 0$, we have (as $\delta \to 0$) that

$$\lim A_\delta^d = \lim \mathrm{vol}(A_\delta^d) = \lim \mathrm{vol}(\xi + \delta h) = 0.$$

By $(\flat)$, it then follows that

$$\lim \frac{\xi \cdot \gamma_\delta}{h \cdot \gamma_\delta} = 0$$

which is the required contradiction. □

9. Rationally connected fibrations

Definition 9.1. *Let X be a smooth complex projective variety, then*

(1) *X is rational if it is birational to $\mathbb{P}^n_{\mathbb{C}}$ i.e. $\mathbb{C}(X) \cong \mathbb{C}(x_1, \ldots, x_n)$.*
(2) *X is unirational if there is a dominant morphism $\mathbb{P}^m_{\mathbb{C}} \dashrightarrow X$ i.e. if there are inclusions $\mathbb{C} \subset \mathbb{C}(X) \subset \mathbb{C}(x_1, \ldots, x_m)$.*
(3) *X is rationally connected if there is a variety T and a morphism $u : U := \mathbb{P}^1 \times T \to X$ such that*

$$u^{(2)} : U \times_T U \to X \times X \qquad \text{is dominant.}$$

In other words for any two general points p and q there is a rational curve C passing through p and q.

(4) *X is rationally chain connected if there is a family of proper connected algebraic curves $g : U \to T$ whose geometric fibers have only rational components with cycle morphism $u : U \to X$ such that*

$$u^{(2)} : U \times_T U \to X \times X \qquad \text{is dominant.}$$

In other words for any two general points p and q there is a chain of rational curves C passing through p and q.

(5) *X is uniruled if for any general point $p \in X$ there is a rational curve $p \in C \subset X$.*

It is clear that if X is rational then it is unirational and if it is rationally connected then it is rationally chain connected. Note that the cone over an elliptic curve is rationally chain connected but not rationally connected. Next we recall the following result of Kollár Mori and Miyaoka.

Theorem 9.2. *Let X be a smooth projective variety of dimension ≤ 2, then X is rational iff it is unirational iff it is rationally chain connected.*

Proof. Rational implies unirational implies rationally chain connected. Now suppose that X is rationally chain connected, then K_X is not pseudo-effective and so running the minimal model program we arrive at a Mori fiber space $X' \to Z$. If $\dim Z = 0$, then $X' \cong \mathbb{P}^2$ and so X is rational. Otherwise $\dim Z = 1$ and Z is rationally chain connected so that $Z \cong \mathbb{P}^1$. But then X is birational to a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ so that X is rational. $\square$

Theorem 9.3. *If X is a smooth projective variety, then X is rationally connected iff it is rationally chain connected.*

Proof. See [**?**, §IV]. $\square$

Rational curves on a variety X define an interesting equivalence relation.

Definition 9.4. *Let X be a smooth complex projective variety. The maximally rationally connected fibration (MRC fibration) is a morphism $f : X' \to Z$ where X' is birational to X, the fibers of f are rationally chain connected and if $z \in Z$ is general and C is a rational curve intersecting X'_z, then C is contained in X'_z.*

Theorem 9.5 (Graber-Harris-Starr)**.** *Let $f : X \to B$ be a morphism of smooth projective varieties where $\dim B = 1$ and $f_*\mathcal{O}_X = \mathcal{O}_B$. If the general fibers of f are rationally connected, then f has a section.*

Corollary 9.6. *The image of the MRC fibration $X' \to Z$ is not uniruled.*

Theorem 9.7. *Let X be a $\mathbb{Q}$-factorial klt variety such that K_X is not pseudo-effective, then X is uniruled.*

Proof. We run the K_X mmp with scaling. Since K_X is not pseudo-effective, this ends with a Mori fiber space $X' \to Z$. The general fiber X'_z is Fano since $-K_{X'_z} = (-K_X)|_{X'_z}$. Let C be a general curve in X'_z obtained by intersecting general very ample divisors. Since X' is klt, it is normal and hence smooth in codimension 1. But then C is contained in the smooth locus of X'_z and $K_{X'_z} \cdot C < 0$ so that by Bend and Break there is a rational curve through any point of C and hence through a general point of X'_z. Note moreover that $X \dashrightarrow X'$ is an isomorphism in a neighborhood of C and hence $-K_X \cdot C < 0$. By the same argument X is uniruled. □

Theorem 9.8. *Let (X, B) be a klt variety.*

(1) *If $-(K_X + B)$ is ample then X is rationally connected, and*
(2) *if $f : Y \to X$ a resolution, then every fiber of is rationally chain connected.*

Proof. We proceed by induction on the dimension. Suppose that we have already shown (1) in dimension $d-1$ and (2) in dimension d. For simplicity we assume that B is 0.

We begin by proving (1) in dimension d. Suppose that $-K_X$ is ample. Pick $H \sim_{\mathbb{Q}} -K_X$ a general element so that $K_X + H$ is klt. Passing to a $\mathbb{Q}$-factorialization and using (2) in dimension d we may assume that X is $\mathbb{Q}$-factorial. Since K_X is not pseudo-effective, arguing as above we may run the K_X mmp with scaling. This ends with a Mori fiber space $g : X' \to Z$ whose fibers are klt Fano varieties so that by induction on the dimension X'_z is rationally connected. Note that if H' is the strict transform of H then $K_{X'} + H' \sim_{\mathbb{Q}} 0$ is also klt. Let G be an general ample divisor on Z. Since H' is big, we may write $H' \sim \epsilon g^*G + B$ where $0 < \epsilon \ll 1$ and $B \geq 0$. But then $K_{X'} + (1-\delta)H' + \delta(\epsilon g^*G + B) \sim_{\mathbb{Q}} 0$ is klt. But then by Kawamata's canonical bundle formula, we have that

$$0 \sim_{\mathbb{Q}} K_{X'} + (1-\delta)H' + \delta(\epsilon g^*G + B) \sim_{\mathbb{Q}} g^*(K_Z + B_Z + \delta\epsilon G + M_Z$$

where (Z, B_Z) is klt, and M_Z is the pushforward of a nef divisor. More precisely there is a proper birational morphism $\nu : Z' \to Z$ such that

$$\nu^*(K_Z + B_Z + M_Z) = K_{Z'} + B_{Z'} + M_{Z'})$$

where $M_{Z'}$ is nef and $(Z', B_{Z'})$ is sub-klt with $\nu_* B_{Z'} = B_Z$ and $\nu_* M_{Z'} = M_Z$. Note that ν^*G is semi-ample and big and so it is easy to see that $B_{Z'} + M_{Z'} + \frac{1}{2}\epsilon\delta\nu^*G \sim \Delta_{Z'}$ where $(Z', \Delta_{Z'})$ is klt. But then if $\Delta_Z = \nu_*\Delta_{Z'}$, it follows that (Z, Δ_Z) is klt and $-(K_Z + \Delta_Z) \sim_{\mathbb{Q}} \frac{1}{2}\epsilon\delta G$ which is ample. By induction on the dimension Z is rationally chain connected. By a result of Graber-Harris-Starr, it follows that X' is rationally chain connected. To show that X is rationally chain connected, consider the individual steps of the minimal model program $X_i \dashrightarrow X_{i+1}$ and we show that if X_{i+1} is rationally chain connected then so is X_i. Suppose $f_i : X_i \dashrightarrow X_{i+1}$ is a divisorial contraction, then by (2) in dimension d, the fibers of f_i are rationally chain connected and hence so is X_i.

Suppose instead that f_i is a flip and let $X_i \to Z_i$, $X_{i+1} \to Z_i$ be the corresponding contractions. If X_{i+1} is rationally chain connected then so is Z_i. By (2) in dimension d, the fibers of $X_i \to Z_i$ are rationally chain connected and hence X_i is rationally chain connected.

We will now prove (2) in dimension $d+1$. We are free to replace Y by a higher model and hence we may assume that it is a log resolution, in particular $E = \mathrm{Ex}(f)$ is a simple normal crossings divisor. Assume for simplicity that $p \in X$ is an isolated singularity and f is an isomorphism on $X \setminus \{p\}$. Let G be a general divisor through p. Working locally over $p \in X$ there is an effective exceptional divisor Φ such that $-\Phi$ is relatively ample and hence $-\Phi \sim_{\mathbb{Q}} A$ where A is a general ample $\mathbb{Q}$-divisor. We write

$$K_Y + E_t = f^*(K_X + tG) + \Phi + A.$$

W may assume that (Y, E_0) is klt. Let $E = F_1 + \ldots + F_r$ be the irreducible components of the exceptional divisor. Let t_i be the smallest rational number such that $\mathrm{mult}_{F_i}(E_{t_i}) = 1$. Perturbing G and relabelling the F_i we may assume $t_1 < t_2 < \ldots < t_r$. It suffices to show that F_i is rationally chain connected modulo $F_1 + \ldots + F_{i-1}$ meaning that for general $p, q \in F_i$ there is a chain of curves connecting p, q which are either rational or contained in $F_1 + \ldots + F_{i-1}$. Proceeding by induction, it follows that $F_1 + \ldots + F_i$ is rationally chain connected. Note that

$$(K_Y + E_{t_1})|_{F_1} = K_{F_1} + \Delta_{F_1}$$

is klt and $\Delta_{F_1} \geq A|_{F_1}$. By (1) in dimension d, F_1 is rationally chain connected. We now consider $E_{t_2} = E_{t_2}^{\leq 1} + \tau F_1$. By the connectedness lemma $F_1 \cap F_2 \neq \emptyset$. Let

$$(K_Y + E_{t_2}^{\leq 1})|_{F_2} = K_{F_2} + \Delta_{F_2},$$

then $K_{F_2} + \Delta_{F_2} \sim_{\mathbb{Q}} -\tau F_1|_{F_2}$ is dlt but not pseudo-effective. We run the $K_{F_2} + \Delta_{F_2}$ minimal model program with scaling which terminates with a Mori fiber space $g : F_2' \to Z$. Since $F_1|_{F_2}$ is g-ample, it dominates Z. Since $-(K_{F_2} + \Delta_{F_2})$ is g-ample, the fibers of g are rationally chain connected and so (following the arguments above) we see that F_2 is rationally chain connected modulo $F_2 \cap F_1$. Repeating this argument the claim follows. $\square$

Corollary 9.9. *If (X, B) is klt, then*

1. *X is rationally connected iff it is rationally chain connected.*
2. *If $-(K_X + B)$ is nef and big, then X is rationally connected.*

Proof. (1) Assume that X is rationally chain connected. By the previous Theorem 9.8, every fiber of a resolution $Y \to X$ is rationally chain connected and hence Y is rationally chain connected. By Theorem 9.3, Y is rationally connected but then so is X.

(2) Since $-(K_X + B)$ is big we may write $-(K_X + B) \sim_{\mathbb{Q}} A + G$ where A is ample and $G \geq 0$. But then

$$-(K_X + B + \delta G) \sim_{\mathbb{Q}} (1 - \delta)(K_X + B) + \delta A \qquad \text{is ample}$$

and $(X, B+\delta G)$ is klt. By Theorem 9.8 X is rationally chain connected. We conclude by (1).

$\square$

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