Generic vanishing and the birational geometry of irregular varieties

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Introduction

- In these lectures we will discuss in detail recent results on the birational geometry of irregular varieties i.e. with $h^1(\mathcal{O}_X) \neq 0$.
- Let X be a smooth complex projective variety and $a : X \to A$ its Albanese morphism. Let $g = \dim A = h^0(\Omega^1_X) = h^1(\mathcal{O}_X)$.
- Recall that $A = H^0(X, \Omega^1_X)^\vee / H_1(X, \mathbb{Z})$ and if $\omega_1, \ldots, \omega_g$ is a basis of $H^0(X, \Omega^1_X)$, fixing a base point $x_0 \in X$ we may view a as being defined by $a(x) = (\int_{x_0}^x \omega_1, \ldots, \int_{x_0}^x \omega_g)$.
- Of course, these integrals are only defined up to integrals over closed loops in X i.e. up to elements of $H_1(X, \mathbb{Z})$.
- We let $\dim a(X)$ be the Albanese dimension of X and we say that X has maximal Albanese dimension (MAD) if $\dim a(X) = \dim X$.
- The geometry of MAD varieties is very well understood.

Introduction I

- Let X be a smooth complex projective variety of MAD, then X has a good minimal model (Fujino).
- Let X be a smooth complex projective variety of MAD, then $|4K_X|$ defines the Iitaka fibration and $|3K_X|$ is birational if X is of general type (Jiang-Lahoz-Tirabassi).
- Let X be a smooth complex projective variety of MAD, then $\chi(\omega_X) \geq 0$ (Green-Lazarsfeld).
- Let $f : X \to Y$ be a morphism of smooth complex projective varieties with general fiber F. If Y has MAD, then $\kappa(X) \geq \kappa(F) + \kappa(Y)$ (Cao-Paun).
- Recall that $\kappa(X) = -\infty$ if $h^0(mK_X) = 0$ for all $m > 0$ and otherwise $\kappa(X) = \max\{\dim \phi_{|mK_X|}(X) | m \geq 0\}$.

Introduction II

- If (A, Θ) is a PPAV and $G \in |m\Theta|$ then $\mathrm{mult}_p(G) \leq m \dim A$, with equality iff A is a product of elliptic curves (Ein-Lazarsfeld, Hacon).
- Let X be a smooth complex projective variety with $h^0(\omega_X^2) = 1$, then its Albanese morphism is surjective and hence $h^0(\Omega_X^1) \leq \dim X$. If $h^0(\omega_X^2) = 1$ and $h^0(\Omega_X^1) = \dim X$, then X is birational to an abelian variety (Hacon-Chen).
- These surprising results are obtained by combining Generic Vanishing theorems developed by Green-Lazarsfeld, Simpson, Hacon, Popa-Pareschi and others with the Fourier Mukai transform introduced by S. Mukai.
- We will also see that using the theory of F-singularities, some of these results apply to positive characteristics.

Outline of the talk

1. Vanishing Theorems
2. The generic vanishing theorem of Green and Lazarsfeld
3. The Fourier Mukai transform
4. A derived category approach to Generic Vanishing
5. Pluricanonical maps of MAD varieties
6. Characterization of abelian varieties

Kodaira and Kawamata-Viehweg vanishing

- We begin by reviewing some of the highlights of vanishing theorems which play a fundamental role in the higher dimensional birational geometry of complex projective varieties.
- **Kodaira Vanishing (1953)**: Let X be a smooth complex projective variety and L an ample line bundle on X, then $H^i(\omega_X \otimes L) = 0$ for all $i > 0$.

Kodaira and Kawamata-Viehweg vanishing

- **Kawamata-Viehweg vanishing (1982)**: Let X be a smooth complex projective variety and L a nef and big line bundle on X, then $H^i(\omega_X \otimes L) = 0$ for all $i > 0$.
- Recall that L is **nef** iff $L \cdot C = (\deg(L|_C)) \geq 0$ for any curve C on X and L is **big** if $\lim h^0(L^m)/m^{\dim X} > 0$. In other words L is big if $h^0(L^m)$ behaves like a polynomial of degree $n = \dim X$.
- It is well known that if L is nef, then L is big iff $L^{\dim X} > 0$.
- If L is ample, then it is clearly nef and big.
- Pull backs of nef and big line bundles are nef and big (this fails for ample line bundles).

Kawamata-Viehweg and Grauert-Riemanschneider vanishing

- At first sight Kawamata-Viehweg vanishing is a mild technical generalization of Kodaira Vanishing, however the extra flexibility has many important applications. For example Kawamata-Viehweg vanishing easily implies Grauert-Riemanschneider vanishing.
- **Grauert-Riemanschneider vanishing (1970)**: Let X be a smooth complex projective variety and $f : X \to Y$ be a birational morphism, then $R^i f_* \omega_X = 0$ for all $i > 0$.
- To see that Kawamata-Viehweg vanishing implies Grauert-Riemanschneider vanishing, we proceed as follows.

KVV implies GRV

- Let L be a sufficiently ample line bundle on Y, then $R^if_*\omega_X \otimes L$ is globally generated and has vanishing higher cohomologies $H^j(R^if_*\omega_X \otimes L) = 0$ for $i \geq 0$, $j > 0$.
- By an easy spectral sequence argument we have $H^i(\omega_X \otimes f^*L) = H^0(R^if_*\omega_X \otimes L)$.
- By KVV, the left hand side vanishes for $i > 0$. Since $R^if_*\omega_X \otimes L$ is globally generated, it follows that $R^if_*\omega_X \otimes L = 0$ for $i > 0$ and hence $R^if_*\omega_X = 0$.
- One can use the above argument to show that $Rf_*\omega_X$ is a sheaf in degree 0 iff $H^i(\omega_X \otimes f^*L) = 0$ for $i > 0$ and L sufficiently ample (iff $H^{n-i}((f^*L)^\vee) = 0$ for $i > 0$).
- We will apply this simple observation in the context of the FM transform.

Kawamata-Viehweg vanishing II

- In what follows we will need a more flexible version of KVV which can be deduced from the above version via "covering tricks".
- Recall that (X, B) is a **klt pair** if X is a normal variety and $B \geq 0$ is a $\mathbb{R}$-divisor such that $K_X + B$ is $\mathbb{R}$-Cartier and for any (one) log resolution $f : X' \to X$, we have $K_{X'} + B' = f^*(K_X + B)$ where $\lfloor B' \rfloor \leq 0$.
- Thus, if X is smooth and B has snc support, then (X, B) is klt iff $\lfloor B \rfloor = 0$.
- **Kawamata-Viehweg vanishing II**: Let (X, B) be a complex projective klt pair and L a line bundle such that $L \equiv K_X + B + M$ where M is nef and big, then $H^i(L) = 0$ for all $i > 0$.

Multiplier ideal sheaves

- One can even deduce a version of KVV that works for non-klt pairs.
- Assume that X is smooth and $B \geq 0$ is an $\mathbb{R}$-divsor, then we define the **multiplier ideal sheaf** $\mathcal{J}(B)$ as follows:
- Let $f : X' \to X$ be a **log resolution** of (X, B) so that f is proper and birational, X' is smooth, $\mathrm{Ex}(f)$ is a divisor, and $\mathrm{Ex}(f) + f_*^{-1}B$ has simple normal crossings.
- Then $\mathcal{J}(B) = f_*\mathcal{O}_{X'}(K_{X'/X} - \lfloor f^*B \rfloor)$.
- It is not hard to see that the definition is independent of the log resolution and that if $K_{X'/X} - \lfloor f^*B \rfloor = E - F$ where $E, F \geq 0$ are divisors with no common component, then E is exceptional and so $\mathcal{J}(B) \subset f_*\mathcal{O}_{X'}(E) = \mathcal{O}_X$ is an ideal sheaf.

Nadel vanishing

- **Nadel Vanishing**: Let (X, B) be a smooth complex projective pair and L a line bundle such that $L \equiv K_X + B + M$ where M is nef and big, then $H^i(L \otimes \mathcal{J}(B)) = 0$ for all $i > 0$.
- This is an immediate consequence of KVV II.
- Let $f : X' \to X$ be a log resolution and note that $K_{X'/X} - \lfloor f^*B \rfloor + f^*L \equiv K_{X'} + \{f^*B\} + f^*M$.
- Since $(X', \{f^*B\})$ is klt and f^*M is nef and big, $R^i f_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor f^*B \rfloor + f^*L) = 0$ for $i > 0$ (by the relative version of KVV).
- By the projection formula $f_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor f^*B \rfloor + f^*L) = L \otimes \mathcal{J}(B)$.
- By KVV II and an easy spectral sequence argument, $\forall i > 0$ $0 = H^i(\mathcal{O}_{X'}(K_{X'/X} - \lfloor f^*B \rfloor + f^*L)) = H^i(L \otimes \mathcal{J}(B))$.

Kollár vanishing

- **Kollár vanishing (1986)**: Let $f : X \to Y$ be a surjective morphism of projective varieties, X smooth and L nef and big on Y. Then
  (1) $R^i f_* \omega_X$ is torsion free.
  (2) $Rf_* \omega_X = \sum_{i=0}^{k} R^i f_* \omega_X[-i]$ in $D(Y)$ where $k = \dim X - \dim Y$, in particular $H^i(\omega_X) = \oplus_{j=0}^{i} H^{i-j}(R^j f_* \omega_X)$.
  (3) $H^i(R^j f_* \omega_X \otimes L) = 0$ for all $i > 0$.
- One can also deduce appropriate relative versions for klt pairs similarly to KVV II and in fact versions similar to Nadel vanishing.

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The Picard bundle

- Let X be a smooth complex projective variety and $a : X \to A$ its Albanese morphism.
- We have $q(X) = h^0(\Omega^1_X) = h^1(\mathcal{O}_X) = \dim A$.
- Let $\mathrm{Pic}^0(X)$ be the set of topologically trivial line bundles on X.
- From the short exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}^*_X \to 0$ we see that $\mathrm{Pic}^0(X) \cong H^1(\mathcal{O}_X)/H^1(X, \mathbb{Z})$.
- We also have $\mathrm{Pic}^0(X) \cong \mathrm{Pic}^0(A) = \hat{A}$ (where $\hat{A}$ denotes the abelian variety dual to A).
- Let $\mathcal{P}$ be the Poincaré line bundle on $A \times \hat{A}$ so that $\mathcal{P}|_{A\times 0} \cong \mathcal{O}_A$, $\mathcal{P}|_{0\times \hat{A}} \cong \mathcal{O}_{\hat{A}}$ and in fact $\mathcal{P}|_{A\times x} \cong P_x$ where $P_x \in \mathrm{Pic}^0(A)$ is the topologically trivial line bundle corresponding to $x \in \hat{A}$.

Cohomological support loci

- For any coherent sheaf $\mathcal{F}$ on X we define the **cohomological support loci** $V^i(\mathcal{F}) = \{P \in \mathrm{Pic}^0(A) | h^i(\mathcal{F} \otimes a^*P) \neq 0\}$.
- Note that $V^i(\mathcal{F}) \subset \mathrm{Pic}^0(A)$ are closed subsets.

Theorem (Green and Lazarsfeld (1987))

Let X be a smooth complex projective variety, then

1. *each irreducible component of $V^i(\Omega^j_X)$ is a translate of an abelian subvariety of $\mathrm{Pic}^0(A)$,*
2. *the codimension of $V^i(\omega_X) \subset \mathrm{Pic}^0(A)$ is $\geq i - (\dim X - \dim a(X))$, and*
3. *if $\dim X = \dim a(X)$, then we have inclusions $V^0(\omega_X) \supset V^1(\omega_X) \supset V^2(\omega_X) \supset \dots$.*

Deforming sections of $H^i(\Omega^j)$

- Fix $x \in \hat{A}$ corresponding to $P_x \in \mathrm{Pic}^0(A) \cong \mathrm{Pic}^0(X)$ (for ease of notation we defote a^*P_x simply by P_x).
- Fix $v \in H^1(\mathcal{O}_A) \cong T_x\hat{A}$.
- Suppose that $\psi \in H^j(\Omega^i_X \otimes P)$, then we hope to show that if ψ deforms to first order in the direction $v \in H^1(\mathcal{O}_A) \cong T_x\hat{A}$, then ψ deforms to all orders.
- Suppose that T is an irreducible component of $V^i(\Omega^j_X) \subset \hat{A}$, $x \in T$ is a general point, and $v \in H^1(\mathcal{O}_X) \cong T_x\hat{A}$ is tangent to T at x.
- Pick $0 \neq \psi \in H^i(\Omega^j_X \otimes P_x)$, then ψ deforms to first order in the direction of v. But then ψ deforms to all higher orders and so T contains the tangent space to T at x which implies that T is a subtorus of $\hat{A}$.

Deforming sections of $H^i(\Omega^j)$.

- For any ball $Z \subset \mathbb{C}$ centered at the origin, with local coordinate t we consider the linear map $\tau_{x,v} : Z \to \hat{A}$ given by $\tau_{x,v}(t) = x + tv$. We wish to describe $(\mathrm{id}_X \times \tau_{x,v})^* \mathcal{P}$.
- Suppose that U_α is a cover of X via open complex balls, v corresponds to a $\bar{\partial}$ closed $(0,1)$ form which is locally given by $\phi|_{U_\alpha} = \bar{\partial}\lambda_\alpha$ and $g_{\alpha\beta} \in \mathcal{O}^*_{U_\alpha \cap U_\beta}$ are the transition functions corresponding to $P = P_x$.
- The transition functions for $(\mathrm{id}_X \times \tau_{x,v})^* \mathcal{P}$ are given by $\tilde{g}_{\alpha\beta} = \exp(2\pi i t(\lambda_\alpha - \lambda_\beta)) g_{\alpha\beta}$ and the differential corresponds to $\bar{\partial}_{P_{\tau(t)}} = \bar{\partial}_{P_x} + \wedge t\phi$.

Deforming sections of $H^i(\Omega^j)$.

- Note that if $P \in \mathrm{Pic}^0(X)$, then P has a unitary flat connection and hence we can do Hodge Theory with coefficients in P.
- For example, there is a $\mathbb{C}$ anti linear isomorphism $H^i(\Omega^j \otimes P) \cong H^j(\Omega^i \otimes P^\vee)$ induced by conjugation.
- If $\psi \in A^{i,j}(P)$ is $\bar\partial$ closed and ∂ exact, then $\psi = \bar\partial\partial\gamma$ for some $\gamma \in A^{i-1,j-1}(P)$ (Principle of two types).
- We may represent ψ by harmonic representative $\psi_0 \in A^{i,j}(P)$.
- With the above notation, deforming ψ to order k means that we can solve $(\bar\partial_{P_x} + \wedge t\phi)(\sum \psi_i t^i) = 0$ modulo t^{k+1}.
- i.e. deforming to first order is equivalent to solving $\bar\partial_{P_x}\psi_1 = \psi_0 \wedge \phi$ i.e. to showing that $\psi_0 \wedge \phi$ is $\bar\partial_{P_x}$-exact or $[\psi_0 \wedge \phi] = 0 \in H^{i+1}(\Omega^{i+1}_X \otimes P)$

Deforming sections of $H^i(\Omega^j)$.

- Suppose now that ψ deforms to first order so that we are given a harmonic representative ψ_0 and we have $\psi_0 \wedge \phi = \bar{\partial}_{P_x}\psi_1$.
- Since ψ_0 and ϕ are harmonic, we have $\partial_{P_x}(\psi_0 \wedge \phi) = 0$ and by the principle of two types $\psi_0 \wedge \phi = \bar{\partial}_{P_x}\partial_{P_x}\gamma_1$.
- We replace ψ_1 by $\psi_1 = \partial_{P_x}\gamma_1$.
- We now have $\psi_1 \wedge \phi = \partial_{P_x}\gamma_1 \wedge \phi = \partial_{P_x}(\gamma_1 \wedge \phi)$ is ∂_{P_x} exact and $\bar{\partial}_{P_x}$ closed since $\bar{\partial}_{P_x}(\psi_1 \wedge \phi) = \bar{\partial}_{P_x}\psi_1 \wedge \phi = \psi_0 \wedge \phi \wedge \phi = 0$.
- By the principle of two types we have $\psi_1 \wedge \phi = \bar{\partial}_{P_x}\partial_{P_x}\gamma_2$ and we let $\psi_2 = \partial_{P_x}\gamma_2$.
- Repeating this procedure, we obtain $\sum_{i=0}^{\infty}\psi_i t^i$ such that $(\bar{\partial}_{P_x} + \wedge t\phi)(\sum \psi_i t^i) = 0$ modulo t^{k+1} for any $k > 0$.

Linearity of $V^i(\omega_X)$

- The above argument can be made precise with some care. We will not go through the details of the argument, but we reproduce the following crucial statement of Green and Lazarsfeld.

Linearity of $V^i(\omega_X)$

- Let $H^i(P)$ be the set of harmonic elements of $A^{0,i}(P)$ and $D_P^\bullet$ be the complex given by $D_P^i = H^i(P) \otimes \mathcal{O}_Z$ with differentials induced by $\wedge tv$ (where v is a harmonic representative of $v \in H^1(\mathcal{O}_X)$).
- We have an equality of stalks of coherent sheaves $(R^i p_{Z,*}(\mathrm{id}_X \times \tau_{x,v})^*\mathcal{P})_0 = (\mathcal{H}^i(D_P^\bullet))_0$.
- It follows that $(R^i p_{Z,*}(\mathrm{id}_X \times \tau_{x,v})^*\mathcal{P}) = \mathcal{K}^i \otimes \mathcal{O}_Z/\mathfrak{m} \oplus \mathcal{H}^i \otimes \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_Z$ at 0, $\mathcal{K}^i = \ker(H^i(P) \to H^{i+1}(P))$ and $\mathcal{H}^i = \ker(H^i(P) \to H^{i+1}(P))/\mathrm{im}(H^{i-1}(P) \to H^i(P))$ (here $H^i(P) \to H^{i+1}(P)$ is induced by $\wedge v$).
- In particular $H^i(P_{x+tv}) \cong \mathcal{H}^i$ on a punctured neighborhood of $0 \in Z$.

Linearity of $V^i(\omega_X)$

- In particular, it follows that if v is tangent to T at x, then the homomorphisms in $H^{i-1}(P) \to H^i(P) \to H^{i+1}(P)$ vanish,
- and if v is not tangent to T at x, then $H^{i-1}(P) \to H^i(P) \to H^{i+1}(P)$ is exact.
- Notice that by Serre duality $h^i(P) = h^{n-i}(\omega_X \otimes P^\vee)$ where $n = \dim X$. Thus $V^i(\omega_x) = -V^{n-i}(\mathcal{O}_X)$.
- We will now deduce that if T is a component of $V^i(\omega_X)$, then $T \subset \hat{A}$ has codimension $\geq i - (\dim X - \dim a(X))$
- If v is tangent to T, then by Serre duality the maps in the complex $H^{n-i-1}(P) \to H^{n-i}(P) \to H^{n-i+1}(P)$ vanish.

Codimension of $V^i(\omega_X)$

- By Hodge theory, we then have that the maps in the complex $H^0(\Omega_X^{n-i-1} \otimes P) \to H^0(\Omega_X^{n-i} \otimes P) \to H^0(\Omega_X^{n-i+1} \otimes P)$ vanish. (Here the differential is induced by wedging with $\omega = \bar{v} \in H^0(\Omega_X^1)$.)
- Let $S = \hat{T}$ and $p : A \to S$ the morphism of abelian varieties dual to the inclusion $T \to \hat{A}$, then we can identify $\overline{H^1(\mathcal{O}_T)}$ with $H^0(\Omega_S^1)$ and hence the above differentials vanish for any $\omega \in p^* H^0(\Omega_S^1)$
- For a general point $x \in X$, $a^* p^* H^0(\Omega_S^1)$ generates a subspace of $T_x(X)^\vee$ of dimension $\geq \dim S - (\dim X - \dim a(X))$.
- On the other hand, it is easy to see that given $\phi \in H^0(\Omega_X^{n-i})$ viewing $\phi_x \in \mathrm{Hom}(\Omega_X^{n-i} \otimes \mathbb{C}(x) \to \Omega_X^{n-i+1} \otimes \mathbb{C}(x))$, then $\dim \ker(\phi_x) \leq n - i$ and so $i \geq \dim S - (\dim X - \dim a(X))$.

Inclusions of Cohomological support loci

- We now assume that $\dim a(X) = \dim X$ and we aim to show that $V^0(\omega_X) \supset V^1(\omega_X) \supset V^2(\omega_X) \supset \dots$
- Suppose that $P \in V^i(\omega_X)$, then $h^0(\Omega_X^{n-i} \otimes P) = h^{n-i}(P^\vee) = h^i(\omega_X \otimes P) \neq 0$.
- Since $\dim a(X) = \dim X$, for general $x \in X$, the differential $T_x(X) \to H^0(\Omega_A^1)^\vee$ is injective and hence the codifferential $H^0(\Omega_A^1) \to \Omega_X^1 \otimes \mathbb{C}(x)$ is surjective.
- If $0 \neq \phi \in H^0(\Omega_X^{n-i} \otimes P)$, then we may find $\omega \in H^0(\Omega_A^1)$ such that $0 \neq a^*\omega \wedge \phi \in H^0(\Omega_X^{n-i+1} \otimes P)$
- Since $h^{i-1}(\omega_X \otimes P) = h^{n-i-1}(P^\vee) = h^0(\Omega_X^{n-i-1} \otimes P)$, we have shown that $V^{i-1}(\omega_X) \supset V^i(\omega_X)$.

Euler Characteristic of varieties of MAD

- One immediate consequence of GVT is the following.
- **Corollary:** Let X be a smooth projective variety of MAD, then $\chi(\omega_X) \geq 0$.
- **Proof:** We have $\chi(\omega_X) = \chi(\omega_X \otimes P) = h^0(\omega_X \otimes P) \geq 0$ for general $P \in \mathrm{Pic}^0(X)$.
- Note that one could have $\chi(\omega_X) = 0$:
- Let E be an elliptic curve, P a 2 torsion line bundle and L a line bundle of degree 1 and $C \to E$ the double cover corresponding to a general section of L^2.
- Let $b : C \to E$ be the obvious $(\mathbb{Z}/2\mathbb{Z})^2$ cover such that $b_*\mathcal{O}_C = \mathcal{O}_E \oplus P \oplus L^\vee \oplus L^\vee \otimes P$.

Example

- We let $a : X \to A = E \times E \times E$ be a bi-double cover such that $a_*\mathcal{O}_X = \mathcal{O}_E \boxtimes \mathcal{O}_E \boxtimes \mathcal{O}_E \oplus P \boxtimes L^\vee \boxtimes L^\vee \oplus L^\vee \boxtimes P \boxtimes L^\vee \oplus L^\vee \boxtimes L^\vee \boxtimes P$
- This is obtained as a $(\mathbb{Z}/2\mathbb{Z})^4$ quotient of the obvious $(\mathbb{Z}/2\mathbb{Z})^6$ cover $C \times C \times C \to E \times E \times E$
- One can easily check that the singularities of X are canonical.
- Thus if $\mu : X' \to X$ is a resolution, then we have that $V^0(\omega_X) = \{\mathcal{O}_A\} \cup P \times E \times E \cup E \times P \times E \cup E \times E \times P$.
- In particular $\chi(\omega_X) = 0$.

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The Fourier Mukai transform

- Let A be an abelian variety, $\hat{A} \cong \mathrm{Pic}^0(A)$ its dual abelian variety and $\mathcal{P}$ the Poincaré line bundle on $A \times \hat{A}$.
- One can define derived functors $R\hat{S} : D^{\mathrm{b}}_{\mathrm{qc}}(A) \to D^{\mathrm{b}}_{\mathrm{qc}}(\hat{A})$ and $RS : D^{\mathrm{b}}_{\mathrm{qc}}(\hat{A}) \to D^{\mathrm{b}}_{\mathrm{qc}}(A)$ as follows:
- $R\hat{S}(?) = Rp_{\hat{A},*}(\mathcal{P} \otimes Lp_A^*(?))$ and $RS(?) = Rp_{A,*}(\mathcal{P} \otimes Lp_{\hat{A}}^*(?))$.

Theorem (Mukai (1981))

There are isomorphisms of functors on $D^{\mathrm{b}}_{\mathrm{qc}}(A)$ and $D^{\mathrm{b}}_{\mathrm{qc}}(\hat{A})$:

$$RS \circ R\hat{S} \cong (-1_A)^*[-g], \qquad R\hat{S} \circ R\hat{S} \cong (-1_{\hat{A}})^*[-g].$$

Here $g = \dim A$, $[-g]$ is shift g places to the right.

The Fourier Mukai transform

- **Remark:** It is unknown if we also obtain an isomorphism of functors on $D(A)$ and $D(\hat{A})$.
- To prove the theorem, one considers the composition $RS \circ R\hat{S}$ which is induced by the kernel

$$\mathcal{Q} = Rp_{12,*}(p_{13}^*\mathcal{P} \otimes p_{23}^*\mathcal{P})$$

  where p_{ij} is the projection of $A \times A \times \hat{A}$ on to the product of its i-th and j-th factors.
- It is easy to see (by the see saw principle for p_{12}) that $p_{13}^*\mathcal{P} \otimes p_{23}^*\mathcal{P} \cong (m \times \mathrm{id}_{\hat{A}})^*\mathcal{P}$ where $m : A \times A \to A$ is the multiplication map.
- Thus $\mathcal{Q} = Rp_{12,*}(m \times \mathrm{id}_{\hat{A}})^*\mathcal{P} = m^* Rp_{1,*}\mathcal{P}$. (By flat base change.)

The Fourier Mukai transform

- It is well known that for $i < g$, we have $R^i p_{1,*}\mathcal{P} = 0$ and $R^g p_{1,*}\mathcal{P} = \mathbb{C}(0)$. Grant this for the time being.
- But then $\mathcal{Q} \cong \mathcal{O}_\Gamma[-g]$ where Γ is the graph of $-1_A : A \to A$ and so $RS \circ R\hat{S} = (-1_A)^*[-g]$.
- For completeness we now prove $Rp_{1,*}\mathcal{P} = 0$ and $R^g p_{1,*}\mathcal{P} = \mathbb{C}(0)$.

The Fourier Mukai transform

- Note in fact that for $x \neq 0$, $\mathcal{P}|_{x \times \hat{A}} \neq \mathcal{O}_{\hat{A}}$ and so $h^i(\mathcal{P}|_{x \times \hat{A}}) = 0$ for all i and by cohomology and base change, $Rp_{1,*}\mathcal{P}$ is supported on $0 \in A$ and $R^g p_{1,*}\mathcal{P} \cong \mathbb{C}(0)$.
- We may now replace A by the spectrum of the local ring $R = \mathcal{O}_{A,0}$. We pick a regular sequence $x_1, \ldots, x_g$ so that $R/(x_1, \ldots, x_i)$ is a local ring of length $g - i$.
- It suffices to show that $R^j p_* \mathcal{P}|_{A \times Z_i} = 0$ for $j < i$ where $Z_i = V(x_1, \ldots, x_{g-i})$ and $0 \leq i \leq g$.
- The statement is clear for $i = 0$. Consider the following short exact sequence

The Fourier Mukai transform

- $0 \to \mathcal{P}|_{A\times Z_i} \to \mathcal{P}|_{A\times Z_i} \to \mathcal{P}|_{A\times Z_{i-1}} \to 0$ induced by mutliplication by x_{g-1+1}.
- Applying Rp_* we obtain $R^{k-1}p_*\mathcal{P}|_{A\times Z_i} \to R^{k-1}p_*\mathcal{P}|_{A\times Z_i} \to R^{k-1}p_*\mathcal{P}|_{A\times Z_{i-1}} \to R^k p_*\mathcal{P}|_{A\times Z_i} \to R^k p_*\mathcal{P}|_{A\times Z_i}$
- Proceeding by induction, we may assume that $R^{k-1}p_*\mathcal{P}|_{A\times Z_{i-1}} = 0$ for $k < i$ and hence that $R^{k-1}p_*\mathcal{P}|_{A\times Z_i} \to R^{k-1}p_*\mathcal{P}|_{A\times Z_i}$ is an injective homomorphism of Artinian modules.
- But then $R^{k-1}p_*\mathcal{P}|_{A\times Z_i} = 0$. (In fact multiplication by x^l_{g-i+1} is zero on $R^{k-1}p_*\mathcal{P}|_{A\times Z_i}$ for some $l > 0$ and this gives an immediate contradiction.)

Vanishing and non-vanishing

- At first sight the FM transforms appears to be very technical, but it has many easy far reaching applications.
- **Lemma:** If $\mathcal{F} \in \mathrm{Coh}(A)$ is a non-zero coherent sheaf such that $h^i(\mathcal{F} \otimes P) = 0$ for all $i > 0$ and $P \in \hat{A}$, then $h^0(\mathcal{F}) \neq 0$.
- **Proof:** Suppose that $h^0(\mathcal{F}) = 0$, then $h^0(\mathcal{F} \otimes P) = \chi(\mathcal{F} \otimes P) = \chi(\mathcal{F}) = h^0(\mathcal{F}) = 0$ and so by cohomology and base change $R\hat{S}(\mathcal{F}) = 0$.
- By Mukai's Theorem, $\mathcal{F} = (-1_A)^* RS(R\hat{S}(\mathcal{F}))[-g] = 0$ which is a contradiction $\square$
- This is much stronger than the usual Castelnuovo-Mumford regularity (which implies that if $H^i(\mathcal{F}(-jH)) = 0$ for all $i > 0$ and $0 \leq j \leq \dim X$, then $\mathcal{F}$ is generated and so $H^0(\mathcal{F}) \neq 0$).

Applications

Theorem (Green and Lazarsfeld)

Let (A, Θ) be a PPAV and $D \in |m\Theta|$, then $(A, D/m)$ is log canonical (hence $\mathrm{mult}_x D \leq m \dim A$).

- **Proof:** Recall that since A is smooth, then $(A, D/m)$ is log canonical iff $(A, \frac{1-\epsilon}{m} D)$ is klt for $0 < \epsilon \ll 1$ iff $\mathcal{J}(\frac{1-\epsilon}{m} D) = \mathcal{O}_A$ for $0 < \epsilon \ll 1$.
- Since $\Theta - \frac{1-\epsilon}{m} D$ is ample, by Nadel vanishing $H^i(\mathcal{O}_A(\Theta) \otimes \mathcal{J}(\frac{1-\epsilon}{m} D) \otimes P) = 0$ for all $i > 0$ and $P \in \hat{A}$.
- Thus $h^0(\mathcal{O}_A(\Theta) \otimes \mathcal{J}(\frac{1-\epsilon}{m} D) \otimes P) \neq 0$ for all $P \in \hat{A}$.
- But $|\mathcal{O}_A(\Theta) \otimes P|$ is generated by a translate of Θ and so all translates of Θ vanish along the support of $\mathcal{O}_A / \mathcal{J}(\frac{1-\epsilon}{m} D)$, so that this support is empty i.e. $\mathcal{J}(\frac{1-\epsilon}{m} D) = \mathcal{O}_A$ $\square$

Applications

- In particular it follows that $\mathrm{mult}_x D \leq m \dim A$ for any closed point $x \in A$.
- One can push the above argument to show the following.

Theorem (Hacon)

Let (A, Θ) be a PPAV and $D \in |m\Theta|$. If $\mathrm{mult}_x(D) = m \dim A$, then A splits as a product of elliptic curves $E_1 \times \ldots \times E_g$ and $D = m(p_1^*\Theta_1 + \ldots + p_g^*\Theta_g)$.

- There are many more applications that we will explore later.

Fourier Mukai calculus

- Next we describe the relationship between RS, $\otimes P_y$, T_x and D_A where $T_x : A \to A$ denotes the translation by a closed point $x \in A$. We have that the following equalities hold for any $x \in X$ and any isogeny $\phi : A \to B$
- $RS \circ T_x^* \cong (\otimes P_{-x}) \circ RS$ in $D_{\rm qc}(A)$,
- $D_A \circ RS = ((-1_A)^* \circ RS \circ D_{\hat{A}})[g]$ in $D_{\rm qc}(\hat{A})$,
- $\phi^* \circ RS_B = RS_A \circ \hat{\phi}_*$ in $D_{\rm qc}(B)$,
- $\phi_* \circ RS_A = RS_B \circ \hat{\phi}^*$ in $D_{\rm qc}(A)$, and
- $T_x^* \circ D_A \cong D_A \circ T_x^*$ on $D(A)$.

Outline of the talk

1. Vanishing Theorems
2. The generic vanishing theorem of Green and Lazarsfeld
3. The Fourier Mukai transform
4. A derived category approach to Generic Vanishing
5. Pluricanonical maps of MAD varieties
6. Characterization of abelian varieties

A derived category approach to Generic Vanishing

- In this section we will establish a version of generic vanishing theorem via an algebraic proof which avoids Hodge Theory and (partially) generalizes to positive characteristics. This approach is due to Hacon and Pareschi-Popa.
- Recall that if L is an ample line bundle on $\hat{A}$, then there is an isogeny $\phi_L : \hat{A} \to A$ such that $\phi_L(x) = L \otimes T_x^* L^\vee \in \hat{\hat{A}} \cong A$.
- Let $\hat{L} = R^0 S(L)$, then we have $\phi_L^*(\hat{L}^\vee) \cong L^{\oplus h^0(L)}$.
- Thus $\hat{L}^\vee$ is an ample vector bundle of rank $h^0(L)$ with $h^0(\hat{L}^\vee) = 1$.
- If M is an ample line bundle on A and $L = mH$, then $\phi_L^*(M \otimes \hat{L}) = m(m\phi_H^* M - H)$ is ample for any $m \gg 0$.

A derived category approach to Generic Vanishing

Theorem

Let A be an abelian variety over an algebraically closed field k and $\mathcal{F}$ a coherent sheaf on A, then the following are equivalent.

1. $h^i(A, \mathcal{F} \otimes \hat{L}^\vee) = 0$ *for all* $i > 0$ *and* L *sufficiently ample,*
2. $R\hat{S}(D_A(\mathcal{F})) = R^g\hat{S}(D_A(\mathcal{F}))$,
3. $\mathrm{codim}\ \mathrm{Supp} R^i\hat{S}(\mathcal{F}) \geq i$ *for all* $i > 0$.

A derived category approach to Generic Vanishing

- We begin by showing that (1) and (2) are equivalent.
- By Groethendieck duality and the projection formula, we have $D_k(R\Gamma(\mathcal{F}\otimes\hat{L}^\vee))\cong R\Gamma(D_A(\mathcal{F}\otimes\hat{L}^\vee))\cong R\Gamma(D_A(\mathcal{F})\otimes\hat{L})\cong R\Gamma(D_A(\mathcal{F})\otimes p_{A,*}(\mathcal{P}\otimes p_{\hat{A}}^*L))\cong R\Gamma(L_{p_A^*}D_A(\mathcal{F})\otimes\mathcal{P}\otimes p_{\hat{A}}^*L)\cong R\Gamma(R\hat{S}(D_A(\mathcal{F}))\otimes L)$.
- It follows that $D_k(R\Gamma(\mathcal{F}\otimes\hat{L}^\vee))$ is a sheaf if and only if so is $R\Gamma(R\hat{S}(D_A(\mathcal{F}))\otimes L)$.

A derived category approach to Generic Vanishing

- $D_k(R\Gamma(\mathcal{F}\otimes\hat{L}^\vee))$ is a sheaf in degree 0 if and only if $H^i(A,\mathcal{F}\otimes\hat{L}^\vee)=0$ for all $i>0$.
- If L is sufficiently ample, then we may assume that each $R^j\hat{S}(D_A(\mathcal{F}))\otimes L$ are globally generated with vanishing higher cohomologies
- Thus $R^j\Gamma(R\hat{S}(D_A(\mathcal{F}))\otimes L)\cong H^0(R^j\hat{S}(D_A(\mathcal{F}))\otimes L)$ is a sheaf in degree 0 if and only if $H^0(R^j\hat{S}(D_A(\mathcal{F}))\otimes L)=0$ for all $j>0$ or equivalently if and only if $R^j\hat{S}(D_A(\mathcal{F}))=0$ for all $j>0$ (since each $R^j\hat{S}(D_A(\mathcal{F}))\otimes L$ is globally generated).

A derived category approach to Generic Vanishing

- We will now show that (2) implies (3).
- Let $\mathcal{G} = R^g\hat{S}(D_A(\mathcal{F}))$, then we have $R\hat{S}(\mathcal{F}) = D_{\hat{A}}D_{\hat{A}}(R\hat{S}(\mathcal{F})) = (-1_{\hat{A}})^*D_{\hat{A}}R\hat{S}D_A(\mathcal{F}[g]) = (-1_{\hat{A}})^*D_{\hat{A}}(\mathcal{G})$
- Thus $R^i\hat{S}(\mathcal{F}) = (-1_{\hat{A}})^*\mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_X)$ and we conclude by the following well known result:
- **Lemma:** Let $\mathcal{F}$ be a coherent sheaf on a smooth projective variety X, then $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) = 0\ \forall i < \mathrm{codim\ Supp}(\mathcal{F})$, and $\mathrm{codim\ Supp}\ \mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) \geq i \qquad \forall i \geq 0$.

A derived category approach to Generic Vanishing

- Finally we show that (3) implies (1).
- Suppose that $\operatorname{codim} \operatorname{Supp} R^i\hat{S}(\mathcal{F}) \geq i$ for all $i > 0$, then $H^j(R^i\hat{S}(\mathcal{F}) \otimes L^\vee) = 0$ for all $i + j > g$ and any line bundle L.
- From the spectral sequence $H^j(R^i\hat{S}(\mathcal{F}) \otimes L^\vee) \to R^{i+j}\Gamma(R\hat{S}(\mathcal{F}) \otimes L^\vee)$, it follows that $R^l\Gamma(R\hat{S}(\mathcal{F}) \otimes L^\vee) = 0$ for $l > g$.
- We have $R^l\Gamma(R\hat{S}(\mathcal{F}) \otimes L^\vee) = R^l\Gamma(Rp_{\hat{A},*}(p_A^*\mathcal{F} \otimes \mathcal{P}) \otimes L^\vee) = R^l\Gamma(p_A^*\mathcal{F} \otimes \mathcal{P} \otimes p_{\hat{A}}^*L^\vee) = R^l\Gamma(Rp_{A,*}(p_A^*\mathcal{F} \otimes \mathcal{P} \otimes p_{\hat{A}}^*L^\vee))$.
- Since $\widehat{L^\vee} = (-1_A)^*\hat{L}^\vee[-g]$, it follows that $H^l(\mathcal{F} \otimes \hat{L}^\vee) = R^l\Gamma(\mathcal{F} \otimes \hat{L}^\vee) = R^{l+g}\Gamma(p_A^*\mathcal{F} \otimes \mathcal{P} \otimes p_{\hat{A}}^*L^\vee) = 0, \ \forall l > 0$.

A derived category approach to Generic Vanishing

Definition

Let $\mathcal{F}$ be a coherent sheaf on an abelian variety A, then we say that $\mathcal{F}$ is a GV (generic vanishing) sheaf if $\mathcal{F}$ satisfies one of the equivalent statements of Theorem.

Proposition

Let $\mathcal{F}$ be a GV sheaf on an abelian variety A, then $V^i(\mathcal{F}) \supset V^{i+1}(\mathcal{F})$ for all $i \geq 0$.

A derived category approach to Generic Vanishing

Proof: This is an easy consequence of cohomology and base change.

- Suppose that $P \not\in V^i(\mathcal{F})$, then $0 = H^i(A, \mathcal{F} \otimes P)^\vee \cong H^{-i}(D_k R\Gamma(\mathcal{F} \otimes P)) \cong R^{-i}\Gamma(D_A(\mathcal{F}) \otimes P^\vee)$.
- But then, by cohomology and base change, the natural homomorphism $R^{-i-1}\hat{S}(D_A(\mathcal{F})) \otimes k(P^\vee) \to R^{-i-1}\Gamma(D_A(\mathcal{F}) \otimes P^\vee)$ is surjective.
- By assumption $R^{-i-1}\hat{S}(D_A(\mathcal{F})) = 0$ and so $H^{i+1}(A, \mathcal{F} \otimes P)^\vee \cong R^{i+1}\Gamma(D_A(\mathcal{F}) \otimes P^\vee) = 0$, so that $P \not\in V^{i+1}(\mathcal{F})$.
- Thus $V^i(\mathcal{F}) \supset V^{i+1}(\mathcal{F})$. $\square$

A derived category approach to Generic Vanishing

Proposition

Let $0 \neq \mathcal{F}$ be a GV sheaf on an abelian variety A, then $H^0(\mathcal{F} \otimes P) \neq 0$ for some $P \in \hat{A}$.

Proof: Suppose $H^0(\mathcal{F} \otimes P) = 0$ for all $P \in \hat{A}$.
Since $\mathcal{F}$ is GV, then $V^i(\mathcal{F}) = \emptyset$ for all $i \geq 0$.
But then $\mathcal{F} = 0$. $\square$

A derived category approach to Generic Vanishing

Proposition

Let $f : X \to A$ be a morphism from a smooth complex projective variety to an abelian variety, then $R^i f_* \omega_X$ is a GV sheaf for every $i \geq 0$. In particular $\operatorname{codim} V^i(R^j f_* \omega_X) \geq i$.

- **Proof:** This is an easy consequence of Kollár vanishing.
- We must show that $h^j(R^i f_* \omega_X \otimes \hat{L}^\vee) = 0$ for $j > 0$ and any sufficiently ample line bundle L on $\hat{A}$.
- Let $\phi_L : \hat{A} \to A$ and $\hat{X} = \hat{A} \times_A X$ and $\hat{a} : \hat{X} \to \hat{A}$
- Then $R^i a_* \omega_X \otimes \hat{L}^\vee$ is a summand of $\phi_{L,*}(R^i \hat{a}_* \omega_{\hat{X}} \otimes \phi_L^* \hat{L}^\vee) = R^i a_* \omega_X \otimes \hat{L}^\vee \otimes \phi_{L,*} \mathcal{O}_{\hat{A}}$
- But $h^j(R^i \hat{a}_* \omega_{\hat{X}} \otimes \phi_L^* \hat{L}^\vee) = h^j(R^i \hat{a}_* \omega_{\hat{X}} \otimes L)^{\oplus h^0(L)} = 0$ by KV.

A derived category approach to Generic Vanishing

Corollary

Let X be a smooth complex projective variety and $a : X \to A$ its Albanese morphism, then

$$\operatorname{codim} V^i(\omega_X) \geq i - \dim X + \dim a(X).$$

- **Proof:** $Ra_*\omega_X = \sum R^i a_*\omega_X[-i]$ so $H^j(\omega_X \otimes a^*P) = \oplus_{i=0}^k H^{j-i}(R^i a_*\omega_X \otimes P)$ where $k = \dim X - \dim a(X)$ so $V^i(\omega_X) = \cup_{i=0}^k V^{j-i}(R^i a_*\omega_X)$.
- Each $R^i a_*\omega_X$ is a GV sheaf.
- But then $\operatorname{codim} V^{j-i}(R^i a_*\omega_X) \geq j - i$ and so $\operatorname{codim} V^j(\omega_X) \geq j - k = j - \dim X + \dim a(X)$. $\square$

Fibration Theorem

Theorem (Green Lazarsfeld (1991))

Let X be a smooth complex projective variety and W be an irreducible component of $V^i(\omega_X)$ for some i, then there is a morphism $f : X \to Y$ with Y normal of MAD and $\dim Y \leq \dim X - i$ such that

$$W \subset P + f^*\mathrm{Pic}^0(Y), \qquad P \in \mathrm{Pic}^0(X).$$

Proof: Let $A \to S = \hat{W}$ be the morphism dual to the inclusion $W \subset A = \mathrm{Pic}^0(X)$. Consider the Stein factorization $(h \circ g) : X \to Y \to S$.

Fibration Theorem

- We may assume that P is torsion and hence $Rh_*(\omega_X \otimes P) = \sum R^i h_*(\omega_X \otimes P)[-i]$.
- Since $Rg_*(R^i h_*(\omega_X \otimes P)) = g_*(R^i h_*(\omega_X \otimes P))$ is a GV sheaf, then $h^j(g_*(R^i h_*(\omega_X \otimes P)) \otimes g^*\alpha) = 0$ for all $j > 0$ and general $\alpha \in \hat{S}$.
- But then $0 \neq h^i(\omega_X \otimes P \otimes (h \circ g)^*\alpha) = h^0(R^i h_*(\omega_X \otimes P) \otimes g^*\alpha)$.
- Thus $R^i h_*(\omega_X \otimes P) \neq 0$. In particular (as $R^i h_*(\omega_X \otimes P)$ is torsion free), $i \geq \dim X - \dim Y$. $\square$

Structure of CSL

Proposition

Let $\mathcal{F}$ be a GV coherent sheaf on an abelian variety A and $Z \subset V^0(\mathcal{F})$ an irreducible component of codimension k. Then Z is a component of $V^k(\mathcal{F})$. Thus $\dim \mathrm{Supp}(\mathcal{F}) \geq k$.

- **Proof:** We know that $R\hat{S}(D_A(\mathcal{F})) = R^g\hat{S}(D_A(\mathcal{F}))$ is a sheaf.
- Thus $R\hat{S}\mathcal{F} = RHom(\mathcal{G}, \mathcal{O}_A)$ (where $\mathcal{G} = (-1_{\hat{A}})^* R^g(D_A(\mathcal{F}))$).
- Then $\mathrm{Supp}\mathcal{G} = (-1_{\hat{A}})^*\mathrm{Supp}(R^g\hat{S}(D_A(\mathcal{F}))) = (-1_{\hat{A}})^* V^0(\mathcal{F})$.
- Localizing at the generic point of Z, we have $R^i\hat{S}\mathcal{F} = \mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_A) = 0$ for $i < k$ and $\mathrm{codim}\mathrm{Supp}R^i\hat{S}\mathcal{F} \geq i$.
- Thus Z is a component of $R^k\hat{S}\mathcal{F}$ and hence a component of $V^k(\mathcal{F})$ (by cohomology and base change). $\square$

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Pluricanonical maps

- Let X be a smooth complex projective variety, then the m-th pluricanonical map is $\phi_{mK_X} : X \dashrightarrow \mathbb{P}^N = |mK_X|$
- If $s_0, \ldots, s_N$ is a basis of $H^0(mK_X)$, then this map is simply defined by $\phi_{mK_X}(x) = [s_0(x) : \ldots : s_N(x)]$.
- The Kodaira dimension of X is $\kappa(X) = \max\{\dim \phi_{mK_X}(X) | m > 0\}$ (unless $|mK_X| = \emptyset$ for all $m > 0$ in which case we say that $\kappa(X) = -\infty$).
- For example, if $\dim(X) = 1$, then $\kappa(X) = -\infty, 0, 1$ iff $g(X) = 0, 1, \geq 2$.
- If $\kappa(X) = \dim X$, then X has general type and ϕ_{mK_X} is birational for some $m > 0$.

Pluricanonical maps

- If $\dim X = 1, 2, 3$, then ϕ_{mK_X} is birational for $m \geq 3, 5, 77$ (by results of Bombieri and Chen-Chen).
- In general (by a result of Hacon-McKernan, Takayama, Tsuji) there is an integer r_n depending on $n = \dim X$ such that ϕ_{mK_X} is birational for $m \geq r_n$.

Theorem (Lahoz, Jiang, Tirabassi (2011))

Let X be a smooth complex projective variety of MAD and general type. Then ϕ_{mK_X} is birational for all $m \geq 3$.

Pluricanonical maps

- We will give an idea of the proof for $m \geq 6$. A similar argument works for $m \geq 3$ if we assume $\chi(\omega_X) > 0$ (eg. if the Albanese variety is simple). The general case is of course more involved.
- **Proof:** We begin by observing that $h^0(\omega_X^2 \otimes a^*P) \neq 0$ for all $P \in \hat{A}$ (where $a : X \to A$ is the Albanese morphism).
- To this end, consider $\mathcal{F} = a_*(\omega_X^2 \otimes \mathcal{J}(||K_X||))$.
- **Claim:** $h^i(\mathcal{F} \otimes P) = 0$ for all $i > 0$ and $P \in \hat{A}$ and since $\mathcal{F} \neq 0$. (Grant this for now.)
- Then $h^0(\omega_X^2 \otimes \mathcal{J}(||K_X||) \otimes a^*P) = h^0(\mathcal{F} \otimes P) \neq 0$ for all $P \in \hat{A}$. It follows that

Pluricanonical maps

- There is an open subset $U \subset X$ such that if $x \in U$, then for general $P \in \hat{A}$, $\omega_X^2 \otimes \mathcal{J}(||K_X||) \otimes a^*P$ is generated at $x \in X$.
- Note that $\mathcal{J}(||K_X||) \cdot \mathcal{J}(||K_X||) \subset \mathcal{J}(||3K_X||)$.
- From the natural map $H^0(\omega_X^2 \otimes \mathcal{J}(||K_X||) \otimes a^*P) \otimes H^0(\omega_X^2 \otimes \mathcal{J}(||K_X||) \otimes a^*Q) \to H^0(\omega_X^4 \otimes \mathcal{J}(||3K_X||) \otimes a^*P \otimes a^*Q)$ it follows that $\omega_X^4 \otimes \mathcal{J}(||3K_X||) \otimes a^*R$ is generated at x for any $R \in \hat{A}$.
- From the short exact sequence $0 \to \omega_X^4 \otimes \mathcal{J}(||3K_X||) \otimes I_x \to \omega_X^4 \otimes \mathcal{J}(||3K_X||) \to \mathbb{C}(x) \to 0$, it follows easily that $h^i(\omega_X^4 \otimes \mathcal{J}(||3K_X||) \otimes I_x \otimes P) = 0$ for all $i > 0$, $P \in \hat{A}$.

Pluricanonical maps

- Since $\omega_X^4 \otimes \mathcal{J}(||3K_X||) \otimes I_x \neq 0$, it follows that $h^0(\omega_X^4 \otimes \mathcal{J}(||3K_X||) \otimes I_x \otimes P) > 0$ for all $P \in \hat{A}$.
- Possibly shrinking U, we may assume that for general $P \in \hat{A}$, $\omega_X^4 \otimes \mathcal{J}(||3K_X||) \otimes I_x \otimes P$ is generated at any point $x \neq x' \in U$.
- Arguing as above, $\omega_X^6 \otimes \mathcal{J}(||3K_X||) \otimes I_x \otimes P$ is generated at $x \neq x' \in U$ for any $P \in \hat{A}$. □
- We must now verify the claim that $h^i(a_*(\omega_X^2 \otimes \mathcal{J}(||K_X||)) \otimes P) = 0$ for all $i > 0$ and $P \in \hat{A}$.
- Recall that $\mathcal{J}(||K_X||)) = \cup_{m \geq 0} \mathcal{J}(\frac{1}{m}|mK_X|) = \mathcal{J}(\frac{1}{m}|mK_X|)$ for some $m > 0$ sufficiently divisible.

Pluricanonical maps

- To compute $\mathcal{J}(\frac{1}{m}|mK_X|)$, let $\mu : X' \to X$ be a log resolution of $|mK_X|$ so that X' is smooth, $\mathrm{Ex}(\mu)$ is a divisor, $\mu^*|mK_X| = F_m + M_m$ where M_m is base point free and F_m is a divisor such that $F_m + \mathrm{Ex}(\mu)$ has SNC support.
- Then $\omega_X^2 \otimes \mathcal{J}(\frac{1}{m}|mK_X|) = \mu_*\mathcal{O}_{X'}(K_{X'} + \mu^*K_X - \lfloor F_m/m \rfloor)$.
- Since $f^*K_X - \lfloor F_m/m \rfloor \equiv \{F_m/m\} + M_m/m$, where $(X', \{F_m/m\})$ is klt and M_m/m is nef and big, we have $R^j\mu_*\mathcal{O}_{X'}(K_{X'} + \mu^*K_X - \lfloor F_m/m \rfloor) = 0$ for $j > 0$.
- Thus for all $i > 0$ and $P \in \mathrm{Pic}^0(X)$
  $H^i(\omega_X^2 \otimes \mathcal{J}(\frac{1}{m}|mK_X|) \otimes P) =$
  $H^i(\mathcal{O}_{X'}(K_{X'} + \mu^*K_X - \lfloor F_m/m \rfloor) \otimes \mu^*P) = 0.$

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Characterization of abelian varieties

Theorem (Chen-Hacon (2002))

Let X be a smooth complex projective variety and $a : X \to A$ be its Albanese morphism.

1. *If $P_2(X) \leq 1$, then a is surjective and in particular $q(X) \leq \dim X$.*
2. *If $P_2(X) = 1$ and $q(X) = \dim X$, then X is birational to an abelian variety.*

- The above result has a long history. Kawamata showed that if $\kappa(X) = 0$, then the Albanese morphism a is surjective and if moreover $q(X) = \dim(X)$, then X is birational to an abelian variety.

$C_{n,m}$ Conjecture

- The first statement is a consequence of Kawamata's results on the $C_{n,m}$ conjecture:
- $C_{n,m}$ **Conjecture:** Let $g : V \to W$ be a surjective morphism of smooth complex projective varieties with connected fibers. Let F be a very general fiber. Then $\kappa(V) \geq \kappa(W) + \kappa(F)$.
- Recall that if $P_m(V) > 0$ for some $m > 0$, then we let

$$\kappa(V) = \max\{\dim(\phi_{mK_V}(V))\}.$$

- If instead $P_m(V) = 0$ for all $m > 0$, then we say that $\kappa(V) = -\infty$.
- We say that V is of general type if $\kappa(V) = \dim V$.

$C_{n,m}$ Conjecture

Kawamata proved several important cases of the above conjecture.

Theorem (Kawamata (1981))

Let $g : V \to W$ be a surjective morphism of smooth complex projective varieties with connected fibers. Let F be a very general fiber. If W is of general type, then Then

$$\kappa(V) \geq \dim(W) + \kappa(F).$$

Therefore, if X is a smooth complex projective variety with $\kappa(X) = 0$, it is easy to see that for any surjective morphism $f : X \to Y$ with connected very general fiber F, we have $\kappa(F) \geq 0$ (simply because if $G \in |mK_X|$, then $G|_F \in |mK_F|$) and then by the above theorem, we have that Y is not of general type.

Characterization of abelian varieties

- Consider now the Albanese morphism $a : X \to A$. Let W be its image.
- If
$$K = \{a \in A | a + W \subset W\}^0,$$
then $W \to V := W/K$ is the Iitaka fibration of W and $V \subset A/K$ is of general type.
- If $\dim V > 0$, then this is a contradiction as explained above.
- If $\dim V = 0$, then $V = A/K$ and so $W = A$, i.e. $a : X \to A$ is surjective.
- One then shows that if $\kappa(X) = 0$ and $q(X) = \dim(X)$ then the generically finite map $X \to A$ is étale in codimension 1 and hence birationally étale so that X is birational to an abelian variety.

Proof

- We will now explain the proof of the above theorem of Chen-Hacon.
- Assume that $P_2(X) = 1$ and let $W = a(X)$ and $V = W/K$ as above. We denote $f : X \to V$ the induced morphism.
- Consider now the asymptotic multiplier ideal sheaf

$$\mathcal{J} = \mathcal{J}(||K_X||) = \cup_{m>0}\mathcal{J}(\frac{1}{m}|mK_X|).$$

- Recall that if $\mu : X' \to X$ is a log resolution of $|mK_X|$ so that $\mu^*|mK_X| = F_m + M_m$ where μ is a proper birational morphism, X' is smooth, $\mathrm{Ex}(\mu)$ is a divisor, F_m is the fixed divisor of $\mu^*|mK_X|$, $F_m + \mathrm{Ex}(\mu)$ has simple normal crossings support and M_m is base point free.
- Then $\mathcal{J}(\frac{1}{m}|mK_X|) = \mu_*\mathcal{O}_{X'}(K_{X'/X} - \lfloor F_m/m \rfloor)$.

Proof

- Notice that if μ is a log resolution for $|mK_X|$ and $|pmK_X|$, then from the inclusion $|mK_X|^{\times p} \to |pmK_X|$, one easily sees that $pF_m \geq F_{pm}$ and hence that $\mathcal{J}(\frac{1}{m}|mK_X|) \subset \mathcal{J}(\frac{1}{pm}|pmK_X|)$.
- Since X is Noetherian, $\cup_{m>0}\mathcal{J}(\frac{1}{m}|mK_X|) = \mathcal{J}(\frac{1}{k}|kK_X|)$ for any $k > 0$ sufficiently divisible and in particular $\mathcal{J}(||K_X||)$ is a coherent sheaf.

Claim

We have that $H^i(f_*(\omega_X^2 \otimes \mathcal{J}(||K_X||)) \otimes P) = 0$ for all $i > 0$ and $P \in \mathrm{Pic}^0(A/K)$.

Proof

- To see the claim, let $f' = f \circ \nu$, then since
$$\nu^* K_X - \lfloor F_m/m \rfloor \equiv \{F_m/m\} + M_m/m$$
and $(X', \{F_m/m\})$ is klt whilst M_m/m is semiample, it follows that $R^i \nu_*(\omega_{X'} \otimes \nu^* \omega_X(-\lfloor F_m/m \rfloor)) = 0$ for $i > 0$
- Thus $H^i(f_*(\omega_X^2 \otimes \mathcal{J}(||K_X||)) \otimes P) =$ $H^i(f'_*(\omega_{X'}(\nu^* K_X - \lfloor F_m/m \rfloor)) \otimes P)$.
- We may assume that dim $V > 0$ and hence that V is of general type.
- From the proof of Kawamata's Theorem, it follows that for some $m > 0$ we have $m\nu^* K_X \sim f'^* H + E$ where H is ample on A/K and $E \geq 0$. It is easy to see that $F_m \leq E$. But then $\nu^* K_X - \lfloor F_m/m \rfloor \equiv \{F_m/m\} + \delta(E - F_m) + (1-\delta) M_m/m + \delta f'^* H/m$.

Proof

- Since $(X', \{F_m/m\} + \delta(E - F_m))$ is klt for $0 < \delta \ll 1$, and $H/m + P$ is ample, by Kollár vanishing we have that $H^i(f'_*(\omega^2_{X'}(-\lfloor F_m/m \rfloor)) \otimes P) = 0$. The claim is proven.
- **Claim:** We have that $H^0(f_*(\omega^2_X \otimes \mathcal{J}(||K_X||)) \otimes P) = 1$ for all $P \in \mathrm{Pic}^0(A/K)$.
- To see the claim, assume that μ is a log resolution of $|2K_X|$ and $|mK_X|$.
- Notice that $F_m/m \leq F_2/2 \leq F_2$ (with room to spare!).
- Thus $H^0(f_*(\omega^2_X \otimes \mathcal{J}(||K_X||))) = H^0(f'_*(\omega^2_{X'}(-\lfloor F_m/m \rfloor))) \supset H^0(f'_*(\omega^2_{X'}(-F_2))) = H^0(f'_*(\omega^2_{X'}))$

Proof

- But then $H^0(f_*(\omega_X^2 \otimes \mathcal{J}(||K_X||))) = H^0(f'_*(\omega_{X'}^2)) = P_2(X) = 1$.
- We next observe that $H^0(f'_*(\omega_{X'}^2 \otimes \mathcal{J}(||K_X||)) \otimes P) = \chi(f'_*(\omega_{X'}^2 \otimes \mathcal{J}(||K_X||)) \otimes P) = \chi(f'_*(\omega_{X'}^2 \otimes \mathcal{J}(||K_X||))) = h^0(f'_*(\omega_{X'}^2 \otimes \mathcal{J}(||K_X||))) = 1$.
- The claim is proven.

Proof

- Let $R\hat{S}$ denote the Fourier Mukai transform from A/K to $\widehat{A/K}$.
- By cohomology and base change, one sees that $R\hat{S}(f'_*(\omega^2_{X'}\otimes\mathcal{J}(||K_X||))) = R^0\hat{S}(f'_*(\omega^2_{X'}\otimes\mathcal{J}(||K_X||)))$ is a line bundle L of index $\dim A/K$ (i.e. $-L$ is ample) and hence $f'_*(\omega^2_{X'}\otimes\mathcal{J}(||K_X||))$ is a vector bundle of rank $h^{\dim A/K}(L)$.
- In particular $\mathrm{Supp}(f'_*(\omega^2_{X'}\otimes\mathcal{J}(||K_X||))) = A/K$ and so $X \to A$ is surjective and (1) is proven.

Proof

- We now turn to the proof of (2). From what we have already seen, we may assume that $a : X \to A$ is generically finite on to its image. We begin by proving the following.
- **Claim:** $V^0(\omega_X) = \{\mathcal{O}_X\}$.
- Recall that if P is an isolated point of $V^0(\omega_X)$, then $h^g(\omega_X \otimes P) \neq 0$. Since $g = \dim A = \dim X$, it follows that $h^0(P^\vee) \neq 0$ and hence that $P = \mathcal{O}_X$.
- Suppose now that $\{\mathcal{O}_X\} \neq Z$ is an irreducible component of $V^0(\omega_X)$, then $\dim Z > 0$.

Proof

- If $\mathcal{O}_X \in Z$, then as Z is a subtorus of $\hat{A}$, we have $Z = -Z$ and hence nontrivial maps
$$|K_X + P| \times |K_X - P| \to |2K_X|$$
for any $P \in Z$.
- Since $\dim |2K_X| = 0$, it follows that $|K_X + P|$ is non-empty for only finitely many P. Thus $\dim Z = 0$ contradicting our assumption.

Proof

- Therefore we may assume that $\mathcal{O}_X \notin Z$
- We may write $Z = P + T$ where T is a subtorus (i.e. $\mathcal{O}_X \in T$) and $P \in \mathrm{Pic}^0(X)$ is a torsion point.
- Let $S = \hat{T}$, and consider $p : A \to S$ the induced morphism.
- Let $k = q(X) - \dim T$ be the codimension of T in $\hat{A}$. We have $H^k(\omega_X \otimes Q) \neq 0$ for any $Q \in Z$.
- For any $\alpha \in T = \hat{S}$ we have

$$H^k(\omega_X \otimes P \otimes a^* p^* \alpha) = \oplus_{j=0}^{k} H^{k-j}(S, R^j(p \circ a)_*(\omega_X \otimes P) \otimes \alpha).$$

- The left hand side does not vanish for any $\alpha \in \hat{S}$ (since $Z = P + (p \circ a)_* \hat{S}$).

Proof

- By the generic vanishing Theorem, we have $0 = H^{k-j}(S, R^j(p \circ a)_*(\omega_X \otimes P) \otimes \alpha)$ for any $j < k$ and $\alpha \in \hat{S}$ general.
- Hence $H^0(S, R^k(p \circ a)_*(\omega_X \otimes P) \otimes \alpha) \neq 0$ for any $\alpha \in \hat{S}$.
- In particular $R^k(p \circ a)_*(\omega_X \otimes P) \neq 0$.
- Let $X \to W \to S$ be the Stein factorization and F the general fiber of $X \to W$, then $\dim F = k$.
- Since $R^k(p \circ a)_*(\omega_X \otimes P) \neq 0$, it follows easily that $h^0(F, (P|_F)^\vee) = h^k(F, \omega_F \otimes P|_F) \neq 0$.
- But then $P|_F \cong \mathcal{O}_F$ and so $R^k(p \circ a)_*(\omega_X \otimes P^\vee) \neq 0$. It follows that $V^0(R^k(p \circ a)_*(\omega_X \otimes P^\vee)) \neq \emptyset$.

Proof

- Notice that if $\alpha \in \hat{S}$ is an isolated point of $V^0(R^k(p \circ a)_*(\omega_X \otimes P^\vee)) \subset \hat{S}$, then it is also a point in $V^{\dim S}(R^k(p \circ a)_*(\omega_X \otimes P^\vee))$.
- However $k + \dim S = \dim A$ and then $h^{\dim A}(\omega_X \otimes P^\vee \otimes (p \circ a)^*\alpha) \neq 0$.
- Thus $h^0(P \otimes (p \circ a)^*\alpha^\vee) \neq 0$ so that $P = \otimes(p \circ a)^*\alpha$ which is a contradiction.
- Therefore there is a positive dimensional subtorus $T' \subset T$ and a point $P' \in \mathrm{Pic}^0(X)$ such that $T' + P', T' - P' \subset V^0(\omega_X)$.
- Considering the map $|K_X + Q| \times |K_X - Q| \to |2K_X|$ for any $Q \in T' + P'$ shows that $\dim |2K_X| > 0$ which is impossible. $\square$

Proof

- Therefore $V^0(\omega_X) = \{\mathcal{O}_A\}$.
- Consider the inclusion $\mathcal{O}_A \to a_*\omega_X$ and let $\mathcal{F} = a_*\omega_X/\mathcal{O}_A$.
- Since $H^g(\mathcal{O}_A) \to H^g(a_*\omega_X)$ is an isomorphism (dual to $H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_A)$), it follows that $V^g(\mathcal{F}) = \emptyset$ and hence $R^g\hat{S}(\mathcal{F}) = 0$.
- Since $V^i(a_*\omega_X) \subset \{\mathcal{O}_A\}$, $R\hat{S}(a_*\omega_X) = R^g\hat{S}(a_*\omega_X)[-g]$.
- Since $k(0_{\hat{A}}) = R^g\hat{S}(\mathcal{O}_A) \to R^g\hat{S}(a_*\omega_X)$ is nonzero, it is injective and hence an isomorphism (as $R^g\hat{S}(\mathcal{F}) = 0$).
- But then $R\hat{S}(\mathcal{O}_A) \to R\hat{S}(a_*\omega_X)$ is an isomorphism and hence so is $\mathcal{O}_A \to a_*\omega_X$.
- Thus the degree of a is one and hence $X \to A$ is birational. $\square$

Other Birational characterization results

Theorem (Hacon-Pardini, Pirola (2001))

Let X be a smooth complex projective surface with $p_g = q = 3$ then X is birational to a theta divisor or to the quotient $C_2 \times C_3/(\mathbb{Z}/2\mathbb{Z})$ where C_i are curves of genus i.

Theorem (Hacon-Pardini (2004))

Let X be a variety of maximal Albanese dimension. if $\chi(\omega_X) = 1$ then the $q(X) \leq 2\dim X$ and if $q(X) = 2\dim X$, then X is birational to a product of curves of genus 2.

Other Birational characterization results

Theorem (Hacon-Pardini, Hacon, Hacon-Chen)

If $P_3(X) \leq 4$, then $X \to A$ is surjective and if moreover $q(X) = \dim X$ then X is classified up to birational isomorphism.

Theorem (Jiang-Lahoz-Tirabassi (2013))

Let $X \subset A$ be a subvariety of an abelian variety and $\mu : X' \to X$ a resolution. Then X is a product of Theta divisors if and only if X is normal and $\chi(\omega_{X'}) = 1$.

Outline of the talk

1. Vanishing Theorems
2. The generic vanishing theorem of Green and Lazarsfeld
3. The Fourier Mukai transform
4. A derived category approach to Generic Vanishing
5. Pluricanonical maps of MAD varieties
6. Characterization of abelian varieties

Failure of vanishing and GV

- It is well known that Kodaira vanishing fails in positive characteristics (Mumford, Raynaud, Mukai, Lauritzen-Rao....)
- Therefore, one expects that generic vanishing will also fail in positive characteristic.

Theorem (Hacon-Kovacs (2013), Filipazzi (2016))

Let p be a prime number and $n \geq 2$ then there exists a smooth complex projective variety X defined over an algebraically closed field of characteristic p with $\dim X = n$ and a generically finite morphism $a : X \to A$ to an abelian variety such that $Rp_{\hat{A},*}((a \times p_{\hat{A}})^* \mathcal{P})$ is not a sheaf (and in particular generic vanishing fails).

Failure of GV

- The idea is as follows. Fix $n = 2$ and p a prime.
- Let X be a smooth projective surface defined over an algebraically closed field of char $p > 0$ and L a line bundle such that $H^1(K_X + L) \neq 0$ (Mukai).
- It may happen that $q(X) = 0$, so consider a generic projection $X \to \mathbb{P}^2$ and $A \to \mathbb{P}^2$ where A is an abelian surface.
- Consider $\alpha : X' = X \times_{\mathbb{P}^2} A \to A$, then α is generically finite.
- X' is smooth (as the projections are generic).
- Note that $0 \neq H^1(K_X + L) \subset H^1(K_{X'} + L')$ (if $p \not| \deg(a)$).
- Replacing (X, L) by (X', L'), we may assume that $a : X \to A$ is finite.

Failure of GV

- Now, $0 \neq H^1(K_X + L) = H^1(a_*(K_X + L))$.
- We must show that if $m \gg 0$ and H is ample on $\hat{A}$, then $H^1(\omega_X \otimes a^*\widehat{mH}^\vee) = H^1(a_*\omega_X \otimes \widehat{mH}^\vee) \neq 0$
- Let $\phi_{mH} : \hat{A} \to A$. Assume that p doesn't divide $h^0(mH)$ and hence p does not divide the degree of ϕ_{mH}.
- Let $\hat{X} = X \times_A \hat{A}$, then it suffices to show that $H^1(\omega_{\hat{X}} \otimes \hat{a}^*\phi^*_{mH}\widehat{mH}^\vee) = \oplus_{h^0(mH)} H^1(\omega_{\hat{X}} \otimes m\hat{a}^*H) \neq 0$.
- For $m \gg 0$, assume that there is a smooth curve $C \sim \hat{L} - m\hat{a}^*H$ (where $\hat{L}$ is the pull back of L to $\hat{X}$).
- This can be achieved after taking another finite cover.

Failure of GV

- Clearly $h^1(\omega_C(\hat{a}^*(mH)|_C)) = 0$.
- From the short exact sequence

$$0 \to \omega_{\hat{X}}(\hat{a}^*(mH)) \to \omega_{\hat{X}}(\hat{L}) \to \omega_C(\hat{a}^*(mH)|_C) \to 0$$

it follows that
$h^1(\omega_{\hat{X}}(\hat{a}^*(mH))) \geq h^1(\omega_{\hat{X}}(\hat{L})) \geq h^1(\omega_X(L)) > 0$.

GV in positive characteristics

- Of course, if we assume that X lifts to the second Witt vectors W_2, then by the celebrated theorem of Deligne-Illusie, Kodaira vanishing still holds on X and hence we expect that Generic Vanishing also holds.
- If $\dim X = 2$ and X lifts to W_2, then Yuan Wang has shown that Generic Vanishing holds.
- In general one can still recover a version of generic vanishing by using the theory of F singularities.

GV in positive characteristics

- Let $F : X \to X$ be the Frobenius morphism, by Groethendieck duality there is an isomorphism $F_* Hom(\omega_X, \omega_X) \cong Hom(F_*\omega_X, \omega_X)$.
- Let $\Phi : F_*\omega_X \to \omega_X$ be the homomorphism corresponding to $F_*(\mathrm{id}_{\omega_X})$, and $\Phi^e : F^e_*\omega_X \to \omega_X$ the homomorphism obtained by iterating this procedure e times.
- Note that $\mathrm{Im}(\Phi) \supset \mathrm{Im}(\Phi^2) \supset \mathrm{Im}(\Phi^3) \supset \dots$
- By a result of Gabber, we have $\cap_{e>0}\mathrm{Im}(\Phi^e) = \mathrm{Im}(\Phi^{e_0})$ for some $e_0 > 0$.
- In particular the sheaves $F^e_*\omega_X$ satisfy the Mittag-Leffler condition so that $H^i(\varprojlim F^e_*\omega_X) = \varprojlim H^i(F^e_*\omega_X)$.

GV in positive characteristics

- Therefore if L is ample on X, we have $H^i((\varprojlim F^e_*\omega_X)\otimes L) = \varprojlim H^i(F^e_*(\omega_X \otimes L^{p^e}))$.
- Since F is finite, $H^i(F^e_*(\omega_X \otimes L^{p^e})) \cong H^i(\omega_X \otimes L^{p^e}) = 0$ for all $i > 0$ and hence $H^i((\varprojlim F^e_*\omega_X)\otimes L) = 0$ for all $i > 0$.
- One could hope that $\varprojlim F^e_*\omega_X$ could play the role played by ω_X in characteristic 0. However, since $\varprojlim F^e_*\omega_X$ is not even quasi-coherent, this leads to many difficulties.
- For example it is hoped that if $f : X \to Y$ is a birational map and X is smooth then $\varprojlim R^j f_*\omega_X = 0$ for $j > 0$ (which would generalize Grauert Riemanschneider vanishing).
- In practice one often considers $S^0(\omega_X \otimes L) = \mathrm{Im}(H^0(F^e_*(\omega_X \otimes L^{p^e})) \to H^0(\omega_X \otimes L))$ and $\omega_X \otimes \sigma = \mathrm{Im}(\Phi^e)$ for $e \gg 0$.

GV in positive characteristics

Theorem (Hacon-Patakfalvi (2013))

Let $\Omega_{e+1} \to \Omega_e$ be an inverse system of coherent sheaves on an abelian veriety A such that for any sufficiently ample line bundle L on $\hat{A}$ and any $e \gg 0$ we have $H^i(\Omega_e \otimes \hat{L}^\vee) = 0$ for $i > 0$, then $\Lambda := \mathrm{hocolim} R\hat{S}(D_A(\Omega_e))$ is a quasi-coherent sheaf in degree 0 (i.e. $\Lambda = \mathcal{H}^0(\Lambda)$). If Ω_e satisfies the Mittag Leffler property, then $\Omega := \varprojlim \Omega_e = (-1_A)^* D_A RS(\Lambda)[-g]$.

The proof of the above statement is omitted because it is similar to the derived category proof of generic vanishing. It is somewhat more involved due to some technical issues arising from the fact that we are not dealing with coherent sheaves.

GV in positive characteristics

- Let $a : X \to A$ a morphism from a smooth projective variety to an abelian variety.
- The homomorphism $\Phi^e : F^e_* \omega_X \to \omega_X$ gives rise to a homomorphism $F^e_* a_* \omega_X = a_* F^e_* \omega_X \to a_* \omega_X$.
- If $\Omega_e = F^e_* a_* \omega_X$, then Ω_e satisfies the ML.
- **Claim:** If $\Omega_e = F^e_* a_* \omega_X$, then $H^i(\Omega_e \otimes \hat{L}^\vee \otimes P) = 0$ for $e \gg 0$, $i > 0$ and $P \in \hat{A}$.
- **Proof.** To see this, note that $H^i(\Omega_e \otimes \hat{L}^\vee \otimes P) \cong H^i(F^e_* \Omega_0 \otimes \hat{L}^\vee \otimes P) \cong H^i(F^e_*(\Omega_0 \otimes F^{e,*}\hat{L}^\vee \otimes P^{p^e})) \cong H^i(\Omega_0 \otimes F^{e,*}\hat{L}^\vee \otimes P^{p^e})$.
- By Cohomology and base change, the above condition is equivalent to $R^i\hat{S}(\Omega_0 \otimes F^{e,*}\hat{L}^\vee) = 0$ for $i \neq 0$.

GV in positive characteristics

- This is equivalent to $\hat{\phi}_{L,*}R^i\hat{S}(\Omega_0 \otimes F^{e,*}\hat{L}^\vee) = R^i\hat{S}(\phi_L^*(\Omega_0 \otimes F^{e,*}\hat{L}^\vee)) = 0$ for $i \neq 0$.
- Arguing as above, this is equivalent to $H^i(\phi_L^*(\Omega_0 \otimes F^{e,*}\hat{L}^\vee) \otimes P) = 0$ for $i \neq 0$.
- But $\phi_L^* F^{e,*}\hat{L}^\vee = F^{e,*}\phi_L^*\hat{L}^\vee = \oplus_{h^0(L)} L^{p^e}$ and so it suffices to show that $H^i(\Omega_0 \otimes L^{p^e} \otimes P) = 0$ for $i \neq 0$ and $P \in \hat{A}$.
- Since $e \gg 0$, this follows immediately from Fujita vanishing $\square$
- Somewhat surprisingly, one can use the above theorem to prove some interesting consequences concerning the birational geometry of irregular varieties.

GV in positive characteristics

Corollary

Let $y \in \hat{A}$ is a closed point, then
$\Lambda \otimes k(y) \cong \varinjlim H^0(\Omega_e \otimes P_y^\vee)^\vee \cong \varinjlim H^0(\Omega_0 \otimes P_y^{-p^e})$.

Proof: Since $\Lambda = \mathcal{H}^0(\Lambda) = \varinjlim \mathcal{H}^0(\Lambda_e)$ and by cohomology and base change and Serre duality $\mathcal{H}^0(\Lambda_e) \otimes k(y) \cong \mathcal{H}^0(R\hat{S}D_A(\Omega_e)) \otimes k(y) \cong R^0\Gamma(D_A(\omega_e) \otimes P_y) \cong H^0(\Omega_e \otimes P_y^\vee)^\vee$. □

Corollary

If $H^0(\Omega_e \otimes P) = 0$ for all $P \in \hat{A}$, then $\Lambda = 0$ and hence $\Omega = 0$.

Proof: Immediate. □

GV in positive characteristics

Proposition

Suppose that $F^e_*\Omega_0 \to \Omega_0$ is surjective for $e \gg 0$ (eg. $\Omega_0 = S^0 a_*\omega_X$). If $V^0(\Omega_0)$ is contained in finitely many translates of an abelian subvariety $\hat{B} \subset \hat{A}$, then $T_x^*\Omega \cong \Omega$ for any $x \in \widehat{\hat{A}/\hat{B}}$. Thus the supports of Ω and Ω_0 are fibered by the projection $A \to B$.

Proof: The coherent sheaf $\mathcal{H}^0(\Lambda_0)$ is supported on $V^0(\Omega_0)$ and hence on finitely many fibers of $\pi : \hat{A} \to \hat{A}/\hat{B}$.

Thus $\mathcal{H}^0(\Lambda_0) \otimes \pi^* P \cong \mathcal{H}^0(\Lambda_0)$ for any $P \in \widehat{\hat{A}/\hat{B}}$.
For $e > 0$ we also have $\mathcal{H}^0(\Lambda_e) \otimes \pi^* P \cong \hat{F}^{e,*}\mathcal{H}^0(\Lambda_0) \otimes \pi^* P \cong \hat{F}^{e,*}(\mathcal{H}^0(\Lambda_0) \otimes \pi^* Q) \cong \hat{F}^{e,*}\mathcal{H}^0(\Lambda_0) \cong \mathcal{H}^0(\Lambda_e)$.

GV in positive characteristics

- Therefore $\Lambda \otimes \pi^* P \cong \mathcal{H}^0(\Lambda) \otimes \pi^* P \cong \varinjlim \mathcal{H}^0(\Lambda_e) \otimes \pi^* P \cong \varinjlim \mathcal{H}^0(\Lambda_e) \cong \Lambda$.
- Since $T^*_{-x} RS(?) \cong RS(? \otimes P_x)$, it follows that $T^*_x \Omega = \Omega$ for any $x \in \widehat{\hat{A}/\hat{B}}$. $\square$

Let $a : X \to A$ and $V^0_S(X) := \{P \in \hat{A} | S^0(\omega_X \otimes a^* P) \neq 0\}$, then it is easy to see that $V^0_S(X) \subset \overline{V^0_S(X)}$ is the complement of countably many closed subsets.

GV in positive characteristics

Proposition

If $P \in V^0_S(X)$, then $P^p \in V^0_S(X)$.

Proof: By definition, if $P \in V^0_S(X)$, then the image of the compositions $\ldots \to H^0(F^3_*(\omega_X \otimes P^{p^3})) \to H^0(F^2_*(\omega_X \otimes P^{p^2})) \to H^0(F_*(\omega_X \otimes P^p)) \to H^0(\omega_X \otimes P)$ is non-zero.
Ignoring the last term in the above sequence and using the isomorphism $H^0(F^e_*(\omega_X \otimes P^{p^e})) \cong H^0(F^{e-1}_*(\omega_X \otimes P^{p^{e-1}}))$, one sees that the image of the compositions
$\ldots \to H^0(F^2_*(\omega_X \otimes P^{p^3})) \to H^0(F_*(\omega_X \otimes P^{p^2})) \to H^0(\omega_X \otimes P^p)$ is non-zero.
Thus $P^p \in V^0_S(X)$. $\square$

GV in positive characteristics

Corollary (Hacon-Patakfalvi)

If A is ordinary (or has no supersingular factors) then $V^0_S(X)$ is a finite union of torsion translats of abelian subverieties of $\hat{A}$.

This is an immediate consequence of the following.

Theorem (Pink-Rossler)

If A has no supersingular factors and $X \subset A$ is a subscheme such that $p(X) = X$ (resp. $p(X) \subset X$), then X (resp. any maximal dimensional irreducible components of X) is a finite union of torsion translates of sub abelian varieties of A.

GV in positive characteristics

- Recall that A is ordinary (resp. supersingular) if $F^* : H^1(\mathcal{O}_A) \to H^1(\mathcal{O}_A)$ is bijective (resp. 0).
- It is expected that if A is defined over $\mathbb{Z}$, then its reduction mod p is ordinary for infinitely many primes p.
- This result is known for elliptic curves.
- It is also known that if A is defined over $\mathbb{Z}$, then its reduction mod p has no supersingular factors for infinitely many primes p.
- Pink and Rossler use this to re-prove that if X is a smooth complex projective variety then $V^i(\Omega^j_X)$ is a finite union of torsion translates of sub abelian varieties of A.

Characterization of ordinary AV

- Consider the trace map $F^e_*\omega_X \to \omega_X$ and tensor it by ω_X^{m-1} for $m \geq 1$.
- We obtain $\Phi^e : F^e_*(\omega_X^{1+(m-1)p^e}) \to \omega_X^m$.
- Define $S^0(\omega_X^m) = \cap_{e>0}\mathrm{Im}(H^0(F^e_*(\omega_X^{1+(m-1)p^e})) \to H^0(\omega_X^m))$.
- Let $S(\omega_X) = \oplus_{m\geq 0}S^0(\omega_X^m) \subset \oplus_{m\geq 0}H^0(\omega_X^m) = R(\omega_X)$.
- Then $S(\omega_X) \subset R(\omega_X)$ is an ideal.
- In fact, if $f \in S^0(\omega_X^m)$ and $g \in H^0(\omega_X^l)$, then $f = \Phi^e(f_e)$ and by the projection formula $fg = \Phi^e(f_e g^{p^e})$.
- We define $\kappa_S(X) = \max\{k|\overline{\lim}\dim S^0(\omega_X^m)/m^k > 0\}$. (If $\dim S^0(\omega_X^m) = 0$ for all $m > 0$, then let $\kappa_S(X) = -\infty$.)

Characterization of ordinary AV

- Since $S(\omega_X) \subset R(\omega_X)$ is an ideal, it follows that if $\kappa_S(X) \neq -\infty$, then $\kappa_S(X) = \kappa(X)$.
- If A is an abelian variety, then $\kappa_S(A) = 0$ iff A is ordinary.

Theorem (Hacon-Patakfalvi)

If $\kappa_S(X) = 0$, then the Albanese map $a : X \to A$ is surjective (and in particular $b_1(X) \leq 2 \dim X$). If $\kappa_S(X) = 0$, $b_1(X) \leq 2 \dim X$ and p does not divide the degree of $a : X \to A$, then X is birational to an abelian variety.

Here $b_i(X) = \dim_{\mathbb{Q}_l} H^i_{et}(X, \mathbb{Q}_l)$, $l \neq p$ and $b_2(X) = 2 \dim A$.

Characterization of ordinary AV

- We illustrate the proof of the first statement.
- **Claim:** There exists a neighborhood $0 \in U \subset \hat{A}$ such that for any $0 \neq P \in U$, $h^0(\omega_X \otimes a^*P) = 0$.
- Suppose not and let $0 \in T \subset \hat{A}$ be a positive dimensional component of $V^0(\omega_X)$.
- The map $g : T^{g+1} \to \hat{A}$ has positive dimensional fibers, in particular $Z = g^{-1}0$ is positive dimensional.
- For $(P_1, \ldots, p_{g+1}) \in Z$ consider the map $H^0(\omega_X \otimes a^*P_1) \times \ldots \times H^0(\omega_X \otimes a^*P_1) \to H^0(\omega_X^{g+1})$.
- Since $\kappa(X) = 0$, $h^0(\omega_X^{g+1}) = 1$ and so the unique divisor in $|(g+1)K_X|$ can be decomposed in only finitely many ways.
- This contradicts $\dim Z > 0$.

Characterization of ordinary AV

- Therefore (by cohomology and base change) $\mathcal{H}^0(\Lambda_e)$ has a summand $\mathcal{A}_e$ which is an Artinian module supported at 0. It follows that $\mathcal{A} := \varinjlim \mathcal{A}_e$ is a summand of Λ
- **Claim:** $\mathcal{A} = \varinjlim \mathcal{A}_e \neq 0$.
- By cohomology and base cahnge $\mathcal{A}_e \otimes k(0) = H^0(F^e_* a_*\omega_X)^\vee$ and $\varinjlim H^0(F^e_* a_*\omega_X)^\vee \neq 0$ since $\varprojlim H^0(F^e_* a_*\omega_X) \neq 0$.
- Since $\Omega = \varprojlim F^e_* a_*\omega_X = D_A(RS(\Lambda))$, we have $\mathrm{Supp}(S^0 a_*\omega_X) = \mathrm{Supp}(\Omega) \supset \mathrm{Supp}(D_A(R_S(\mathcal{A})))$.
- Let $\mathcal{A}'_e = \mathrm{Im}(\mathcal{A}_e \to \Lambda)$, then $\mathcal{A} := \varinjlim \mathcal{A}'_e$ and the induced maps $\mathcal{A}'_e \to \mathcal{A}'_{e+1}$ are injective.
- Let $V_e = R^0S(\mathcal{A}'_e)$ be the corresponding unipotent vector bundle, then the induced map $V_e \to V_{e+1}$ is injective.

Characterization of ordinary AV

- It follows that $V_{e+1}^{\vee} \to V_e^{\vee}$ are surjective and hence $\varprojlim V_e^{\vee} \neq 0$ is supported on A.
- But since $\varprojlim V_e^{\vee}$ is a summand of Ω, it follows that Ω (and hence also $S^0 a_* \omega_X$) is supported on the whole of A.
- Thus $X \to A$ is surjective.
- We will now show that if moreover $a : X \to A$ is generically finite and the degree of a is not divisible by p, then $X \to A$ is birational.
- There is a splitting $\omega_A \to a_* \omega_X \to \omega_A$ compatible with the traces of the frobenii.

Characterization of ordinary AV

- It follows that A is an ordinary abelian variety (as $\kappa_S(A) = \kappa_S(X) = 0$).
- We write $a_*\omega_X = \omega_A \oplus \mathcal{M}$. Then $V^0(\mathcal{M})$ contains no torsion points.
- Thus the support of $\Lambda_{\mathcal{M}_e} := R\hat{S}(D_A(F^e_*\mathcal{M}))$ contains no torsion points.
- Let $\Lambda_{\mathcal{M}_\infty} = \varinjlim \Lambda_{\mathcal{M}_e}$, then $\mathrm{im}(\Lambda_{\mathcal{M}_e} \to \Lambda_{\mathcal{M}_\infty}) = 0$ so that $\Lambda_{\mathcal{M}_\infty} = 0$.
- But then $\varprojlim F^e_*\mathcal{M} = 0$ and hence $\varprojlim F^e_* a_*\omega_X = \varprojlim F^e_*\omega_A$.
- From the surjection $\varprojlim F^e_* a_*\omega_X \to S^0 a_*\omega_X$ we deduce $S^0 a_*\omega_X = \omega_A$.
- Since $S^0 a_*\omega_X$ equals $a_*\omega_X$ over the open subset $U \subset A$ where a is étale, it follows that the degree of a is 1. □