Math 6310, Assignment 4 **Due in class: Monday, October 26**

1. Let x, y be group elements such that x and y commute with $[x,y] = xyx^{-1}y^{-1}$. Prove that for each $n \geqslant 1$, one has
$$[x^n, y] = [x,y]^n \qquad \text{and} \qquad x^n y^n = (xy)^n [x,y]^{\binom{n}{2}}.$$

2. Let $\mathbb{F}$ be a field with more than three elements. Prove that the commutator subgroup of $\mathrm{SL}_2(\mathbb{F})$ contains all matrices of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, for $a \in \mathbb{F}$.

3. For p, q distinct primes, prove that a group of order $p^2 q$ is solvable.

4. Let p be a prime integer. What is the number of Sylow p-subgroups of $\mathrm{GL}_2(\mathbb{F}_p)$?

5. For subgroups H, K of G we use $[H,K]$ for the subgroup generated by $hkh^{-1}k^{-1}$ with $h \in H$ and $k \in K$. Prove:
   (a) $[H,K] \lhd \langle H,K \rangle$, where $\langle H,K \rangle$ is the subgroup generated by H and K.
   (b) $[G,H]H \lhd G$.
   (c) $[G,H]H$ is the smallest normal subgroup of G that contains H.

6. Let G be a group. Set $Z_0 = \{e\}$ and recall that Z_{i+1} is the inverse image of $Z(G/Z_i)$ under the canonical surjection $G \longrightarrow G/Z_i$, giving us
$$Z_0 \lhd Z_1 \lhd Z_2 \lhd \cdots.$$
Set $C_0 = G$ and inductively define $C_{i+1} = [C_i, G]$. This gives us
$$C_0 \rhd C_1 \rhd C_2 \rhd \cdots.$$
   (a) If $Z_k = G$ for some integer k, show that $C_i \subseteq Z_{k-i}$ for all $1 \leqslant i \leqslant k$.
   (b) If $C_k = \{e\}$ for some integer k, show that $C_{k-i} \subseteq Z_i$ for all $1 \leqslant i \leqslant k$.

   Hence $Z_k = G$ if and only if $C_k = \{e\}$. In this case, G is *nilpotent*; the least k is the *nilpotency class* of G.

7. Let G be a nilpotent group, and let H be the set of elements of finite order.
   (a) Prove that H is a subgroup of G. Hint: Use induction on the nilpotency class of G.
   (b) Prove that every finite subset of H generates a finite subgroup.
   (c) Prove that elements of H of relatively prime order commute.

8. Let G be the group of upper triangular matrices $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$ in $\mathrm{GL}_2(\mathbb{R})$. Show that G is solvable. Is it nilpotent?