SIAM J. Math. Anal.
Vol. 47, No. 4, pp. 3035–3063

STOCHASTIC SWITCHING IN INFINITE DIMENSIONS WITH APPLICATIONS TO RANDOM PARABOLIC PDE*

SEAN D. LAWLEY†, JONATHAN C. MATTINGLY†, AND MICHAEL C. REED†

Abstract. We consider parabolic PDEs with randomly switching boundary conditions. In order to analyze these random PDEs, we consider more general stochastic hybrid systems and prove convergence to, and properties of, a stationary distribution. Applying these general results to the heat equation with randomly switching boundary conditions, we find explicit formulae for various statistics of the solution and obtain almost sure results about its regularity and structure. These results are of particular interest for biological applications as well as for their significant departure from behavior seen in PDEs forced by disparate Gaussian noise. Our general results also have applications to other types of stochastic hybrid systems, such as ODEs with randomly switching right-hand sides.

Key words. random PDEs, hybrid dynamical systems, switched dynamical systems, piecewise deterministic Markov process, ergodicity

AMS subject classifications. 35R60, 37H99, 46N20, 60H15, 92C30

DOI. 10.1137/140976716

1. Introduction. The primary motivation for this paper is to study parabolic partial differential equations (PDEs) with randomly switching boundary conditions. More precisely, given an elliptic differential operator, L, on a domain $D \subset \mathbb{R}^d$, we want to study the stochastic process $u(t, x)$ that solves $\partial_t u = Lu$ in D subject to boundary conditions that switch at random times between two given boundary conditions.

This type of random PDE is an example of a stochastic hybrid system. The word "hybrid" is used because these stochastic processes involve both continuous dynamics and discrete events. In this example, the continuous dynamics are the different boundary value problems corresponding to the different boundary conditions for the given PDE, and the discrete events are when the boundary condition switches.

In general, a stochastic hybrid system is a continuous-time stochastic process with two components: a continuous component $(X_t)_{t\geq 0}$ and a jump component $(J_t)_{t\geq 0}$. The jump component, J_t, is a jump process on a finite set, and for each element of its state space we assign some continuous dynamics to X_t. In between jumps of J_t, the component X_t evolves according to the dynamics associated with the current state of J_t. When J_t jumps, the component X_t switches to follow the dynamics associated with the new state of J_t.

An ordinary differential equation (ODE) with a switching right-hand side is the type of stochastic hybrid system that is most commonly used in applications. Such ODE switching systems have been used extensively in applied areas such as control theory, computer science, and engineering (for example, [46], [8], [3], and [31]). More recently, these systems have been used in diverse areas of biology (for example, molecular biology [10], [37], [9], ecology [47], and epidemiology [21]). Furthermore, such

*Received by the editors July 9, 2014; accepted for publication (in revised form) June 5, 2015; published electronically August 11, 2015.

http://www.siam.org/journals/sima/47-4/97671.html

†Mathematics Department, Duke University, Durham, NC 27708 (lawley@math.duke.edu, jonm@math.duke.edu, reed@math.duke.edu). The research of the first author was supported in part by NSF grant DMS-0943760. The research of the second author was supported in part by NSF grant DMS-0854879. The research of the third author was supported in part by NSF grants EF-1038593 and DMS-0943760 and NIH grant R01 ES019876.

ODE switching systems have also recently been the subject of much mathematical study [29], [12], [6], [5], [2], [24], [25], [4].

Comparatively, stochastic hybrid systems stemming from PDEs have received little attention. While deterministic PDEs coupled to random boundary conditions have been studied, the random boundary conditions have typically been assumed to involve some Gaussian noise forcing [1], [18], [42], [41], [15]. The randomness enters our PDE system in a fundamentally different way than in Stochastic PDEs which are driven by additive space-time white noise (or even spatially smoother Gaussian fields). There, the fine scales are often asymptotically independent of each other [35], [36]. Here, there is a single piece of randomness which dictates the fine structure. Hence, the fine scales, though not asymptotically deterministic, are asymptotically perfectly correlated. (See Proposition 4.5 for more details.)

We were led to study such random PDEs by various biological applications. One application is to insect respiration. Essentially all insects breathe via a network of tubes that allow oxygen and carbon dioxide to diffuse to and from their cells [44]. Air enters and exits this network through valves (called spiracles) in the exoskeleton, which regulate air flow by opening and closing quite irregularly in time. This leads naturally to the following model problem. Let $u(x,t)$ satisfy the heat equation $\partial_t u = D\partial_x^2 u$ on $[0,L]$. $x = 0$ corresponds to tissue where the oxygen is absorbed, so $u(0,t) = 0$. $x = L$ corresponds to a spiracle, so there the boundary condition switches between $\partial_x u(L,t) = 0$ (spiracle closed) and $u(L,t) = b > 0$ (spiracle open). Suppose that the boundary conditions switch at exponential rates r_0 and r_1. We would like to calculate the long-term statistics of the solution $u(t,x)$; in particular, we would would like to know how the oxygen absorption at the tissue, $\partial_x u(0,t)$, depends on D and the switching rates. This model problem is fully developed in section 4.3.

A second such problem arises in understanding the concentration of neurotransmitters in the extracellular space in the brain. Imagine that the axonal projections from a nucleus of cells make a dense, random set of terminals in projection region, P, in the brain. For example, there is a dense set of terminals of serotonin neurons in the striatum that come from the dorsal raphe nucleus [22]. Action potentials arrive as a Poisson process at a terminal, and when they do, neurotransmitter is released at a high rate into the extracellular space for a very short amount of time. At other times, the neurotransmitter is absorbed back into the terminal. We would like to calculate the long-term statistics of the neurotransmitter concentration in the exterior domain that consists of P with the terminal volumes removed. On the large scale, this is a homogenization problem. But to solve it, one first has to understand the local switching problem. The solution of the heat equation in the exterior domain, $u(x,t)$, satisfies $\partial_n u(x,t) = c >> 0$ for a short time after an action potential has arrived, and $u(x,t) = 0$ at other times, for points x on the boundary of a terminal. Thus, as in the previous paragraph, we are switching between Dirichlet and Neuman boundary conditions at random times. These questions are extremely important for neuroscience because it is now known that some groups of neurons affect distant locations of the brain by firing more or less and thus changing the ambient concentration of the neurotransmitter in the extracellular space in the distant region, a phenomenon called "volume transmission" [38], [23]. An analysis of this problem, using the techniques developed in this paper, will be the subject of future work.

Our paper is organized as follows. In section 2, we consider more general stochastic hybrid systems from the viewpoint of iterated random functions. (See [16] or [27], [28] for a review of iterated random functions.) Assuming that the continuous dynamics are contracting on average, we prove convergence to a stationary distribution and describe the structure and properties of this distribution. In section 3, we apply these general results to the random PDE problems described above. We show that the mean of the process satisfies the PDE and that the mean of the stationary distribution satisfies the time homogeneous version of the PDE. Then in section 4, we apply our results from sections 2 and 3 to the one-dimensional heat equation with randomly switching boundary conditions. We find explicit formulae for various statistics of the solution and obtain almost sure results about its regularity and structure. There, we also show that our general results have applications to other types of stochastic hybrid systems, such as ODEs with randomly switching right-hand sides. Finally, we end section 4 by explaining that our results can be applied to the question in insect physiology mentioned above.

We conclude this introduction by giving two examples that motivated our study. We return to these examples in section 4. Consider the heat equation on the interval $[0, L]$ with an absorbing boundary condition at 0 and a randomly switching boundary condition at L. Let the switching be controlled by a Markov jump process, J_t, on $\{0, 1\}$ with r_0 and r_1 the respective rates for leaving states 0 and 1. In the following two examples, we consider different possible boundary conditions at L.

Example 1. Suppose the boundary condition at L switches between an inhomogeneous Dirichlet condition and a Neumann no flux condition. More precisely, consider the stochastic process $u(t, x) \in L^2[0, L]$ that solves

$$\begin{aligned} \partial_t u &= D\Delta u \quad \text{in } (0, L), \\ u(0, t) = 0 \quad &\text{and} \quad J_t u_x(L, t) + (1 - J_t)(u(L, t) - b) = 0. \end{aligned}$$

We show in section 4.1 that as $t \to \infty$, the process $u(t, x)$ converges in distribution to an $L^2[0, L]$-valued random variable whose expectation is a linear function. Letting $\gamma = L\sqrt{(r_0 + r_1)/D}$ and $\rho = r_0/r_1$, we will show that the slope of this function is

$$\left(1 + \frac{\rho}{\gamma}\tanh(\gamma)\right)^{-1} \frac{b}{L}.$$

Example 2. Suppose the boundary condition at L switches between an inhomogeneous Dirichlet condition and a homogeneous Dirichlet condition. More precisely, consider the stochastic process $u(t, x) \in L^2[0, L]$ that solves

$$\begin{aligned} \partial_t u &= D\Delta u \quad \text{in } (0, L), \\ u(0, t) = 0 \quad &\text{and} \quad J_t u(L, t) + (1 - J_t)(u(L, t) - b) = 0. \end{aligned}$$

We show in section 4.2 that as $t \to \infty$, the process $u(t, x)$ converges in distribution to an $L^2[0, L]$-valued random variable whose expectation is a linear function. Letting $p = r_0/(r_0 + r_1)$, we will show that the slope of this function is

$$(1 - p)\frac{b}{L}.$$

The expectations for Examples 1 and 2 are quite different. In Example 2, the expectation is the solution to the time homogenous PDE with boundary conditions given by the average of the two possible boundary conditions. We will see in section 4.2 that this simple result holds because the process switches between boundary conditions of the same type and the corresponding semigroups commute. Moreover, because the boundary conditions are the same type, we will be able to compute individual and joint statistics of the Fourier coefficients of the stationary solution and show that this solution almost surely has a very specific structure and regularity.

In both examples, the expectation is a linear function with slope given by b/L multiplied by a factor less than one. While in Example 2 this factor is simply the proportion of time the boundary condition is inhomogeneous, the factor in Example 1 is an unexpected expression involving the hyperbolic tangent. Furthermore, while the factor in Example 1 still depends on the proportion of time the boundary condition is inhomogeneous, it also depends on how often the boundary conditions switch. Observe that if we keep this proportion fixed by fixing the ratio r_0/r_1 and take the frequency of switches small by letting $r_0 + r_1$ go to 0, then the slope for Example 1 approaches the same slope as in the Example 2. And if we keep the ratio r_0/r_1 fixed but let the $r_0 + r_1$ go to infinity, then the slope for Example 1 approaches b/L. Some biological implications of this result are discussed in section 4.3.

2. Abstract setting. We first consider stochastic hybrid systems in a separable Banach space X. Under certain contractivity assumptions, we prove that the process converges in distribution at large time and we show that the limiting distribution satisfies certain invariance properties. Although applicable to a range of stochastic hybrid systems, the contents of this section will prove particularly useful when we consider PDEs with randomly switching boundary conditions in sections 3 and 4.

2.1. Discrete-time process. We first define the set Ω of all possible switching environments and equip it with a probability measure $\mathbb{P}$ and associated expectation $\mathbb{E}$. Let μ_0 and μ_1 be two probability distributions on the positive real line. Define each switching environment, $\omega \in \Omega$, as the sequence $\omega = (\omega_1, \omega_2, \dots)$, where each ω_k is a pair of nonnegative real numbers, (τ_0^k, τ_1^k), drawn from $\mu_0 \times \mu_1$. That is, (τ_0^k, τ_1^k) is an $\mathbb{R}^2$-valued random variable drawn from the product measure $\mu_0 \times \mu_1$. We take $\mathbb{P}$ to be the infinite product measure generated by $\mu_0 \times \mu_1$. To summarize some notation,

$$\omega = (\omega_1, \omega_2, \omega_3, \dots) = \big((\tau_0^1, \tau_1^1), (\tau_0^2, \tau_1^2), (\tau_0^3, \tau_1^3), \dots\big) \in \Omega. \tag{2.1}$$

For each $t \geq 0$, let $\Phi_t^0(x)$ and $\Phi_t^1(x)$ be two mappings from a separable Banach space X to itself. Make the following assumptions on Φ_t^i for each $i \in \{0,1\}$, $t \geq 0$, $x, y \in X$, and with τ_i an independent draw from μ_i:

(a) $\Phi_t^0(x) = x = \Phi_t^1(x)$ if $t = 0$.
(b) $t \mapsto \Phi_t^i(x) \in X$ is continuous.
(c) $\mathbb{E}|\Phi_{\tau_i}^i(x)| < \infty$.
(d) $|\Phi_t^i(x) - \Phi_t^i(y)| \leq K_i(t)|x - y|$ for some $K_i(t)$.
(e) $\mathbb{E}K_0(\tau_1)\mathbb{E}K_1(\tau_1) < \infty$ and $\mathbb{E}\log(K_0(\tau_1)\mathbb{E}K_1(\tau_1)) < 0$.

For each $\omega \in \Omega$, $x \in X$, and natural number k, define the compositions

$$G_\omega^k(x) := \Phi_{\tau_1^k}^1 \circ \Phi_{\tau_0^k}^0(x) \qquad \text{and} \qquad F_\omega^k(x) := \Phi_{\tau_0^k}^0 \circ \Phi_{\tau_1^k}^1(x).$$

For each $\omega \in \Omega$, $x \in X$, and natural number $n > 0$, we define the forward maps φ^n and γ^n and the backward maps φ^{-n} and γ^{-n} by the following compositions of G and F:

$$\begin{aligned} \varphi_\omega^n(x) &= G_\omega^n \circ \cdots \circ G_\omega^1(x) \qquad \text{and} \qquad \gamma_\omega^n(x) = F_\omega^n \circ \cdots \circ F_\omega^1(x), \\ \varphi_\omega^{-n}(x) &= G_\omega^1 \circ \cdots \circ G_\omega^n(x) \qquad \text{and} \qquad \gamma_\omega^{-n}(x) = F_\omega^1 \circ \cdots \circ F_\omega^n(x). \end{aligned} \tag{2.2}$$

To make our notation complete, we define $\varphi^0(x) = x = \gamma^0(x)$.

Remark 1. The maps φ^n and γ^n are iterated random functions. (See [16] for a review.) Assumptions (d) and (e) above ensure that G^k and F^k are contracting on average. Thus, $\{\varphi^n\}_{n\geq 0}$ and $\{\gamma^n\}_{n\geq 0}$ are Markov chains with invariant probability distributions given by the distributions of the almost sure limits of φ^{-n} and γ^{-n} as $n\to\infty$, respectively. Moreover, the distributions of the Markov chains φ^n and γ^n converge at a geometric rate to these invariant distributions. These results are immediately attained by applying theorems in, for example, [16], [28], [19]. Nonetheless, we prove the following proposition to make our results more self-contained.

PROPOSITION 2.1. *Define*

$$Y_1(\omega) := \lim_{n\to\infty} \varphi_\omega^{-n}(x) \quad \textit{and} \quad Y_0(\omega) := \lim_{n\to\infty} \gamma_\omega^{-n}(x). \tag{2.3}$$

These limits exist almost surely and are independent of $x \in X$.

Remark 2. A random set which attracts all initial data started at "$-\infty$" and is forward-invariant under the dynamics is called a random pullback attractor [28], [14], [13], [34]. When that attractor consists of a single point almost surely, is called a random point attractor. In this case, the single point can be viewed as a random variable. Random variables such as these are often called random pullback attractors, or "pullbacks" for short, because they take an initial condition x and pull it back to the infinite past [13], [40], [33]. Since when the random attractor is a single point one can associate to each realization of random "forcing" a single attracting solution which gives the asymptotic behavior, this is also ofter referred to as the "one force, one solution" paradigm [43], [33], [34].

Proof. We will show that the sequence $\varphi^{-n}(x)$ is almost surely Cauchy. Let $x_1, x_2 \in X$ and $n \geq m$. Using the triangle inequality repeatedly, we obtain

$$\begin{aligned} |\varphi^{-n}(x_1) - \varphi^{-m}(x_2)| &\leq \sum_{i=m+1}^{n} |G^1 \circ \cdots \circ G^i(x_1) - G^1 \circ \cdots \circ G^{i-1}(x_2)| \\ &\leq \sum_{i=m+1}^{n} |G^i(x_1) - x_2| \left(\prod_{j=1}^{i-1} K_0(\tau_0^j) K_1(\tau_1^j) \right). \end{aligned} \tag{2.4}$$

Assumptions (c), (d), and (e) give that $\mathbb{E}|G^i(x_1) - x_2| < \infty$, and thus a simple application of the Borel–Cantelli lemma gives the existence of an almost surely finite random constant $C_1(\omega)$ such that for all $i \in \mathbb{N}$

$$|G^i(x_1) - x_2| < C_1(\omega)(i)^2. \tag{2.5}$$

Let $Z_j := K_0(\tau_0^j)K_1(\tau_1^j)$. A standard argument (see, for example, [16, Lemmas 5.2 and 5.4]) gives the existence of constants $\epsilon > 0$, $A > 0$, and $0 < r < 1$ such that for all $i \in \mathbb{Z}$, we have $\mathbb{P}(\sum_{j=1}^{i} \log Z_j > -i\epsilon) < Ar^i$, by assumption (e). Thus, another application of the Borel–Cantelli lemma gives the existence of an almost surely finite random constant $C_2(\omega)$ such that for all $i \in \mathbb{N}$

$$\sum_{j=1}^{i} \log Z_j \leq -i\epsilon + C_2(\omega). \tag{2.6}$$

Plugging the bounds from (2.5) and (2.6) into (2.4) gives

$$|\varphi^{-n}(x_1) - \varphi^{-m}(x_2)| \leq \sum_{i=m+1}^{n} C_1(\omega)(i)^2 e^{-i\epsilon + C_2(\omega)}.$$

Therefore, $\varphi^{-n}(x_1)$ is almost surely Cauchy, and thus Y_1 exists almost surely. Since x_1 and x_2 were arbitrary, Y_1 is independent of the x used in its definition. The proof for Y_0 is similar. ∎

The random variables Y_1 and Y_0 satisfy the following invariance properties.

Proposition 2.2. *Let τ_0 and τ_1 be independent draws from μ_0 and μ_1. Then*

$$Y_0 =_d \Phi^0_{\tau_0}(Y_1) \quad \textit{and} \quad Y_1 =_d \Phi^1_{\tau_1}(Y_0), \tag{2.7}$$

where $=_d$ denotes equal in distribution.

Proof. Let $y \in X$ and observe that for any $n \in \mathbb{N}$, we have that

$$\gamma^{-n}(y) =_d \Phi^0_{\tau_0}\left(\varphi^{n-1}(\Phi^1_{\tau_1}(y))\right).$$

Taking the limit as $n \to \infty$ yields

$$\lim_{n\to\infty} \gamma^{-n}(y) =_d \lim_{n\to\infty} \Phi^0_{\tau_0}\left(\varphi^{n-1}(\Phi^1_{\tau_1}(y))\right) = \Phi^0_{\tau_0}\left(\lim_{n\to\infty} \varphi^{n-1}(\Phi^1_{\tau_1}(y))\right) \tag{2.8}$$

since $\Phi^0_t(x)$ is continuous in x for each t. Recalling that the definitions of Y_0 and Y_1 in (2.3) are independent of x by Proposition 2.1, we have that (2.8) becomes $Y_0 =_d \Phi^0_{\tau_0}(Y_1)$. The proof that $Y_1 =_d \Phi^1_{\tau_1}(Y_0)$ is similar. ∎

Proposition 2.3. *Suppose there exists a nonempty set $S \subset X$ so that for all $t \geq 0$, $\Phi^i_t : S \to S$ for $i = 0$ and 1. Then Y_0 and Y_1 are in the closure, $\bar{S}$, almost surely.*

Proof. If $x \in S$, then $\varphi^{-n}(x) \in S$ almost surely for all $n \geq 0$ by assumption. Thus, $\lim_{n\to\infty} \varphi^{-n}(x) = Y_1 \in \bar{S}$ almost surely. But by Proposition 2.1, the random variable Y_1 is independent of the initial x used in its definition, so $Y_1 \in \bar{S}$ almost surely. The proof for Y_0 is similar. ∎

2.2. Continuous-time process. To define the continuous time process, we need more notation. Much of the following notation is standard in renewal theory. For each $\omega \in \Omega$ and natural number n, define

$$S_n := \sum_{k=1}^{n} \left(\tau_0^k + \tau_1^k\right)$$

with $S_0 := 0$. Define $S'_{n+1} := S_n + \tau_0^{n+1}$ for $n \geq 0$. Observe that $S'_n < S_n < S'_{n+1} < S_{n+1}$ by definition. Define

$$N_t := \max\{n \geq 0 : S_n \leq t\}.$$

We also define the state process J_t for $t \geq 0$ by

$$J_t := \begin{cases} 0 & S_{N_t} \leq t < S'_{N_t+1}, \\ 1 & S'_{N_t+1} \leq t. \end{cases} \tag{2.9}$$

Finally, for $t \geq 0$, define the elapsed time since the last switch, often called the age process, by

$$a_t := J_t(t - S'_{N_t+1}) + (1 - J_t)(t - S_{N_t}).$$

We are now ready to define our continuous-time X-valued process. For $u_0 \in X$, $\omega \in \Omega$, and $t \geq 0$, define

$$u(t,\omega) = J_t \Phi^1_{a_t} \circ \Phi^0_{\tau_0^{N_t+1}}(\varphi^{N_t}(u_0)) + (1 - J_t)\Phi^0_{a_t}(\varphi^{N_t}(u_0)). \tag{2.10}$$

2.3. Convergence in distribution to mixture of pullbacks. In this section, we will find the limiting distribution of $u(t)$ as $t \to \infty$. In order to describe this limiting distribution, we will need to define three more random variables. Define a^0 and a^1 to be two random variables with the following cumulative distribution functions:

$$\mathbb{P}(a^0 \leq x) = \frac{\mathbb{E}\min(\tau_0, x)}{\mathbb{E}\tau_0} \quad \text{and} \quad \mathbb{P}(a^1 \leq x) = \frac{\mathbb{E}\min(\tau_1, x)}{\mathbb{E}\tau_1}.$$

We will see in Lemma 2.6 that the distributions of a^0 and a^1 can be thought of as the limiting distributions of the age process conditioned on either $J_t = 0$ or 1. Let ξ be a Bernoulli random variable with parameter $p := (\mathbb{E}\tau_1)/(\mathbb{E}\tau_0+\mathbb{E}\tau_1)$, the probability that $J_t = 1$ at large time. Assume a^0, a^1, and ξ are all chosen to be mutually independent and independent of (τ_0^k, τ_1^k) for every k. Recall that a measure μ on the real line is said to be arithmetic if there exist a $d > 0$ so that $\mu(\{0, d, 2d, \dots\}) = 1$.

THEOREM 2.4. *Suppose that Φ_t^0 and Φ_t^1 satisfy assumptions* (a)-(e) *of section* 2.1. *Let $u(t)$ be defined as in* (2.10)*, and a^0, a^1, and ξ as in the above paragraph. If the switching time distributions, μ_0 and μ_1, are nonarithmetic, then we have the following convergence in distribution as $t \to \infty$:*

$$(u(t), J_t) \to_d (\bar{u}, \xi) \quad \text{as } t \to \infty,$$

where $\bar{u} := \xi\Phi^1_{a^1}(Y_0) + (1-\xi)\Phi^0_{a^0}(Y_1)$.

The pullbacks Y_0 and Y_1 give the invariant distributions of the discrete Markov processes φ^n and γ^n defined in (2.2) (see Proposition 2.1 and Remark 1). Thus, Theorem 2.4 describes the limiting distribution of the (not necessarily Markovian) continuous process $(u(t), J_t)$ in terms of the invariant distributions of related Markov processes. Stated colloquially, Theorem 2.4 says that to go from the invariant distributions of the discrete Markov processes to the limiting distribution of the continuous process, one must do the following: first flip a coin with parameter p to decide which map (Φ^0 or Φ^1) is being applied and then apply that map (say it's Φ^i) to pullback Y_{1-i} for time a^i, where a^i is the amount of time since the last switch given that Φ^i is currently being applied.

In the context of piecewise deterministic Markov processes given by switching ordinary differential equations, the authors of [5] relate the invariant measure of the continuous Markov process to the embedded discrete Markov chain. In particular, in Proposition 2.4 of [5], the authors show that the ergodic probability measures of the continuous and discrete processes form a one to one correspondence, and hence the continuous process is stable if and only if the discrete process is stable.

In the following corollary, we show that if the switching time distributions are exponential so that the continuous process $(u(t), J_t)$ is Markov, then its limiting distribution actually is the invariant distribution of the embedded discrete Markov process (if the initial map is either Φ^1 or Φ^0 with probability p or $1-p$). The corollary follows immediately from Proposition 2.2 and Theorem 2.4 since the age of a Poisson process is exponentially distributed.

COROLLARY 2.5. *Suppose the switching time distributions, μ_0 and μ_1, are exponential with respective rate parameters r_0 and r_1. If ξ is Bernoulli with parameter $r_0/(r_0+r_1)$, then we have the following convergence in distribution as $t \to \infty$:*

$$(u(t), J_t) \to_d (\bar{u}, \xi) \quad \text{as } t \to \infty,$$

where $\bar{u} := \xi Y_1 + (1-\xi)Y_0$.

Proof of Theorem 2.4. In light of Proposition 2.2, it is enough to prove the desired convergence in distribution for $\bar{u} := \Phi^1_{a^1} \circ \Phi^0_{\tau_0}(Y_1) + (1-\xi)\Phi^0_{a^0}(Y_1)$, where τ_0 is an independent draw from μ_0. Recall that if Z is a random variable taking values in some metric space and a Borel subset S of that metric space satisfies $\mathbb{P}(Z \in \partial S) = 0$ where ∂S denotes the boundary of S, then S is called a continuity set of Z. Let A, B, C, and D be continuity sets of $\xi a^1 + (1-\xi)a^0$, $\xi\tau_0$, ξ, and Y_1, respectively. We will show that

$$
\begin{aligned}
(2.11)\quad & \mathbb{P}(a_t \in A, J_t\tau_0^{N_t+1} \in B, J_t \in C, \varphi^{N_t}(u_0) \in D) \\
& \qquad \to \mathbb{P}(\xi a^1 + (1-\xi)a^0 \in A, \xi\tau_0 \in B, \xi \in C, Y_1 \in D) \quad \text{as} \quad t \to \infty.
\end{aligned}
$$

Once this convergence is shown, the conclusion of the theorem quickly follows. To see this, assume the convergence in (2.11) holds. Define the $(\mathbb{R}^3 \times X)$-valued random variable $Y_t := (a_t, J_t\tau_0^{N_t+1}, J_t, \varphi^{N_t})$, where we have suppressed the u_0 dependence. We will usually suppress this dependence since the limiting random variables don't depend on the initial u_0 (see Proposition 2.1). Since X is assumed to be separable, the product $\mathbb{R}^3 \times X$ is separable and thus we can apply Theorem 2.8 in [7] to obtain that Y_t converges in distribution to $(\xi a^1 + (1-\xi)a^0, \xi\tau_0, \xi, Y_1)$ as $t \to \infty$.

Define the function $g : \mathbb{R}^3 \times X \to X$ by $g(a,t,j,y) = j\Phi^1_a \circ \Phi^0_t(y) + (1-j)\Phi^0_t(y)$ and observe that $u(t) = g(a_t, \tau_0^{N_{t+1}}, J_t, \varphi^{N_t})$ and $\bar{u} = g(a^1, \tau_0, \xi, Y)$. Further define the function $h : (\mathbb{R}^3 \times X) \to (X \times \mathbb{R})$ by $h(a,t,j,y) = (g(a,t,j,y), j)$. The function h is continuous because g is continuous, and thus the conclusion of the theorem follows from the continuous mapping theorem (see, for example, Theorem 3.2.4 in [20]). Therefore, it remains only to show the convergence in (2.11). Our proof relies on the auxiliary Lemmas 2.6–2.13, whose proofs are given in subsection 2.4.

In what follows, we will make extensive use of indicator functions. For ease of reading, we will often denote the indicator $1_A = 1_A(\omega)$ by $\{A\} = \{A\}(\omega)$.

For each $t \geq 0$, define $\mathcal{F}_t$ to be the σ-algebra generated by S_{N_t} and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+1}^{\infty}$. Since a_t, $\tau_0^{N_t+1}$, and J_t are measurable with respect to $\mathcal{F}_t$, the tower property of conditional expectation and the triangle inequality give

$$
\begin{aligned}
&\Big|\mathbb{E}\{a_t \in A, J_t\tau_0^{N_t+1} \in B, J_t \in C, \varphi^{N_t} \in D\} \\
&\qquad\qquad - \mathbb{E}\{\xi a^1 + (1-\xi)a^0 \in A, \xi\tau_0 \in B, \xi \in C, Y_1 \in D\}\Big| \\
&\quad \leq \Big|\mathbb{E}\left[\{a_t \in A, J_t\tau_0^{N_t+1} \in B, J_t \in C\}\mathbb{E}[\{\varphi^{N_t} \in D\}|\mathcal{F}_t]\right] \\
&\qquad\qquad - \mathbb{E}\left[\{a_t \in A, J_t\tau_0^{N_t+1} \in B, J_t \in C\}\mathbb{E}[\{Y_1 \in D\}]\right]\Big| \\
&\qquad\qquad\qquad + \Big|\mathbb{E}\left[\{a_t \in A, J_t\tau_0^{N_t+1} \in B, J_t \in C\}\mathbb{E}[\{Y_1 \in D\}]\right] \\
&\qquad\qquad\qquad\qquad - \mathbb{E}\{\xi a^1 + (1-\xi)a^0 \in A, \xi\tau_0 \in B, \xi \in C\}\{Y_1 \in D\}\Big|.
\end{aligned}
$$

By Lemma 2.11, we have that $\mathbb{E}\left[\{\varphi^{N_t} \in D\}|\mathcal{F}_t\right] \to \mathbb{E}\left[\{Y_1 \in D\}\right]$ almost surely as $t \to \infty$. Therefore, the first term goes to 0 by the dominated convergence theorem. Since Y_1 is independent of ξ, a^1, a^0, and τ_0, the second term is bounded above by

$$
\begin{aligned}
&(2.12) \\
&\Psi := \left|\mathbb{E}\{a_t \in A, J_t\tau_0^{N_t+1} \in B, J_t \in C\} - \mathbb{E}\{\xi a^1 + (1-\xi)a^0 \in A, \xi\tau_0 \in B, \xi \in C\}\right|.
\end{aligned}
$$

To show that Ψ goes to 0 as $t \to \infty$, we consider the four possible cases for the inclusion of 0 and 1 in C. If both 0 and 1 are not in C, then Ψ is 0 for all $t \geq 0$ since J_t and ξ are each almost surely 0 or 1.

Suppose $0 \in C$ and $1 \notin C$. Then the indicator function in the first term of Ψ is only nonzero if $J_t = 0$. Hence, we can replace $\{J_t \in C\}$ by $(1-J_t)$ and $\{J_t \tau_0^{N_t+1} \in B\}$ by $\{0 \in B\}$. Similarly, in the second term we replace $\{\xi \in C\}$ by $(1-\xi)$, $\{\xi\tau_0 \in B\}$ by $\{0 \in B\}$, and $\{\xi a^1 + (1-\xi)a^0 \in A\}$ by $\{a^0 \in A\}$. Thus, Ψ becomes

$$\begin{aligned}\Psi &= |\mathbb{E}\{a_t \in A, 0 \in B\}(1-J_t) - \mathbb{E}\{a^0 \in A, 0 \in B\}(1-\xi)| \\ &\leq |\mathbb{E}\{a_t \in A\}(1-J_t) - \mathbb{E}\{a^0 \in A\}(1-\xi)|.\end{aligned}$$

By Lemma 2.6, this term goes to 0 as $t \to \infty$.

Suppose $1 \in C$ and $0 \notin C$. Then the indicator function in the first term of Ψ is only nonzero if $J_t = 1$. Thus, after performing replacements similar to those above, Ψ becomes

$$\Psi = |\mathbb{E}\{a_t \in A, \tau_0^{N_t+1} \in B\}J_t - \mathbb{E}\{a^1 \in A, \tau_0 \in B\}\xi|.$$

Define $\mathcal{F}_t'$ to be the σ-algebra generated by S'_{N_t+1}, $\tau_1^{N_t+1}$, and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+2}^{\infty}$. Observe that J_t and a_t are both measurable with respect to $\mathcal{F}_t'$. Therefore, by the tower property of conditional expectation and the triangle inequality, we have that

$$\begin{aligned}&|\mathbb{E}\{a_t \in A, \tau_0^{N_t+1} \in B\}J_t - \mathbb{E}\{a^1 \in A, \tau_0 \in B\}\xi| \\ &\quad\leq \left|\mathbb{E}\left[\{a_t \in A\}J_t\mathbb{E}\left[\{\tau_0^{N_t+1} \in B\}|\mathcal{F}_t'\right]\right] - \mathbb{E}\left[\{a_t \in A\}J_t\mathbb{E}\left[\{\tau_0 \in B\}\right]\right]\right| \\ &\qquad\qquad + |\mathbb{E}[\{a_t \in A\}J_t\mathbb{E}[\{\tau_0 \in B\}]] - \mathbb{E}\{a^1 \in A, \tau_0 \in B\}\xi|.\end{aligned}$$

Lemma 2.12 gives us that $J_t\mathbb{E}[\{\tau_0^{N_t+1} \in B\}|\mathcal{F}'_t] = J_t\mathbb{E}[\{\tau_0^1 \in B\}|\mathcal{F}'_t]$ almost surely, and Lemma 2.13 gives that $\mathbb{E}[\{\tau_0^1 \in B\}|\mathcal{F}'_t] \to \mathbb{E}[\{\tau_0 \in B\}]$ almost surely as $t \to \infty$. Therefore, the first term goes to 0 as $t \to \infty$ by the dominated convergence theorem. Finally, since τ_0 is independent of ξ and a^1, we have the following bound on the second term:

$$|\mathbb{E}[\{a_t \in A\}J_t\mathbb{E}[\{\tau_0 \in B\}]] - \mathbb{E}\{a^1 \in A, \tau_0 \in B\}\xi| \leq |\mathbb{E}\{a_t \in A\}J_t - \mathbb{E}\{a^1 \in A\}\xi|.$$

This goes to 0 as $t \to \infty$ by Lemma 2.6.

Finally, if both $0 \in C$ and $1 \in C$, then Ψ becomes

$$\begin{aligned}\Psi =& |\mathbb{E}\{a_t \in A, J_t\tau_0^{N_t+1} \in B\} - \mathbb{E}\{\xi a^1 + (1-\xi)a^0 \in A, \xi\tau_0 \in B\}| \\ \leq& |\mathbb{E}\{a_t \in A, \tau_0^{N_t+1} \in B\}J_t - \mathbb{E}\{a^1 \in A, \tau_0 \in B\}\xi| \\ &+ |\mathbb{E}\{a_t \in A, 0 \in B\}(1-J_t) - \mathbb{E}\{a^0 \in A, 0 \in B\}(1-\xi)|.\end{aligned}$$

We've already shown that each of these terms go to zero as $t \to \infty$, so the proof is complete. ∎

2.4. The lemmas. We now state and prove all of the lemmas that are needed for Theorem 2.4. This first lemma calculates the limiting distribution of the age process. It can be interpreted as first flipping a coin to determine if J_t is 0 or 1 and then choosing from the limiting distribution of the age conditioned on J_t.

LEMMA 2.6. *For any continuity set A of $\xi a^1 + (1-\xi)a^0$, we have that as $t \to \infty$*

$$|\mathbb{E}\{a_t \in A\}J_t - \mathbb{E}\{a^1 \in A\}\xi| + |\mathbb{E}\{a_t \in A\}(1-J_t) - \mathbb{E}\{a^0 \in A\}(1-\xi)| \to 0.$$

Proof. We will first show the desired convergence for sets of a special form and then extend to a continuity set. Let $x \geq 0$ and consider the alternating renewal process that is said to be "on" when $0 \leq t - S'_{N_t+1} \leq x$ and "off" otherwise. Formally, we define the "on/off" state process

$$b_t = \begin{cases} 1 & \text{if } 0 \leq t - S'_{N_t+1} \leq x, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the lengths of time that the process is on are $\{\min(\tau_1^k, x)\}_{k=1}^\infty$. Similarly, the lengths of time that the process is off are τ_0^1 and $\{\tau_0^k + (\tau_1^{k-1} - x)^+\}_{k=2}^\infty$, where as usual $(y)^+$ is equal to y if $y \geq 0$, and 0 otherwise. Since the distribution of $\min(\tau_1^k, x) + \tau_0^k + (\tau_1^{k-1} - x)^+$ is nonarithmetic, and since $\mathbb{E}[\min(\tau_1^k, x) + \tau_0^k + (\tau_1^{k-1} - x)^+] < \infty$, we can apply Theorem 3.4.4 in [39] to obtain

$$\lim_{t\to\infty} \mathbb{P}(b_t = 1) = \frac{\mathbb{E}\min(\tau_1, x)}{\mathbb{E}[\min(\tau_1^k, x) + \tau_0^k + (\tau_1^{k-1} - x)^+]}. \tag{2.13}$$

Informally, this intuitive result states that the probability that the alternating renewal process is on at large time is just the expected length of an on bout divided by the sum of the expected lengths of an off bout and an on bout. Since $\mathbb{E}[\min(\tau_1^k, x) + \tau_0^k + (\tau_1^{k-1} - x)^+] = \mathbb{E}\tau_0 + \mathbb{E}\tau_1$ and since the distribution of a^1 is chosen so that $\mathbb{E}\tau_1\mathbb{P}(a^1 \leq x) = \mathbb{E}\min(\tau_1, x)$, (2.13) simplifies to

$$\lim_{t\to\infty} \mathbb{P}(b_t = 1) = \frac{\mathbb{E}\tau_1\mathbb{P}(a^1 \leq x)}{\mathbb{E}\tau_0 + \mathbb{E}\tau_1}.$$

Therefore,

$$\begin{aligned} \mathbb{E}[\{a_t \leq x\}J_t] &= \mathbb{P}(a_t \leq x, J_t = 1) = \mathbb{P}(0 \leq t - S_{N_t+1} \leq x) \\ &= \mathbb{P}(b_t = 1) \xrightarrow[t\to\infty]{} \frac{\mathbb{E}\tau_1\mathbb{P}(a^1 \leq x)}{\mathbb{E}\tau_0 + \mathbb{E}\tau_1} = \mathbb{E}[\{a^1 \leq x\}\xi]. \end{aligned} \tag{2.14}$$

The last equality holds because ξ and a^1 are independent and $\mathbb{E}\xi = \mathbb{E}\tau_1/(\mathbb{E}\tau_0 + \mathbb{E}\tau_1)$.

Further, since the switching time distributions, μ_0 and μ_1, are nonarithmetic, we can apply Theorem 3.4.4 in [39] to conclude that

$$|\mathbb{P}(J_t = 0) - \mathbb{P}(\xi = 0)| \to 0 \quad \text{as } t \to \infty. \tag{2.15}$$

Now observe that

$$|\mathbb{E}\{J_t a_t \leq x\} - \mathbb{E}\{\xi a^1 \leq x\}| \leq |\mathbb{P}(J_t = 0) - \mathbb{P}(\xi = 0)| + |\mathbb{E}\{a_t \leq x\}J_t - \mathbb{E}\{a^1 \leq x\}\xi|.$$

This bound and the convergence in (2.14) and (2.15) gives that $|\mathbb{E}\{J_t a_t \leq x\} - \mathbb{E}\{\xi a^1 \leq x\}| \to 0$ as $t \to \infty$. Thus, $J_t a_t \to_d \xi a^1$. Further, it's easy to see that any continuity set of $\xi a^1 + (1-\xi)a^0$ must be a continuity set of ξa^1. Thus, for any continuity set A of $\xi a^1 + (1-\xi)a^0$, we have that $|\mathbb{E}\{J_t a_t \in A\} - \mathbb{E}\{\xi a^1 \in A\}| \to 0$ by the Portmanteau theorem (see, for example, [7]). Thus,

$$\begin{aligned} &|\mathbb{E}\{a_t \in A\}J_t - \mathbb{E}\{a^1 \in A\}\xi| \\ &\quad \leq |\mathbb{E}\{J_t a_t \in A\} - \mathbb{E}\{\xi a^1 \in A\}| + |\mathbb{P}(J_t = 0) - \mathbb{P}(\xi = 0)| \to 0 \quad \text{as } t \to \infty. \end{aligned}$$

The analogous argument shows that $|\mathbb{E}\{a_t \in A\}(1 - J_t) - \mathbb{E}\{a^0 \in A\}(1 - \xi)| \to 0$ as $t \to \infty$, and the proof is complete. $\square$

The next three lemmas are general results that are all relatively standard. We return to lemmas specific to our problem in Lemma 2.10.

LEMMA 2.7. *Suppose $X_t \to X_\infty$ a.s. as $t \to \infty$ and $|X_t| \le B$ a.s., where B is a random variable satisfying $\mathbb{E}B < \infty$. If $\mathcal{F}_t \subset \mathcal{F}_s$ for $0 \le s \le t$ is a right-contiuous filtration, and $\mathcal{F}_\infty := \cap_{t\ge 0}\mathcal{F}_t$, then*

$$\mathbb{E}[X_t|\mathcal{F}_t] \to \mathbb{E}[X_\infty|\mathcal{F}_\infty] \quad \textit{almost surely as } t \to \infty.$$

Proof. We first show the convergence for an $X_\infty = X_t$ independent of t. Let X_∞ be any integrable random variable, and for $t \le 0$ define

$$M_t := \mathbb{E}\left[X_\infty|\mathcal{F}_{-t}\right].$$

We claim that $\{M_t\}_{t=0}^{-\infty}$ is a backward martingale with respect to the filtration $\{\mathcal{M}\}_{t=0}^{-\infty}$, where $\mathcal{M}_t := \mathcal{F}_{-t}$. For $s \le t \le 0$, we have that $\mathcal{F}_{-s} \subset \mathcal{F}_{-t}$ and, therefore by the tower property of conditional expectation,

$$\mathbb{E}[M_t|\mathcal{F}_{-s}] = \mathbb{E}\left[\mathbb{E}\left[X_\infty|\mathcal{F}_{-t}\right]|\mathcal{F}_{-s}\right] = \mathbb{E}\left[X_\infty|\mathcal{F}_{-s}\right] = M_s.$$

Since by definition of conditional expectation $M_t \in \mathcal{F}_{-t}$, and since $M_t \le B$ almost surely where $\mathbb{E}B < \infty$, we have that M_t is indeed a backward martingale. Because of the continuity properties of $\mathcal{F}_t$, we know we have a separable version of M_t. By the backward martingale convergence theorem (see [17, Theorem 4.2s, p. 354], for example), $M_{-\infty} := \lim_{t\to-\infty} M_t$ exists almost surely and in $L^1(\Omega)$.

We claim that $M_{-\infty} = \mathbb{E}\left[X_\infty|\mathcal{F}_\infty\right]$. Since for $t \le T \le 0$ we have that $M_t \in \mathcal{F}_{-t} \subset \mathcal{F}_{-T}$, it follows that $M_{-\infty} \in \mathcal{F}_{-T}$. Since $T \le 0$ was arbitrary, $M_{-\infty} \in \mathcal{F}_\infty$.

Let $A \in \mathcal{F}_\infty$. Then

$$|\mathbb{E}M_{-t}1_A - \mathbb{E}M_{-\infty}1_A| \le \mathbb{E}|M_{-t}1_A - M_{-\infty}1_A| \le \mathbb{E}|M_{-t} - M_{-\infty}| \to 0 \quad \text{as } t \to \infty$$

since $M_{-t} \to M_{-\infty}$ in $L^1(\Omega)$. But

$$\mathbb{E}M_{-t}1_A = \mathbb{E}[\mathbb{E}\left[X_\infty|\mathcal{F}_t\right]1_A] = \mathbb{E}[\mathbb{E}\left[X_\infty 1_A|\mathcal{F}_t\right]] = \mathbb{E}X_\infty 1_A.$$

Therefore, $\mathbb{E}X1_A = \mathbb{E}M_{-\infty}1_A$, and so we conclude that $M_{-\infty} = \mathbb{E}\left[X_\infty|\mathcal{F}_\infty\right]$.

We now show the convergence for the case where X_t depends on t. Let $T \ge 0$ and define $B_T := \sup\{|X_t - X_s| : t, s > T\}$. $B_T \le 2B$, so B_T is integrable. Thus,

$$\limsup_{t\to\infty} \mathbb{E}\left[|X_t - X_\infty|\mathcal{F}_t\right] \le \lim_{t\to\infty} \mathbb{E}\left[B_T|\mathcal{F}_t\right] = \mathbb{E}\left[B_T|\mathcal{F}_\infty\right].$$

By assumption, $B_T \to 0$ a.s. as $T \to \infty$, so by Jensen's inequality

$$|\mathbb{E}\left[X_t|\mathcal{F}_t\right] - \mathbb{E}\left[X_\infty|\mathcal{F}_t\right]| \le \mathbb{E}\left[|X_t - X_\infty||\mathcal{F}_t\right] \to 0.$$

Therefore,

$$|\mathbb{E}\left[X_t|\mathcal{F}_t\right] - \mathbb{E}\left[X_\infty|\mathcal{F}_\infty\right]| \le |\mathbb{E}\left[X_t|\mathcal{F}_t\right] - \mathbb{E}\left[X_\infty|\mathcal{F}_t\right]| + |\mathbb{E}\left[X_\infty|\mathcal{F}_t\right] - \mathbb{E}\left[X_\infty|\mathcal{F}_\infty\right]|.$$

We've just shown that the first term goes to 0, and we've shown that the second term goes to 0 since X_∞ doesn't depend on t, so the proof is complete. ∎

LEMMA 2.8. *If $X_n \to X_\infty$ a.s. as $n \to \infty$ and $N_t \to \infty$ a.s. as $t \to \infty$, then*

$$X_{N_t} \to X_\infty \quad \textit{a.s. as } t \to \infty.$$

Proof. Let $A := \{X_n \nrightarrow X_\infty\}$ and $B := \{N_t \nrightarrow \infty\}$. Then

$$\mathbb{P}(X_{N_t} \nrightarrow X_\infty) \le \mathbb{P}(A \cup B) \le \mathbb{P}(A) + \mathbb{P}(B) = 0. \qquad \square$$

We now give some standard definitions. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A measurable map $\pi : \Omega \to \Omega$ is said to be *measure preserving* if $\mathbb{P}(\pi^{-1}A) = \mathbb{P}(A)$ for all $A \in \mathcal{F}$. Let π be a given measure preserving map. A set $A \in \mathcal{F}$ is said to be π-*invariant* if $\pi^{-1}A = A$, where two sets are considered to be equal if their symmetric difference has probability 0. A random variable X is said to be π-*invariant* if $X = X \circ \pi$ almost surely.

LEMMA 2.9. *Let $\pi : \Omega \to \Omega$ be a measure preserving map. If X is π-invariant, then so is every set in its σ-algebra.*

Proof. See, for example, [20, Exercise 7.1.1]. $\square$

We remind the reader that we suppress the u_0 dependence and write $\varphi^{N_t}(u_0) = \varphi^{N_t}$. We will usually suppress this dependence since the limiting random variables don't depend on the initial u_0 (see Proposition 2.1).

LEMMA 2.10. *For each $t \ge 0$, define $\mathcal{F}_t$ to be the σ-algebra generated by S_{N_t} and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+1}^\infty$. If D is a Borel set of X, then for each $t \ge 0$*

$$\mathbb{E}\left[\{\varphi^{N_t} \in D\} | \mathcal{F}_t\right] = \mathbb{E}\left[\{\varphi^{-N_t} \in D\} | \mathcal{F}_t\right] \quad a.s.$$

Remark 3. To see why this lemma should be true, observe that (a) the random variables φ^{N_t} and φ^{-N_t} are equal after a reordering of the first N_t-many ω_k's and that (b) the random variables generating $\mathcal{F}_t$ don't depend on the order of the first N_t-many ω_k's.

Proof. Fix a $t \ge 0$ and let $A \in \mathcal{F}_t$. By the definition of conditional expectation, we have that

$$\int_\Omega \mathbb{E}\left[\{\varphi^{N_t} \in D\} | \mathcal{F}_t\right](\omega)\{A\}(\omega)\, d\mathbb{P} = \int_\Omega \{\varphi^{N_t} \in D\}(\omega)\{A\}(\omega)\, d\mathbb{P}.$$

Define $\sigma_t : \Omega \to \Omega$ to be the permutation that inverts the order of the first N_t-many ω_k's, that is, $(\sigma_t(\omega))_k = \omega_{N_t-k+1}$ for $k \in \{1, \dots, N_t\}$ and $(\sigma_t(\omega))_k = \omega_k$ for $k > N_t$. Observe that $N_t(\omega) = N_t(\sigma_t(\omega))$ and thus $\varphi^{N_t}(\omega) = \varphi^{-N_t}(\sigma_t(\omega))$. Also, S_{N_t} and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+1}^\infty$ are σ_t-invariant, so A is σ_t-invariant by Lemma 2.9. Thus,

$$\int_\Omega \{\varphi^{N_t} \in D\}(\omega)\{A\}(\omega)\, d\mathbb{P} = \int_\Omega \{\varphi^{-N_t} \in D\}(\sigma_t(\omega))\{A\}(\sigma_t(\omega))\, d\mathbb{P}.$$

Since σ_t is measure preserving and by the definition of conditional expectation,

$$\begin{aligned}\int_\Omega \{\varphi^{-N_t} \in D\}(\sigma_t(\omega))\{A\}(\sigma_t(\omega))\, d\mathbb{P} &= \int_\Omega \{\varphi^{-N_t} \in D\}(\omega)\{A\}(\omega)\, d\mathbb{P} \\ &= \int_\Omega \mathbb{E}\left[\{\varphi^{-N_t} \in D\} | \mathcal{F}_t\right](\omega)\{A\}(\omega)\, d\mathbb{P}.\end{aligned}$$

Putting this all together,

$$\int_\Omega \mathbb{E}[\{\varphi^{N_t} \in D\} | \mathcal{F}_t]\{A\}\, d\mathbb{P} = \int_\Omega \mathbb{E}[\{\varphi^{-N_t} \in D\} | \mathcal{F}_t]\{A\}\, d\mathbb{P}.$$

Since A was an arbitrary element of $\mathcal{F}_t$, the proof is complete. $\square$

Recall that the random variable Y_1 is defined by $Y_1 := \lim_{n\to\infty} \varphi^{-n}(x)$ and is independent of the choice of $x \in X$, by Proposition 2.1.

LEMMA 2.11. *For each $t \geq 0$, define $\mathcal{F}_t$ to be the σ-algebra generated by S_{N_t} and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+1}^{\infty}$. If D is a Y_1-continuity set, then with probability one*

$$\mathbb{E}\left[\{\varphi^{-N_t} \in D\}|\mathcal{F}_t\right] \to \mathbb{E}\{Y_1 \in D\} \quad \text{as } t \to \infty \tag{2.16}$$

$$\text{and} \quad \mathbb{E}\left[\{\varphi^{N_t} \in D\}|\mathcal{F}_t\right] \to \mathbb{E}\{Y_1 \in D\} \quad \text{as } t \to \infty \tag{2.17}$$

Proof. In light of Lemma 2.10, it suffices to show the convergence in (2.16).

Since $\varphi^{-n} \to Y_1$ almost surely as $n \to \infty$ and since $N_t \to \infty$ almost surely as $t \to \infty$, we have that $\varphi^{-N_t} \to Y_1$ almost surely by Lemma 2.8. We claim that $\{\varphi^{-N_t} \in D\} \to \{Y_1 \in D\}$ almost surely. Since $\varphi^{-N_t} \to Y_1$ almost surely and $\mathbb{P}(Y_1 \in \partial D) = 0$, there exists a set $S \subset \Omega$ of full measure such that if $\omega \in S$, then $\varphi^{-N_t}(\omega) \to Y_1(\omega) \notin \partial D$ as $t \to \infty$. Let $\omega \in S$. If $Y_1(\omega) \in D$, then $Y_1(\omega)$ must be in the interior of D and hence there exists some $r > 0$ such that the ball of radius r centered at $Y_1(\omega)$ is contained in the interior of D. Since $\varphi^{-N_t}(\omega) \to Y_1(\omega)$, there must exist some T such that $\varphi^{-N_t}(\omega)$ is within r of $Y_1(\omega)$ for all $t \geq T$ and hence $\varphi^{-N_t}(\omega) \in D$ for all $t \geq T$. Thus, $\{\varphi^{-N_t} \in D\}(\omega) \to \{Y_1 \in D\}(\omega) = 1$. A similar argument shows that $\{\varphi^{-N_t} \in D\}(\omega) \to \{Y_1 \in D\}(\omega) = 0$ if $\omega \in S$ is such that $Y_1(\omega) \notin D$. Hence, $\{\varphi^{-N_t} \in D\} \to \{Y_1 \in D\}$ on S which has full measure, so the convergence is almost sure.

Define $\mathcal{F}_\infty := \cap_{t\geq 0}\mathcal{F}_t$ and observe that $\mathcal{F}_t \subset \mathcal{F}_s$ for $t \geq s \geq 0$. Thus, by Lemma 2.7,

$$\mathbb{E}\left[\{\varphi^{-N_t} \in D\}|\mathcal{F}_t\right] \to \mathbb{E}\left[\{Y_1 \in D\}|\mathcal{F}_\infty\right] \quad \text{almost surely as } t \to \infty.$$

To complete the proof, we will show that for every $A \in \mathcal{F}_\infty$, $\mathbb{P}(A) = 0$ or 1. To show this, we will show that $\mathcal{F}_\infty$ is contained in the exchangeable σ-algebra and then apply the Hewitt–Savage zero-one law. Let $n \in \mathbb{N}$, $A \in \mathcal{F}_\infty$, and π_n be an arbitrary permutation of $\omega_1, \dots, \omega_n$. Define $\pi_t : \Omega \to \Omega$ by

$$\pi_t(\omega) = \begin{cases} \pi_n(\omega) & N_t \geq n, \\ \omega & N_t < n. \end{cases}$$

Since S_{N_t} and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+1}^{\infty}$ are π_t-invariant, A is π_t-invariant by Lemma 2.9 as $A \in \mathcal{F}_\infty \subset \mathcal{F}_t$. Therefore,

$$\mathbb{P}(A\Delta\pi_n^{-1}A, N_t \geq n) = \mathbb{P}(A\Delta\pi_t^{-1}A, N_t \geq n) \leq \mathbb{P}(A\Delta\pi_t^{-1}A) = 0.$$

Hence,

$$\begin{aligned}\mathbb{P}(A\Delta\pi_n^{-1}A) &= \mathbb{P}(A\Delta\pi_n^{-1}A, N_t \geq n) + \mathbb{P}(A\Delta\pi_n^{-1}A, N_t < n) \\ &\leq \mathbb{P}(A\Delta\pi_n^{-1}A, N_t < n) \leq \mathbb{P}(N_t < n).\end{aligned}$$

Since t was arbitrary, and because $\mathbb{P}(N_t < n) \to 0$ as $t \to \infty$ since $N_t \to \infty$ almost surely, we conclude that $\mathbb{P}(A\Delta\pi_n^{-1}A) = 0$. Since π_n was an arbitrary finite permutation, we conclude that $\mathcal{F}_\infty$ is contained in the exchangeable σ-algebra. By the Hewitt–Savage zero-one law, $\mathcal{F}_\infty$ only contains events that have probability 0 or 1. Thus, $\{Y_1 \in D\}$ is trivially independent of $\mathcal{F}_\infty$, and therefore $\mathbb{E}\left[\{Y_1 \in D\}|\mathcal{F}_\infty\right] = \mathbb{E}\{Y_1 \in D\}$. $\square$

LEMMA 2.12. *For each $t \geq 0$, define $\mathcal{F}'_t$ to be the σ-algebra generated by S'_{N_t+1}, $\tau_1^{N_t+1}$, and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+2}^{\infty}$. Then*

$$J_t \mathbb{E}\left[\{\tau_0^{N_t+1} \in B\}|\mathcal{F'}_t\right] = J_t \mathbb{E}\left[\{\tau_0^1 \in B\}|\mathcal{F'}_t\right] \quad \textit{almost surely.}$$

Remark 4. Recall that J_t is either 0 if $S_{N_t} \leq t < S'_{N_t+1}$ or 1 if $S'_{N_t+1} \leq t$. Hence, this lemma states that $\mathbb{E}\left[\{\tau_0^{N_t+1} \in B\}|\mathcal{F'}_t\right] = \mathbb{E}\left[\{\tau_0^1 \in B\}|\mathcal{F'}_t\right]$ if $J_t = 1$.

Remark 5. The proof of this lemma is very similar to the proof of Lemma 2.10.

Proof. If ω is such that $J_t = 0$, then the equality is trivially satisfied. Let $A \in \mathcal{F}'_t$. Since $\{\omega \in \Omega : J_t(\omega) = 1\} \in \mathcal{F}'_t$, we have by the definition of conditional expectation that

$$\int_\Omega \mathbb{E}\left[\{\tau_0^{N_t+1} \in B\}|\mathcal{F}'_t\right] \{A, J_t = 1\}\, d\mathbb{P} = \int_\Omega \{\tau_0^{N_t+1} \in B\}(\omega)\{A, J_t = 1\}(\omega)\, d\mathbb{P}.$$

Define $\sigma_t : \Omega \to \Omega$ by

$$(\sigma_t(\omega))_k = \begin{cases} (\tau_0^{N_t+1}, \tau_1^1) & \text{if } k = 1 \text{ and } J_t = 1, \\ (\tau_0^1, \tau_1^{N_t+1}) & \text{if } k = N_t + 1 \text{ and } J_t = 1, \\ \omega_k & \text{otherwise.} \end{cases}$$

That is, σ_t switches τ_0^1 and $\tau_0^{N_t+1}$ if $J_t = 1$ and otherwise does nothing. Since S'_{N_t+1}, $\tau_1^{N_t+1}$, and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+2}^{\infty}$ are all σ_t-invariant, we have that A is σ_t-invariant by Lemma 2.9. Also observe that $\{J_t = 1\}$ is σ_t-invariant. Thus,

$$\int_\Omega \{\tau_0^{N_t+1} \in B\}(\omega)\{A, J_t = 1\}(\omega)\, d\mathbb{P} = \int_\Omega \{\tau_0^1 \in B\}(\sigma_t(\omega))\{A, J_t = 1\}(\sigma_t(\omega))\, d\mathbb{P}.$$

Since σ_t is measure preserving, and by the definition of conditional expectation, we have that

$$\begin{aligned}\int_\Omega \{\tau_0^1 \in B\}(\sigma_t(\omega))\{A, J_t = 1\}(\sigma_t(\omega))\, d\mathbb{P} &= \int_\Omega \{\tau_0^1 \in B\}(\omega)\{A, J_t = 1\}(\omega)\, d\mathbb{P} \\ &= \int_\Omega \mathbb{E}\left[\{\tau_0^1 \in B\}|\mathcal{F}'_t\right] \{A, J_t = 1\}\, d\mathbb{P}.\end{aligned}$$

Putting all this together,

$$\int_\Omega \mathbb{E}[\{\tau_0^{N_t+1} \in B\}|\mathcal{F}'_t]\{A, J_t = 1\}\, d\mathbb{P} = \int_\Omega \mathbb{E}[\{\tau_0^1 \in B\}|\mathcal{F}'_t]\{A, J_t = 1\}\, d\mathbb{P}.$$

This implies that $\mathbb{E}[\{\tau_0^{N_t+1} \in B\}|\mathcal{F'}_t] = \mathbb{E}[\{\tau_0^1 \in B\}|\mathcal{F'}_t]$ almost surely on $\{J_t = 1\}$. To see this, let $\epsilon > 0$ define $\Lambda := \{\omega \in \Omega : \mathbb{E}[\{\tau_0^{N_t+1} \in B\}|\mathcal{F'}_t] - \mathbb{E}[\{\tau_0^1 \in B\}|\mathcal{F'}_t] \geq \epsilon\}$. This set is in $\mathcal{F}'_t$, so by the above calculation we have that

$$0 = \int_{\Lambda \cap \{J_t = 1\}} \mathbb{E}[\{\tau_0^{N_t+1} \in B\}|\mathcal{F'}_t] - \mathbb{E}[\{\tau_0^1 \in B\}|\mathcal{F'}_t]\, d\mathbb{P} \geq \epsilon \mathbb{P}(\Lambda \cap \{J_t = 1\}).$$

So $\mathbb{P}(\Lambda \cap \{J_t = 1\}) = 0$. The same argument with $\Lambda' := \{\omega \in \Omega : \mathbb{E}[\{\tau_0^1 \in B\}|\mathcal{F}'_t] - \mathbb{E}[\{\tau_0^{N_t+1} \in B\}|\mathcal{F}'_t] \geq \epsilon\}$ completes the proof of the claim. Therefore, $J_t \mathbb{E}[\{\tau_0^{N_t+1} \in B\}|\mathcal{F}'_t] = J_t \mathbb{E}[\{\tau_0^1 \in B\}|\mathcal{F}'_t]$ almost surely. $\square$

LEMMA 2.13. *For each $t \geq 0$, define $\mathcal{F}'_t$ to be the σ-algebra generated by S'_{N_t+1}, $\tau_1^{N_t+1}$, and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+2}^{\infty}$. Then*

$$\mathbb{E}\left[\{\tau_0^1 \in B\}|\mathcal{F}'{}_t\right] \to \mathbb{E}\{\tau_0 \in B\} \quad \textit{almost surely as } t \to \infty.$$

Remark 6. The proof of this lemma is very similar to the proof of Lemma 2.11.

Proof. Define $\mathcal{F}'{}_\infty := \cap_{t\geq 0}\mathcal{F}'_t$ and observe that $\mathcal{F}'{}_s \supset \mathcal{F}'{}_t$ for $0 \leq s \leq t$. Thus, by Lemma 2.7

$$\mathbb{E}\left[\{\tau_0^1 \in B\}|\mathcal{F}'{}_t\right] \to \mathbb{E}\left[\{\tau_0^1 \in B\}|\mathcal{F}'_\infty\right] \quad \text{almost surely.}$$

We claim that for each $A \in \mathcal{F}'_\infty$, $\mathbb{P}(A) = 0$ or 1. To show this, we will show that $\mathcal{F}'_\infty$ is contained in the exchangeable σ-algebra and then apply the Hewitt–Savage zero-one law. Let $n \in \mathbb{N}$, $A \in \mathcal{F}'_\infty$, and π_n be an arbitrary permutation of $(\tau_0^1, \tau_1^1), \ldots, (\tau_0^n, \tau_1^n)$.

Define $\pi_t : \Omega \to \Omega$ by

$$\pi_t(\omega) = \begin{cases} \pi_n(\omega) & N_t \geq n, \\ \omega & N_t < n. \end{cases}$$

Since S'_{N_t+1}, $\tau_1^{N_t+1}$, and $\{(\tau_0^k, \tau_1^k)\}_{k=N_t+2}^{\infty}$ are π_t-invariant, A is π_t-invariant by Lemma 2.9 as $A \in \mathcal{F}'_\infty \subset \mathcal{F}'_t$. Therefore,

$$\mathbb{P}(A\Delta\pi_n^{-1}A, N_t \geq n) = \mathbb{P}(A\Delta\pi_t^{-1}A, N_t \geq n) \leq \mathbb{P}(A\Delta\pi_t^{-1}A) = 0.$$

Hence,

$$\begin{aligned} \mathbb{P}(A\Delta\pi_n^{-1}A) &= \mathbb{P}(A\Delta\pi_n^{-1}A, N_t \geq n) + \mathbb{P}(A\Delta\pi_n^{-1}A, N_t < n) \\ &\leq \mathbb{P}(A\Delta\pi_n^{-1}A, N_t < n) \leq \mathbb{P}(N_t < n). \end{aligned}$$

Since t was arbitrary, we conclude that $\mathbb{P}(A\Delta\pi_n^{-1}A) = 0$ because $\mathbb{P}(N_t < n+1) \to 0$ as $t \to \infty$ since $N_t \to \infty$ almost surely. Since π_n was an arbitrary finite permutation, we conclude that $\mathcal{F}'_\infty$ is contained in the exchangeable σ-algebra. By the Hewitt–Savage zero-one law, $\mathcal{F}'_\infty$ contains only events that have probability 0 or 1. Thus, $\{\tau_0^1 \in C\}$ is trivially independent of $\mathcal{F}'_\infty$, and so we conclude that $\mathbb{E}\left[\{\tau_0^1 \in C\}|\mathcal{F}'_\infty\right] = \mathbb{E}\{\tau_0^1 \in C\} = \mathbb{E}\{\tau_0 \in C\}$. $\square$

3. PDEs with randomly switching boundary conditions. We now use our results from section 2 to study parabolic PDEs with randomly switching boundary conditions. Our results apply to a range of specific problems, so in section 3.1 we explain how to cast a problem in our framework. In section 3.2, we collect assumptions, and in section 3.3, we prove theorems about the mean of the process.

3.1. General setup. Our results can be applied to the following type of random PDE. Suppose we are given a strongly elliptic, symmetric, linear, second order differential operator L on a domain $D \subset \mathbb{R}^d$ with smooth coefficients which do not depend on t. Assume that the domain D is bounded with a smooth boundary. We consider the stochastic process $u(t, x)$ that solves

$$\partial_t u = Lu \quad \text{in } D \tag{3.1}$$

subject to boundary conditions that switch at random times between two given boundary conditions, (a) and (b). We allow (a) and (b) to be different types; for example, one can be Dirichlet and the other Neumann. For the sake of presentation, we assume (a) are homogenous, but our analysis is easily modified to include the case where (a) are inhomogenous.

We formulate this problem in the setting of section 2 as alternating flows on the Hilbert space $L^2(D)$. We define

$$Au := Lu \quad \text{if } u \in D(A) \qquad \text{and} \qquad Bu := Lu \quad \text{if } u \in D(B),$$

where $D(A)$ is chosen so that A generates the contraction C_0-semigroup that maps an initial condition to the solution of (3.1) at time t subject to boundary conditions (a), and $D(B)$ is chosen so that B generates the contraction C_0-semigroup that maps an initial condition to the solution of (3.1) at time t subject to the homogenous version of boundary conditions (b). We then choose $h(t) : [0, \infty) \to D(L)$ to satisfy $\partial_t h = Lh$ with boundary conditions (b) and initial condition $h(0) = 0$. Then the H-valued process defined in (2.10) in section 2 with $\Phi^1_t(f) = e^{At}f$ and $\Phi^0_t(f) = e^{Bt}f + h(t)$ corresponds to this random PDE.

3.2. Assumptions. We now formalize the setup from section 3.1. Let H be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let A and B be two self-adjoint operators on H, one with strictly negative spectrum and one with nonpositive spectrum. Hence, A and B generate contraction C_0-semigroups, which we denote, respectively, by e^{At} and e^{Bt}. Assume $A = B$ on $D(A) \cap D(B) \neq \emptyset$. Assume that there exists a continuous function $h(t) : [0, \infty) \to H$ satisfying $h(0) = 0$ and $\frac{d}{dt}\langle \phi, h(t) \rangle = \langle B\phi, h(t) \rangle$ for all $\phi \in D(A) \cap D(B)$. Recalling notation from section 2.1, let the switching time distributions, μ_0 and μ_1, be continuous distributions on the positive real line.

Let $u(t, \omega)$ be the H-valued process defined in (2.10) in section 2.2 with

$$\Phi^1_t(f) = e^{At}f \qquad \text{and} \qquad \Phi^0_t(f) = e^{Bt}f + h(t).$$

It's easy to check that Φ^1_t and Φ^0_t satisfy assumptions (a)-(e) from section 2.1.

Assume there exists a deterministic $M = M(u_0)$ so that with probability one, $\|u(t)\| \leq M$ for each $t \geq 0$, where $\|x\| := \sqrt{\langle x, x \rangle}$.

For every $0 < s \leq t$, define $\eta(s, t)$ to be the random variable that gives the number of switches that occur on the interval (s, t). Formally, we define $\eta(s, t)$ by taking the supremum over partitions σ of the interval (s, t), $s = \sigma_0 < \sigma_1 < \cdots < \sigma_k < \sigma_{k+1} = t$,

$$\eta(s,t)(\omega) := \sup_{\sigma} \sum_{i=0}^{k} |J_{\sigma_{i+1}}(\omega) - J_{\sigma_i}(\omega)|, \tag{3.2}$$

where J_t is as in (2.9). Assume that μ_0 and μ_1 are such that for every $t > 0$, we have that as $s \to 0$,

$$\mathbb{P}\left(\eta(t, t+s) = 1\right) = O(s) \quad \text{and} \quad \mathbb{P}\left(\eta(t, t+s) \geq 2\right) = o(s).$$

3.3. The mean satisfies the PDE. In what follows, fix $\phi \in D(A) \cap D(B)$, which will serve as our test function. The following theorem states that the mean of our process satisfies the weak form of the PDE. We note that we use $\mathbb{E}$ to denote the Bochner integral of Hilbert space-valued random variables.

THEOREM 3.1. *For each $\phi \in D(A) \cap D(B)$ and $t > 0$, we have that*

$$\frac{d}{dt}\langle \phi, \mathbb{E}u(t) \rangle = \langle A\phi, \mathbb{E}u(t) \rangle.$$

To prove this theorem, we need a few lemmas. Our first lemma states that each realization of our stochastic process satisfies the weak form of the PDE away from switching times.

LEMMA 3.2. *Let $\omega_0 \in \Omega$ be given. If $t_0 > 0$ is such that $t_0 \neq S_k(\omega_0)$ and $t_0 \neq S'_k(\omega_0)$ for every k, then for all t in some neighborhood of t_0,*

$$\frac{d}{dt}\langle \phi, u(t,\omega_0)\rangle = \langle A\phi, u(t,\omega_0)\rangle.$$

Proof. By the definition of $u(t,\omega)$ and the assumption that A and B are self-adjoint, we can write the inner product of ϕ and u as

$$\begin{aligned}\langle \phi, u(t)\rangle &= \langle \phi, e^{Aa_t}u(S'_{N_t+1})\rangle J_t + \langle \phi, e^{Ba_t}u(S_{N_t}) + h(a_t)\rangle(1-J_t)\\ &= \langle e^{Aa_t}\phi, u(S'_{N_t+1})\rangle J_t + \langle e^{Ba_t}\phi, u(S_{N_t})\rangle(1-J_t) + \langle \phi, h(a_t)\rangle(1-J_t).\end{aligned}$$

We now calculate $\frac{d}{dt}e^{Aa_t}\phi$ and $\frac{d}{dt}e^{Ba_t}\phi$, where $\frac{d}{dt}$ means the limit in H of the difference quotients. Since t_0 is such that $t_0 \neq S_k(\omega_0)$ and $t_0 \neq S'_k(\omega_0)$ for all k, there exists a neighborhood $J(\omega_0) = J$ of t_0 so that no switches occur in J. Therefore, S_{N_t}, S'_{N_t+1}, and J_t are constant on J. And since e^{At} is a C_0-semigroup and $\phi \in D(A)$, we have that for all $t \in J$

$$\frac{d}{dt}e^{Aa_t}\phi = \frac{d}{dt}e^{A(t-S'_{N_t+1})}\phi = \frac{d}{dt}e^{At}e^{-AS'_{N_t+1}}\phi = Ae^{At}e^{-AS'_{N_t+1}}\phi = Ae^{Aa_t}\phi.$$

Similarly, $\frac{d}{dt}e^{Ba_t}\phi = Be^{Ba_t}\phi$. Since strongly convergent sequences in H are weakly convergent, and again since S_{N_t}, S'_{N_t+1}, and J_t are constant on J, we have that for all $t \in J$

$$\frac{d}{dt}\langle \phi, u(t)\rangle = \langle Ae^{Aa_t}\phi, u(S'_{N_t+1})\rangle J_t + \big(\langle Be^{Ba_t}\phi, u(S_{N_t})\rangle + \frac{d}{dt}\langle \phi, h(a_t)\rangle\big)(1-J_t).$$

Since A and B are self-adjoint, $A = B$ on $D(A)\cap D(B)$, and $\frac{d}{dt}\langle \phi, h(t)\rangle = (B\phi, h(t))$, we conclude that for all $t \in J$,

$$\begin{aligned}\frac{d}{dt}\langle \phi, u(t)\rangle &= \langle A\phi, e^{Aa_t}u(S'_{N_t+1})\rangle J_t + \langle B\phi, e^{Ba_t}u(S_{N_t}) + h(a_t)\rangle(1-J_t)\\ &= \langle A\phi, u(t)\rangle J_t + \langle B\phi, u(t)\rangle(1-J_t) = \langle A\phi, u(t)\rangle. \qquad \square\end{aligned}$$

The next lemma states that our process satisfies a weak continuity condition.

LEMMA 3.3. *For every $\epsilon > 0$ and $t > 0$, there exists a $\delta(\epsilon, t) > 0$ so that if $|t-s| < \delta(\epsilon,t)$, then*

$$|\langle \phi, u(t,\omega) - u(s,\omega)\rangle 1_{\eta(s,t)=1}| < \epsilon \quad a.s.$$

Proof. Let s and t be given and let ρ be the minimum of s and t. Observe that if there are no switches between s and t and $J_\rho = 0$, then

$$|\langle \phi, u(t,\omega) - u(s,\omega)\rangle| = \left|\langle \phi, [e^{A|t-s|} - I]u(\rho,\omega)\rangle\right| \leq \|e^{A|t-s|}\phi - \phi\|M,$$

since A is self-adjoint and $\|u(t)\| \leq M$ a.s. by assumption. Similarly, suppose there are no switches between s and t and $J_\rho = 1$. If $M_2 = \max_{\xi\leq 2t}\|h(\xi)\|$ and $|t-s| < t$, then we have by the mean value theorem

$$\begin{aligned}\big|\langle\phi, u(t,\omega)-u(s,\omega)\rangle\big| &\le \big|\langle\phi,[e^{B|t-s|}-I][u(\rho,\omega)-h(a_\rho)]\rangle\big| + \big|\langle\phi, h(a_t)-h(a_s)\rangle\big| \\ &\le \big|\langle[e^{B|t-s|}-I]\phi, u(\rho,\omega)-h(a_\rho)\rangle\big| + |t-s|\max_{\xi\le 2t}\frac{d}{dt}\big|\langle\phi, h(\xi)\rangle\big| \\ &\le \|e^{B|t-s|}\phi-\phi\|\big(M+M_2\big)+|t-s|\|B\phi\|M_2.\end{aligned}$$

Since e^{At} and e^{Bt} are both C_0-semigroups, we can choose a $0<\delta(\epsilon,t)<t$ so that if $|t-s|<\delta(\epsilon,t)$, then

$$\max\{\|e^{A|t-s|}\phi-\phi\|, \|e^{B|t-s|}\phi-\phi\|, |t-s|\} < \frac{\epsilon}{M+M_2+\|B\phi\|M_2}.$$

Let $\omega\in\Omega$ be given and assume $|t-s|<\delta(\epsilon,t)$. If ω is such that $\eta(s,t)(\omega)\neq 1$, then the result is immediate. Suppose $\eta(s,t)(\omega)=1$. If σ denotes the switching time between s and t, then

$$|\langle\phi, u(t,\omega)-u(s,\omega)\rangle| \le \big|\langle\phi, u(t,\omega)-u(\sigma,\omega)\rangle\big| + \big|\langle\phi, u(\sigma,\omega)-u(s,\omega)\rangle\big| < 3\epsilon. \qquad \square$$

Proof of Theorem 3.1. We seek to differentiate $\mathbb{E}\langle\phi, u(t)\rangle$ with respect to t. Define

$$f(t,\omega)=\langle\phi, u(t,\omega)\rangle.$$

Let $h_n\to 0$ as $n\to\infty$. For a given $t_0>0$, define the difference quotient

$$\begin{aligned} g_n(\omega) &:= \frac{1}{h_n}\left(f(t_0+h_n,\omega)-f(t_0,\omega)\right) \\ &= g_n(\omega)1_{\eta(t_0+h_n,t_0)=0} + g_n(\omega)1_{\eta(t_0+h_n,t_0)=1} + g_n(\omega)1_{\eta(t_0+h_n,t_0)\ge 2} \\ &= \Psi_0+\Psi_1+\Psi_2,\end{aligned}$$

where η is defined in (3.2). We will handle each of these terms differently.

We first consider Ψ_0. Assume that ω is such that t_0 is not a switching time. By Lemma 3.2,

$$\frac{1}{h_n}\left(f(t_0+h_n,\omega)-f(t_0,\omega)\right) \to \frac{d}{dt}f(t_0,\omega)=\langle A\phi, u(t_0)\rangle \text{ as } n\to\infty.$$

Also observe that for such an ω, we have that $1_{\eta(t_0+h_n,t_0)=0}(\omega)=1$ for n sufficiently large. Since μ_0 and μ_1 are continuous distributions, this set of ω's has probability 1, and thus

$$\Psi_0=\frac{1}{h_n}\left(f(t_0+h_n,\omega)-f(t_0,\omega)\right)1_{\eta(t_0+h_n,t_0)=0} \to \langle A\phi, u(t_0)\rangle \quad \text{a.s. as } n\to\infty.$$

We now apply the bounded convergence theorem to Ψ_0. Let n and ω be given. If $\eta(t_0+h_n,t_0)(\omega)\neq 0$, then $|\Psi_0|=0$, trivially. If $\eta(t_0+h_n,t_0)(\omega)=0$, then $f(t,\omega)$ is differentiable in t for all $t\in(t_0,t_0+h_n)$. Therefore, we can employ the mean value theorem to obtain

$$\begin{aligned}\left|\frac{1}{h_n}\left(f(t_0+h_n,\omega)-f(t_0,\omega)\right)\right| &\le \sup_{t\in(t_0,t_0+h_n)}\left|\frac{d}{dt}f(t,\omega)\right| \\ &= \sup_{t\in(t_0,t_0+h_n)}|\langle A\phi, u(t,\omega)\rangle| \le \|A\phi\|M,\end{aligned}$$

since $\|u(t)\| \leq M$ by assumption. Thus, $|\Psi_0| \leq \|A\phi\| M$ almost surely, and so by the bounded convergence theorem, $\mathbb{E}\Psi_0 \to \mathbb{E}\langle A\phi, u(t_0)\rangle$ as $n \to \infty$.

To complete the proof, we need only show that Ψ_1 and Ψ_2 both tend to 0 in mean as $n \to \infty$. We first work on Ψ_1. Observe that

$$\begin{aligned}\mathbb{E}|\Psi_1| &= \mathbb{E}\left|\frac{1}{h_n}\left(f(t_0+h_n,\omega) - f(t_0,\omega)\right) 1_{\eta(t_0+h_n,t_0)=1}\right| \\ &\leq \frac{1}{h_n} \operatorname{ess\,sup}_\omega \left|(f(t_0+h_n,\omega) - f(t_0,\omega)) 1_{\eta(t_0+h_n,t_0)=1}\right| \mathbb{E}\left(1_{\eta(t_0+h_n,t_0)=1}\right).\end{aligned}$$

It follows from Lemma 3.3 that $\operatorname{ess\,sup}_\omega \left|(f(t_0+h_n,\omega) - f(t_0,\omega)) 1_{\eta(t_0+h_n,t_0)=1}\right| \to 0$ as $n \to \infty$. Since by assumption $\mathbb{P}\left(\eta(t_0+h_n,t_0) = 1\right) = O(h_n)$, we conclude that $\mathbb{E}|\Psi_1| \to 0$ as $n \to \infty$.

Finally, we consider Ψ_2. By the assumption that $\|u(t)\| \leq M$,

$$\begin{aligned}\mathbb{E}|\Psi_2| &= \mathbb{E}\left|\frac{1}{h_n}\left(f(t_0+h_n,\omega) - f(t_0,\omega)\right) 1_{\eta(t_0+h_n,t_0)\geq 2}\right| \\ &\leq \frac{2\|\phi\| M}{h_n} \mathbb{P}\left(\eta(t_0+h_n,t_0) \geq 2\right).\end{aligned}$$

By assumption, $\mathbb{P}\left(\eta(t_0+h_n,t_0) \geq 2\right) = o(h_n)$, and hence $\mathbb{E}|\Psi_2| \to 0$ as $n \to \infty$.

Therefore,

$$\frac{\mathbb{E}\langle\phi, u(t_0+h_n)\rangle - \mathbb{E}\langle\phi, u(t_0)\rangle}{h_n} = \mathbb{E}g_n \to \mathbb{E}\langle A\phi, u(t_0)\rangle \text{ as } n \to \infty.$$

Since h_n was an arbitrary sequence tending to 0 and t_0 was an arbitrary positive number, we conclude that $\frac{d}{dt}\mathbb{E}\langle\phi, u(t)\rangle = \mathbb{E}\langle A\phi, u(t)\rangle$ for all $t > 0$.

Since taking the inner product against ϕ or $A\phi$ are both bounded linear operators on H, we can exchange expectation with inner product to obtain

$$\frac{d}{dt}\langle\phi, \mathbb{E}u(t)\rangle = \frac{d}{dt}\mathbb{E}\langle\phi, u(t)\rangle = \mathbb{E}\langle A\phi, u(t)\rangle = \langle A\phi, \mathbb{E}u(t)\rangle. \qquad \square$$

We now show that the mean at large time satisfies the homogeneous PDE.

THEOREM 3.4. *Let $\phi \in D(A) \cap D(B)$. Then $\mathbb{E}u(t) \to \mathbb{E}\bar{u}$ weakly in H as $t \to \infty$, where $\bar{u}$ is as in Theorem* 2.4. *Furthermore, $\mathbb{E}\bar{u}$ satisfies*

$$\langle A\phi, \mathbb{E}\bar{u}\rangle = 0.$$

Remark 7. We will often assume the differential operator and the domain to be sufficiently regular so that this theorem implies $\mathbb{E}\bar{u}$ is a C^∞ function satisfying the PDE pointwise.

Proof. Since the switching time distributions, μ_0 and μ_1, are assumed to be continuous distributions, they are nonarithmetic. Hence, by Theorem 2.4, $\mathbb{E}g(u(t)) \to \mathbb{E}g(\bar{u})$ as $t \to \infty$ for every continuous and bounded $g : H \to \mathbb{R}$. For any $\eta \in H$, the function $\langle\eta, \cdot\rangle : H \to \mathbb{R}$ is continuous, and since by assumption, $\|u(t)\| \leq M$ a.s., it follows that

$$\mathbb{E}\langle\eta, u(t)\rangle \to \mathbb{E}\langle\eta, \bar{u}\rangle \quad \text{as } t \to \infty. \tag{3.3}$$

Since taking the inner product against η is a bounded linear operator on H, we can exchange expectation with the inner product in (3.3) above. Hence, $\mathbb{E}u(t) \to \mathbb{E}\bar{u}$ weakly in H as $t \to \infty$.

Of course it follows that in particular

$$\langle \phi, \mathbb{E}u(t)\rangle \to \langle \phi, \mathbb{E}\bar{u}\rangle \quad \text{and} \quad \langle A\phi, \mathbb{E}u(t)\rangle \to \langle A\phi, \mathbb{E}\bar{u}\rangle \quad \text{as } t \to \infty.$$

By Theorem 3.1, $\frac{d}{dt}\langle \phi, \mathbb{E}u(t)\rangle = \langle A\phi, \mathbb{E}u(t)\rangle$. Thus, $\langle \phi, \mathbb{E}u(t)\rangle$ and $\frac{d}{dt}\langle \phi, \mathbb{E}u(t)\rangle$ both converge as $t \to \infty$, and so we conclude that $\frac{d}{dt}\langle \phi, \mathbb{E}u(t)\rangle$ must actually converge to 0. Hence, $\langle A\phi, \mathbb{E}\bar{u}\rangle = 0$. ∎

4. Examples.

In this section, we apply our results from sections 2 and 3 to the heat equation on the interval $[0, L]$. We impose an absorbing Dirichlet boundary condition at $x = 0$ and a stochastically switching boundary condition at $x = L$. In Example 1, we consider switching between a Dirichlet and a Neumann boundary condition at $x = L$. In Example 2, we consider switching between two Dirichlet boundary conditions at $x = L$. As in section 3, we use $\mathbb{E}$ to denote the Bochner integral of $L^2[0, L]$-valued random variables and not the pointwise expectation of random functions.

4.1. Example 1: Dirichlet/Neumann switching.

Consider the stochastic process that solves

$$\partial_t u = D\Delta u \quad \text{in } (0, L) \tag{4.1}$$

and at exponentially distributed times switches between the boundary conditions

$$\begin{cases} u(0,t) = 0, \\ u_x(L,t) = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} u(0,t) = 0, \\ u(L,t) = b > 0. \end{cases}$$

To cast this problem in the setting of previous sections, we set our Hilbert space to be $L^2[0, L]$ and define the operators

$$\begin{aligned} Au &:= \Delta u \quad \text{if } u \in D(A) := \left\{ \phi \in H^2(0,L) : \frac{\partial \phi}{\partial \mathbf{n}}(L) = 0 = \phi(0) \right\}, \\ Bu &:= \Delta u \quad \text{if } u \in D(B) := H_0^1(0,L) \cap H^2(0,L). \end{aligned}$$

We set $c = \frac{b}{L}x \in L^2[0, L]$ and let our switching time distributions, μ_0 and μ_1, be exponential with respective rate parameters r_0 and r_1. Let $u(t, \omega)$ be the H-valued process defined in (2.10) with

$$\Phi_t^1(f) = e^{At} f \qquad \text{and} \qquad \Phi_t^0(f) = e^{Bt}(f - c) + c. \tag{4.2}$$

We are interested in studying the large time distribution of $u(t)$. By Corollary 2.5, we have that $u(t)$ converges in distribution as $t \to \infty$ to the $L^2[0, L]$-valued random variable $\bar{u}$ defined in the statement of the corollary. By the definitions of Y_0 and Y_1 in (2.3), it is immediate that $\bar{u}$ is almost surely smooth, and using Proposition 2.3, it follows that $\bar{u}(x) \leq \frac{b}{L}x$ almost surely for each $x \in [0, L]$. In this section, we will find the expectation of $\bar{u}$.

PROPOSITION 4.1. *The function $\mathbb{E}\bar{u}$ is affine with slope*

$$\left(1 + \frac{\rho}{\gamma}\tanh(\gamma)\right)^{-1} \frac{b}{L}, \tag{4.3}$$

where $\gamma = L\sqrt{(r_0 + r_1)/D}$ and $\rho = r_0/r_1$.

To prove this proposition, we will use the results from both sections 2 and 3. It is immediate that all of the assumptions in section 3.2 are satisfied, except for one; we need to check that there exists a deterministic M so that $\|u(t)\| \leq M$ almost surely for all $t \geq 0$. We show that and more in the following lemma.

LEMMA 4.2. *Under the assumptions of the current section, we have that*

$$\|u(t)\| \leq L\Big(\max\{\|u_0\|_\infty, b\}\Big)^2,$$

where $\|\cdot\|_\infty$ *denotes the* $L^\infty[0,L]$ *norm. Furthermore,*

$$\|Y_1\|_\infty \leq b \quad \textit{and} \quad \|Y_0\|_\infty \leq b \quad \textit{almost surely.}$$

Proof. First, note that $\|c\|_\infty = \|\frac{b}{L}x\|_\infty = b$. If $f \in L^2[0,L]$, then by the maximum principle, we have that for any $t \geq 0$

$$\|e^{At}f\|_\infty \leq \|f\|_\infty \quad \text{and} \quad \|e^{Bt}(f-c)+c\|_\infty \leq \max\{b, \|f\|_\infty\}. \tag{4.4}$$

Hence, $\max\{\|u(t)\|_\infty, b\}$ is nonincreasing in t, and so the bound on $\|u(t)\|$ is proved.

Since $S := \{f \in L^2[0,L] : \|f\|_\infty \leq b\}$ is a closed set in $L^2[0,L]$, (4.4) and Proposition 2.3 give the desired bounds on $\|Y_1\|_\infty$ and $\|Y_0\|_\infty$. ∎

As in Corollary 2.5, let $\bar{u}$ have the limiting distribution of $u(t)$ as $t \to \infty$. Then by Theorem 3.4, we have that $\mathbb{E}\bar{u} \in L^2[0,L]$ satisfies $\langle \Delta\phi, \mathbb{E}\bar{u}\rangle = 0$ for each $\phi \in C_0^\infty(0,L)$. By the regularity of Δ on $[0,L]$, it follows that $\mathbb{E}\bar{u}$ is not just a weak solution but that it is actually a smooth classical solution, and hence it is the affine function

$$(\mathbb{E}\bar{u})(x) = sx + d$$

for some $s, d \in \mathbb{R}$. By Corollary 2.5 of section 2, we have that

$$sx + d = p\mathbb{E}Y_1 + (1-p)\mathbb{E}Y_0, \tag{4.5}$$

where $p = r_0/(r_0+r_1)$. We will use (4.5) to determine s and d. While both Y_0 and Y_1 are almost surely smooth functions, $\mathbb{E}Y_0$ and $\mathbb{E}Y_1$ are *a priori* only elements of $L^2[0,L]$. It can be shown that $\mathbb{E}Y_0$ and $\mathbb{E}Y_1$ are smooth functions, but we will instead take limits of test functions to avoid evaluating $\mathbb{E}Y_0$ and $\mathbb{E}Y_1$ at specific points in $[0,L]$.

Let $\{\phi_n\}_{n=1}^\infty$ be such that $\phi_n \in C_0^\infty(0,L)$ and $\|\phi_n\|_{L^1} = 1$ for each n and

$$\lim_{n\to\infty} \langle \phi_n, f\rangle = f(0)$$

for each $f \in C[0,L]$. Since the inner product with ϕ_n is a bounded linear functional in $L^2[0,L]$, we can interchange expectation with inner product in (4.5) to obtain

$$d = \lim_{n\to\infty}\left[\langle \phi_n, p\mathbb{E}Y_1 + (1-p)\mathbb{E}Y_0\rangle\right] = \lim_{n\to\infty}\left[p\mathbb{E}\langle\phi_n, Y_1\rangle + (1-p)\mathbb{E}\langle\phi_n, Y_0\rangle\right]. \tag{4.6}$$

We want to exchange the limit with the expectations. To do this, first observe that $Y_1(x)$ and $Y_0(x)$ are each almost surely continuous functions of $x \in [0,L]$ with $Y_0(0) = 0 = Y_1(0)$ almost surely. Thus,

$$\lim_{n\to\infty}\langle\phi_n, Y_0\rangle = 0 \quad \text{and} \quad \lim_{n\to\infty}\langle\phi_n, Y_1\rangle = 0 \quad \text{almost surely.}$$

Using Lemma 4.2 and the assumption that $\|\phi_n\|_{L^1} = 1$ for each n, we have that

$$|\langle \phi_n, Y_0\rangle| \leq b \quad \text{and} \quad |\langle \phi_n, Y_1\rangle| \leq b \quad \text{almost surely.}$$

So we apply the bounded convergence theorem to (4.6) to obtain

$$d = p\mathbb{E} \lim_{n\to\infty} \langle \phi_n, Y_1\rangle + (1-p)\mathbb{E} \lim_{n\to\infty} \langle \phi_n, Y_0\rangle = 0. \tag{4.7}$$

We now find the slope s of $\mathbb{E}\bar{u}$. Denote the orthonormal eigenbasis of A by $\{a_k\}_{k=1}^{\infty}$ and corresponding eigenvalues by $\{-\alpha_k\}_{k=1}^{\infty}$. Since $\sum_{k=1}^{n} \langle a_k, \mathbb{E}Y_1\rangle a_k$ converges to $\mathbb{E}Y_1$ in $L^2[0, L]$ as $n \to \infty$, we have that for any $\phi \in C_0^{\infty}(0, L)$

$$\langle \phi, sx\rangle = \langle \phi, p\mathbb{E}Y_1\rangle + (1-p)\langle \phi, \mathbb{E}Y_0\rangle = p\Big\langle \phi, \sum_{k=1}^{\infty} \langle a_k, \mathbb{E}Y_1\rangle a_k\Big\rangle + (1-p)\langle \phi, \mathbb{E}Y_0\rangle. \tag{4.8}$$

We will need the following lemma, which is an immediate corollary of Proposition 2.2.

LEMMA 4.3. *Under the assumptions of section* 4.1, *we have that for each* $k \in \mathbb{N}$

$$\mathbb{E}[e^{-\alpha_k \tau_1}]\langle a_k, \mathbb{E}Y_0\rangle = \langle a_k, \mathbb{E}Y_1\rangle.$$

Combining this lemma with $sx = p\mathbb{E}Y_1 + (1-p)\mathbb{E}Y_0$ and rearranging terms yields

$$\langle a_k, \mathbb{E}Y_1\rangle = \mathbb{E}[e^{-\alpha_k \tau_1}] \frac{s\langle a_k, x\rangle}{p\mathbb{E}[e^{-\alpha_k \tau_1}] + (1-p)}.$$

Plugging this into (4.8) gives

$$\langle \phi, sx\rangle = p\Big\langle \phi \ , \ \sum_{k=1}^{\infty} \mathbb{E}[e^{-\alpha_k \tau_1}] \frac{s\langle a_k, x\rangle}{p\mathbb{E}[e^{-\alpha_k \tau_1}] + (1-p)} a_k\Big\rangle + (1-p)\langle \phi, \mathbb{E}Y_0\rangle.$$

Solving for s, we find that

$$s = (1-p)\langle \phi, \mathbb{E}Y_0\rangle \Big(\langle \phi, x\rangle - p\Big\langle \phi \ , \ \sum_{k=1}^{\infty} \mathbb{E}[e^{-\alpha_k \tau_1}] \frac{\langle a_k, x\rangle}{p\mathbb{E}[e^{-\alpha_k \tau_1}] + (1-p)} a_k\Big\rangle\Big)^{-1}. \tag{4.9}$$

Let $\{\phi_n\}_{n=1}^{\infty} \in C_0^{\infty}(0, L)$ be such that $\|\phi_n\|_{L^1} = 1$ for each n and $\lim_{n\to\infty}\langle \phi_n, f\rangle = f(L)$ for each $f \in C[0, L]$. We claim that

$$\lim_{n\to\infty} \langle \phi_n, \mathbb{E}Y_0\rangle = b. \tag{4.10}$$

To see this, first note that Y_0 is almost surely smooth and $Y_0(L) = b$ almost surely, so $\lim_{n\to\infty}\langle \phi_n, Y_0\rangle = b$ almost surely. Further, the inner product with ϕ_n is a bounded linear functional so $\langle \phi_n, \mathbb{E}Y_0\rangle = \mathbb{E}\langle \phi_n, Y_0\rangle$. Finally, $\|\phi_n\|_{L^1} = 1$ and $\|Y_0\|_{\infty} \leq b$ almost surely by Lemma 4.2, so the bounded convergence theorem gives (4.10). Now, we want to show that

$$\lim_{n\to\infty} \Big\langle \phi_n \ , \ \sum_{k=1}^{\infty} \frac{\mathbb{E}[e^{-\alpha_k \tau_1}]\langle a_k, x\rangle}{p\mathbb{E}[e^{-\alpha_k \tau_1}] + (1-p)} a_k\Big\rangle = \sum_{k=1}^{\infty} \frac{\mathbb{E}[e^{-\alpha_k \tau_1}]\langle a_k, x\rangle}{p\mathbb{E}[e^{-\alpha_k \tau_1}] + (1-p)} a_k(L). \tag{4.11}$$

To do this, we need to show that $\sum_{k=1}^{\infty} \frac{\mathbb{E}[e^{-\alpha_k \tau_1}]\langle a_k, x\rangle}{p\mathbb{E}[e^{-\alpha_k \tau_1}]+(1-p)} a_k(x)$ converges uniformly in x. Note that for each k

$$a_k(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{(2k-1)\pi x}{2L}\right) \qquad \text{and} \qquad \alpha_k = \frac{D(2k-1)^2\pi^2}{4L^2}.$$

Hence, $\mathbb{E}[e^{-\alpha_k\tau_1}] \leq 1$ and $p\mathbb{E}e^{-\alpha_k\tau_1} + (1-p) \geq 1-p$. Furthermore,

$$\|a_k\|_\infty \leq \sqrt{\frac{2}{L}} \qquad \text{and} \qquad \langle a_k, x\rangle = \frac{4\sqrt{2}L^{3/2}}{\pi^2}\frac{(-1)^{k+1}}{(2k-1)^2}.$$

So for any $N \in \mathbb{N}$

$$\begin{aligned}\left\| \sum_{k=N}^{\infty} \frac{\mathbb{E}[e^{-\alpha_k\tau_1}]\langle a_k, x\rangle}{p\mathbb{E}[e^{-\alpha_k\tau_1}] + (1-p)} a_k(x) \right\|_\infty &\leq \sum_{k=N}^{\infty} \frac{|\langle a_k, x\rangle|}{1-p} \|a_k(x)\|_\infty \\ &= \sum_{k=N}^{\infty} \frac{16L}{(1-p)\pi^2(2k-1)^2} \to 0 \quad \text{as } N \to \infty.\end{aligned}$$

Hence, (4.11) is verified, and thus by (4.9) we have that

$$s = \frac{(1-p)b}{L - p\sum_{k=1}^{\infty}\mathbb{E}[e^{-\alpha_k\tau_1}]\frac{\langle a_k, x\rangle}{p\mathbb{E}e^{-\alpha_k\tau_1}+(1-p)}a_k(L)}.$$

Using the assumptions on τ_0, τ_1, α_k, and a_k, and using a series simplification formula found in Mathematica [45], this becomes

$$s = \left(1 + \frac{\rho}{\gamma}\tanh(\gamma)\right)^{-1}\frac{b}{L},$$

where $\gamma = L\sqrt{(r_0+r_1)/D}$ and $\rho = r_0/r_1$. This expectation is much different than the expectation we obtain when switching between boundary conditions of the same type in the next example below.

4.2. Example 2: Dirichlet/Dirichlet switching. Consider the stochastic process that solves

$$\partial_t u = D\Delta u \quad \text{in } (0, L) \tag{4.12}$$

and at exponentially distributed times switches between the boundary conditions

$$\begin{cases} u(0,t) = 0, \\ u(L,t) = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} u(0,t) = 0, \\ u(L,t) = b > 0. \end{cases}$$

To cast this problem in the setting of previous sections, we set our Hilbert space to be $L^2[0, L]$ and define the operator

$$Bu := \Delta u \quad \text{if } u \in D(B) := H_0^1(0, L) \cap H^2(0, L).$$

We set $c = \frac{b}{L}x \in L^2[0, L]$. Let our switching time distributions, μ_0 and μ_1, be exponential with respective rate parameters r_0 and r_1. Let $u(t, \omega)$ be the H-valued process defined in (2.10) with

$$\Phi_t^1(f) = e^{Bt}f \qquad \text{and} \qquad \Phi_t^0(f) = e^{Bt}(f-c) + c. \tag{4.13}$$

We are interested in studying the large time distribution of $u(t)$. As in Example 1, we can use Corollary 2.5 to obtain that $u(t)$ converges in distribution as $t \to \infty$ to some $L^2[0,L]$-valued random variable $\bar{u}$ defined in the statement of the corollary and use Proposition 2.3 to obtain that $\bar{u}(x) \leq \frac{b}{L}x$ almost surely for each $x \in [0,L]$. And as in Example 1, we can use Theorem 3.4 to find the expectation of $\bar{u}$. However, since this problem switches between boundary conditions of the same type, we will be able to obtain much more information about $\bar{u}$.

Switching between two boundary conditions of the same type is significantly simpler than switching between boundary conditions of different types. This is because the two solution operators that we use when switching between boundary conditions of the same type both employ the same semigroup and thus the same orthonormal eigenbasis. Hence, we only need to consider the projections of the stochastic process in this one basis. In this example, the orthonormal eigenbasis and corresponding eigenvalues for B are for $k \in \mathbb{N}$

$$b_k = \sqrt{\frac{2}{L}} \sin\left(\frac{k\pi}{L}x\right) \qquad \text{and} \qquad -\beta_k = -D(k\pi/L)^2. \tag{4.14}$$

Observe that for each k, the Fourier coefficient $u_k(t) := \langle b_k, u(t) \rangle \in \mathbb{R}$ is the solution to a one-dimensional ODE with a randomly switching right-hand side. Specifically, if J_t is the jump process defined in (2.9), then in between jumps of J_t, the process $u_k(t)$ satisfies

$$\begin{gathered} \frac{d}{dt}u_k = -J_t\beta_k u_k - (1-J_t)\beta_k(u_k - c_k), \\ \text{where} \quad c_k := \langle b_k, c \rangle = \frac{(-1)^{k+1} b\sqrt{2L}}{k\pi}. \end{gathered} \tag{4.15}$$

We can use previous results on one-dimensional ODEs with randomly switching right-hand sides (see [26] or [8]) to determine the marginal distributions of the Fourier coefficients of the stationary $\bar{u}$. For each k, the marginal distributions of the Fourier coefficients of Y_0 and Y_1 are given by

$$\frac{\langle b_k, Y_0 \rangle}{c_k} \sim \text{Beta}\left(\frac{r_1}{\beta_k} + 1, \frac{r_0}{\beta_k}\right) \quad \text{and} \quad \frac{\langle b_k, Y_1 \rangle}{c_k} \sim \text{Beta}\left(\frac{r_1}{\beta_k}, \frac{r_0}{\beta_k} + 1\right). \tag{4.16}$$

Combining this with Corollary 2.5 gives the marginal distributions of the Fourier coefficients of $\bar{u}$.

From (4.16) and Corollary 2.5, we obtain

$$\mathbb{E}\bar{u} = (1-p)\frac{b}{L}x, \tag{4.17}$$

where $p = r_0/(r_0 + r_1)$. Thus, the expectation of the process at large time is merely the solution to the time homogeneous PDE with boundary conditions given by the average of the two boundary conditions that the process switches between.

To further illustrate the usefulness of (4.16), we calculate the L^2-variance of $\bar{u}$. It follows from (4.17) that

$$\mathbb{E}\|\bar{u} - \mathbb{E}\bar{u}\|^2 = \mathbb{E}\|\bar{u}\|^2 - \frac{L}{3}b^2(1-p)^2. \tag{4.18}$$

Now by Corollary 2.5, we have that $\mathbb{E}\|\bar{u}\|^2 = p\mathbb{E}\|Y_1\|^2 + (1-p)\mathbb{E}\|Y_0\|^2$. Combining this with (4.16), we obtain

$$\mathbb{E}\|\bar{u}\|^2 = \sum_{k=1}^{\infty} \frac{r_1(r_1+\beta_k)}{(r_0+r_1)(r_0+r_1+\beta_k)} c_k^2. \tag{4.19}$$

After plugging in our values for β_k, b_k, and c_k in (4.19), using a series simplification formula found in Mathematica [45], and combining with (4.18), we obtain the L^2-variance

$$\mathbb{E}\|\bar{u} - \mathbb{E}\bar{u}\|^2 = \frac{b^2 D r_1 r_0 (\gamma \coth(\gamma) - 1)}{L(r_0+r_1)^3},$$

where $\gamma = L\sqrt{r_0 + r_1/D}$.

While (4.16) is useful, knowing the marginal distributions of the individual Fourier coefficients of Y_0 or Y_1 is of course not enough to find their joint distributions, and the one-dimensional ODE methods used to obtain (4.16) do not give information about these joint distributions. We can, however, use our machinery developed in section 2 to study these joint distributions.

First, we can use Corollary 2.5 and Proposition 2.2 to obtain joint statistics of the components of $\bar{u}$. To illustrate, we will calculate $\mathbb{E}\langle Y_0, b_n\rangle\langle Y_0, b_m\rangle$. Proposition 2.2 gives

$$\mathbb{E}\langle Y_0, b_n\rangle\langle Y_0, b_m\rangle = \mathbb{E}\langle e^{B\tau_0}(e^{B\tau_1}Y_0 - c) + c, b_n\rangle\langle e^{B\tau_0}(e^{B\tau_1}Y_0 - c) + c, b_m\rangle,$$

where τ_0 and τ_1 are independent exponential random variables with rates r_0 and r_1. After recalling some basic facts about exponential random variables and making some algebraic manipulations, we obtain that $\mathbb{E}\langle Y_0, b_n\rangle\langle Y_0, b_m\rangle$ is equal to

$$\frac{(\beta_m+\beta_n+r_1)((\beta_m+\beta_n)(\beta_m+r_1)(\beta_n+r_1) + (2\beta_m\beta_n + (\beta_m+\beta_n)r_1)r_0)}{(\beta_m+\beta_n)(\beta_m+r_1+r_0)(\beta_n+r_1+r_0)(\beta_m+\beta_n+r_1+r_0)} c_m c_n.$$

From this, we can readily compute the covariance of $\langle Y_0, b_n\rangle$ and $\langle Y_0, b_m\rangle$. Other joint statistics of the Fourier coefficients of Y_0 and Y_1 (and hence $\bar{u}$ by Corollary 2.5) are found in analogous ways.

Next, we can use Proposition 2.3 to show that $\bar{u}$ almost surely has a very specific structure.

PROPOSITION 4.4. *Let b_k be as in* (4.14)*, c_k as in* (4.15)*, $\bar{u}$ be as in Corollary* 2.5*, and $\bar{u}_k := \langle b_k, \bar{u}\rangle$. Then for $k < n$ and for almost all $\omega \in \Omega$,*

$$\left(\frac{\bar{u}_k(\omega)}{c_k}\right)^{(n/k)^2} \leq \frac{\bar{u}_n(\omega)}{c_n} \leq 1 - \left(1 - \frac{\bar{u}_k(\omega)}{c_k}\right)^{(n/k)^2}.$$

Proof. For each $k, n \in \mathbb{N}$, let $R_{k,n}$ be the closed planar region enclosed by the following two planar curves:

$$\{P_{k,n}(e^{-Bt}c) : t \geq 0\} \quad \text{and} \quad \{P_{k,n}(c - e^{-Bt}c) : t \geq 0\}.$$

Define $S_{k,n} \subset L^2[0,L]$ by

$$S_{k,n} = \{f \in L^2[0,L] : P_{k,n}(f) \in R_{k,n}\}.$$

It is straightforward to check that $S_{k,n}$ is invariant under Φ_t^0 and Φ_t^1 defined in (4.13) for each $k, n \in \mathbb{N}$. Hence, $\cap_{k,n} S_{k,n}$ is invariant under Φ_t^0 and Φ_t^1, and we have by Proposition 2.3 that Y_0 and Y_1 (and hence $\bar{u}$ by Corollary 2.5) are almost surely contained in $\cap_{k,n} S_{k,n}$.

For $k < n$, observe that $R_{k,n}$ can be written as

$$R_{k,n} = \left\{ (x,y) \in \mathbb{R}^2 : 0 \le \frac{x}{c_k} \le 1 \text{ and } \left(\frac{x}{c_k}\right)^{(n/k)^2} \le \frac{y}{c_n} \le 1 - \left(1 - \frac{x}{c_k}\right)^{(n/k)^2} \right\}.$$

The desired result follows. $\square$

Furthermore, we have the following regularity result on $\bar{u}$. Notice that it implies that as we move to finer and finer spatial scales by taking $k \to \infty$, there is asymptotically only one piece of randomness which determines the fine scale structure.

PROPOSITION 4.5. *Let $r < 1/2$, b_k be as in* (4.14)*, c_k as in* (4.15)*, $Y_0^k := \langle b_k, Y_0 \rangle$, and $Y_1^k := \langle b_k, Y_1 \rangle$. Then for each $\omega \in \Omega$, there exists an $M(\omega)$ so that*

$$1 - \frac{M(\omega)}{k^r} \le \frac{Y_0^k(\omega)}{c_k} \le 1 + \frac{M(\omega)}{k^r} \quad \textit{and} \quad -\frac{M(\omega)}{k^r} \le \frac{Y_1^k(\omega)}{c_k} \le \frac{M(\omega)}{k^r}.$$

Proof. For each k, define

$$A_k := \left\{ \omega \in \Omega : \left| \frac{Y_0^k(\omega)}{c_k} - \mathbb{E}\frac{Y_0^k}{c_k} \right| > \frac{1}{k^r} \right\}.$$

By Chebyshev's inequality and (4.16), we have that

$$\mathbb{P}(A_k) \le \frac{\operatorname{Var}(Y_0^k)}{c_k^2} k^{2r} = \frac{\beta_k r_0 (\beta_k + r_1)}{(\beta_k + r_0 + r_1)^2 (2\beta_k + r_0 + r_1)} k^{2r} \sim k^{2(r-1)} \text{ as } k \to \infty.$$

Thus, if $r < 1/2$, then $\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty$, and we conclude by the Borel–Cantelli lemma that $\mathbb{P}(A_k \text{ infinitely often}) = 0$. Hence, for almost all $\omega \in \Omega$, we can choose an $M(\omega)$ so that for all k,

$$\frac{r_1 + \beta_k}{r_0 + r_1 + \beta_k} - \frac{M(\omega)}{k^r} \le \frac{Y_0^k(\omega)}{c_k} \le \frac{r_1 + \beta_k}{r_0 + r_1 + \beta_k} + \frac{M(\omega)}{k^r}.$$

A similar argument shows that for almost all $\omega \in \Omega$, we can choose an $M(\omega)$ so that for all k,

$$\frac{r_1}{r_0 + r_1 + \beta_k} - \frac{M(\omega)}{k^r} \le \frac{Y_1^k(\omega)}{c_k} \le \frac{r_1}{r_0 + r_1 + \beta_k} + \frac{M(\omega)}{k^r}.$$

Since $\beta_k \sim k^2$ as $k \to \infty$, the desired results follows. $\square$

We can iterate this proposition to obtain the following result, which shows that Y_0^k and Y_1^k depend essentially on only one switching time for large k. Note that we could continue to iterate this proposition to obtain similar bounds. Recall the definition of each $\omega \in \Omega$ in (2.1).

COROLLARY 4.6. *Let $r < 1/2$, b_k be as in* (4.14)*, c_k as in* (4.15)*, $Y_0^k := \langle b_k, Y_0 \rangle$, and $Y_1^k := \langle b_k, Y_1 \rangle$. Then for each $\omega \in \Omega$, there exists an $M_0(\omega)$ depending only on $\{(\tau_0^{k+1}, \tau_1^k)\}_{k \ge 1}$ and an an $M_1(\omega)$ depending only on $\{(\tau_0^k, \tau_1^{k+1})\}_{k \ge 1}$ such that*

$$1 - e^{-\beta_k \tau_0^1}\left(\frac{M_0(\omega)}{k^r} + 1\right) \le \frac{Y_0^k(\omega)}{c_k} \le 1 + e^{-\beta_k \tau_0^1}\left(\frac{M_0(\omega)}{k^r} - 1\right),$$

$$e^{-\beta_k \tau_1^1}\left(1 - \frac{M_1(\omega)}{k^r}\right) \le \frac{Y_1^k(\omega)}{c_k} \le e^{-\beta_k \tau_1^1}\left(1 + \frac{M_1(\omega)}{k^r}\right)$$

Proof. Let ω be given. Define $\sigma : \omega \to \omega$ by

$$\sigma(\omega) = \big((\tau_0^2, \tau_1^1), (\tau_0^3, \tau_1^2), (\tau_0^4, \tau_1^3), \dots\big)\,.$$

Then by the definition of Y_0^k and Y_1^k, we have that

$$\frac{Y_0^k(\omega)}{c_k} = 1 + e^{-\beta_k \tau_0^1}\Big(\frac{Y_1^k(\sigma(\omega))}{c_k} - 1\Big).$$

By Proposition 4.5, there exists an $M(\sigma(\omega))$ so that

$$-\frac{M(\sigma(\omega))}{k^r} \le \frac{Y_1^k(\sigma(\omega))}{c_k} \le \frac{M(\sigma(\omega))}{k^r}.$$

Thus,

$$1 - e^{-\beta_k \tau_0^1}\Big(\frac{M(\sigma(\omega))}{k^r} + 1\Big) \le \frac{Y_0^k(\omega)}{c_k} \le 1 + e^{-\beta_k \tau_0^1}\Big(\frac{M(\sigma(\omega))}{k^r} - 1\Big).$$

The bounds on Y_1^k are proved in a similar way. ∎

4.3. Application to insect physiology. Essentially all insects breathe via a network of tubes that allows oxygen and carbon dioxide to diffuse to and from their cells [44]. Air enters and exits this network through valve-like holes (called spiracles) in the exoskeleton. These spiracles regulate air flow by opening and closing. Surprisingly, spiracles have three distinct phases of activity, each typically lasting for hours. There is a completely closed phase, a completely open phase, and a flutter phase in which the spiracles rapidly open and close [30].

Insect physiologists have proposed at least five major hypotheses to explain the purpose of this behavior [11]. In order to address these competing hypotheses, physiologists would like to understand how much cellular oxygen uptake decreases as a result of the spiracles' closing.

To answer this question, we consider the following model problem. We represent a tube by the interval $[0, L]$ and model the oxygen concentration at a point $x \in [0, L]$ at time t by the function $u(x, t)$. As diffusion is the primary mechanism for oxygen movement in the tubes (see [32]), the function u satisfies the heat equation with some diffusion coefficient D. We impose an absorbing boundary condition at the left endpoint of the interval to represent cellular oxygen absorption where the tube meets the insect tissue. The right endpoint represents the spiracle, and since the spiracle opens and closes, the boundary condition here switches between a no flux boundary condition, $u_x(L, t) = 0$ (spiracle closed), and a Dirichlet boundary condition, $u(L, t) = b > 0$ (spiracle open). We suppose that the spiracle switches from open to closed and from closed to open with exponential rates r_0 and r_1, respectively.

Then, the oxygen concentration $u(x, t)$ is the same process described above in section 4.1. Using the results from that section, if we let $\rho = r_0/r_1$ and $\gamma = L\sqrt{(r_0 + r_1)/D}$, then it follows from Proposition 4.1 that the oxygen flux to the cells at large time is given by

$$\Big(1 + \frac{\rho}{\gamma}\tanh(\gamma)\Big)^{-1}\frac{bD}{L}.$$

This formula is noteworthy because it shows that the cellular oxygen uptake not only depends on the average proportion of time the spiracle is open, but it also depends on the overall rate of opening and closing. In particular, note that if we keep the ratio ρ fixed but let γ become large, then the oxygen uptake approaches $\frac{bD}{L}$. The biological meaning is that the insect can have its spiracles open an arbitrarily small proportion of time and yet receive essentially just as much oxygen as if its spiracles were always open if they open and close with a sufficiently high frequency. This is important biologically, because it is almost certainly the correct explanation for fluttering.

Acknowledgment. JCM would like to thank Yuri Bakhtin for stimulating discussions.

REFERENCES

[1] Y. Bakhtin, *Burgers equation with random boundary conditions*, Proc. Amer. Math. Soc., 135 (2007), p. 2257-2262.
[2] Y. Bakhtin and T. Hurth, *Invariant densities for dynamical systems with random switching*, Nonlinearity, 25 (2012).
[3] M. Balde, U. Boscain, and P. Mason, *A note on stability conditions for planar switched systems*, Internat. J. Control, 82 (2009), pp. 1882–1888.
[4] I. Belykh, V. Belykh, R. Jeter, and M. Hasler, *Multistable randomly switching oscillators: the odds of meeting a ghost*, Eur. Phys. J. Spec. Top., 222 (2013), pp. 2497–2507.
[5] M. Benaim, S. Leborgne, F. Malrieu, and P.A. Zitt, *Qualitative properties of certain piecewise deterministic Markov processes*, preprint.
[6] M. Benaim, S. Leborgne, F. Malrieu, and P. A. Zitt, *Quantitative ergodicity for some switched dynamical systems*, Electron. Commun. Probab., 17 (2012), pp. 1–14.
[7] P. Billingsley, *Convergence of Probability Measures*, 2nd ed., Wiley, New York, 1999.
[8] O. Boxma, H. Kaspi, O. Kella, and D. Perry, *On/off storage systems with state-dependent input, output, and switching rates*, Probab. Engrg. Inform. Sci., 19 (2005), pp. 1–14.
[9] P. Bressloff and J. Newby, *Metastability in a stochastic neural network modeled as a velocity jump Markov process*, SIAM J. Appl. Dyn. Syst., 12 (2013), pp. 1394–1435.
[10] E. Buckwar and M. Riedler, *An exact stochastic hybrid model of excitable membranes including spatio-temporal evolution*, J. Math. Biol., 63 (2011), pp. 1051–1093.
[11] S. Chown, A. Gibbs, S. Hetz, C. Klok, John R. Lighton, and E. Marais, *Discontinuous gas exchange in insects: A clarification of hypotheses and approaches*, Physiol. Biochem. Zool., 79 (2006), pp. 333–343.
[12] B. Cloez and M. Hairer, *Exponential ergodicity for Markov processes with random switching*, Bernoulli, 21 (2015), pp. 505–536.
[13] H. Crauel, *Random point attractors versus random set attractors*, J. London Math. Soc. (2), 63 (2001), pp. 413–427.
[14] H. Crauel and F. Flandoli, *Attractors for random dynamical systems*, Probab. Theory Related Fields, 100 (1994), pp. 365–393.
[15] G. Da Prato and J. Zabczyk, *Evolution equations with white-noise boundary conditions*, Stoch. Stoch. Rep., 42 (1993), pp. 431–459.
[16] P. Diaconis and D. Freedman, *Iterated random functions*, SIAM Rev., 41 (1999), p. 45–76.
[17] J. L. Doob, *Stochastic Processes*, Wiley Classics Library, Wiley, New York, 1990.
[18] J. Duan, S. Luo, and C. Wang, *Hyperbolic equations with random boundary conditions*, in Recent Development in Stochastic Dynamics and Stochastic Analysis, Interdiscip. Inform. Sci. 8, World Scientific, River Edge, NJ, 2010, pp. 1–21.
[19] M. Duflo, *Random Iterative Models*, Appl. Math.(New York) 34, Springer, Berlin, 1997 (in English).
[20] R. Durrett, *Probability: Theory and Examples*, 4th ed., Cambridge University Press, Cambridge, 2010.
[21] J. Farkas, P. Hinow, and J. Engelstadter, *Pathogen evolution in switching environments: A hybrid dynamical system approach*, Math. Biosci., 240 (2012), pp. 70–75.
[22] R. Feldman, K. Meyer, and L. Quenzer, *Principles of Neuropharmacology*, Sinauer, Sunderland, MA, 1997.
[23] K. Fuxe, A. B. Dahlstrom, G. Jonsson, D. Marcellino, M. Guescini, M. Dam, P. Manger, and L. Agnati, *The discovery of central monoamine neurons gave volume transmission to the wired brain*, Prog. Neurobiol., 90 (2010), pp. 82–100.
[24] M. Hasler, V. Belykh, and I. Belykh, *Dynamics of stochastically blinking systems. Part* I: *Finite time properties*, SIAM J. Appl. Dyn. Syst., 12 (2013), pp. 1007–1030.

[25] M. Hasler, V. Belykh, and I. Belykh, *Dynamics of stochastically blinking systems. Part* II*: Asymptotic properties*, SIAM J. Appl. Dyn. Syst., 12 (2013), pp. 1031–1084.
[26] T. Hurth, *Limit Theorems for a One-Dimensional System with Random Switchings*, Master's thesis, Georgia Institute of Technology, Atlanta, 2010.
[27] Y. Kifer, *Ergodic Theory of Random Transformations*, Progr. Probab. Stat. 10, Birkhäuser Boston, Boston, 1986.
[28] Y. Kifer, *Random Perturbations of Dynamical Systems*, Progr. Probab. Stat. 16, Birkhäuser Boston, Boston, 1988.
[29] S. D. Lawley, J. C. Mattingly, and M. C. Reed, *Sensitivity to switching rates in stochastically switched ODEs*, Commun. Math. Sci., 12 (2014), pp. 1343–1352.
[30] J. Lighton, *Discontinuous gas exchange in insects*, Annu. Rev. Ent., 41 (1996), pp. 309–324.
[31] H. Lin and P. J. Antsaklis, *Stability and stabilizability of switched linear systems: A survey of recent results*, IEEE Trans. Automat. Contr., 54 (2009), pp. 308–322.
[32] C. Loudon, *Tracheal hypertrophy in mealworms: Design and plasticity in oxygen supply systems*, J. Exp. Bio., 147 (1989), pp. 217–235.
[33] J. C. Mattingly, *Ergodicity of* 2*D Navier–Stokes equations with random forcing and large viscosity*, Comm. Math. Phys., 206 (1999), pp. 273–288.
[34] J. C. Mattingly, *Contractivity and ergodicity of the random map* $x \mapsto |x - \theta|$, Teor. Veroyatnost. i Primenen., 47 (2002), pp. 388–397.
[35] J. C. Mattingly and T. M. Suidan, *The small scales of the stochastic Navier–Stokes equations under rough forcing*, J. Stat. Phys., 118 (2005), pp. 343–364.
[36] J. C. Mattingly and T. M. Suidan, *Transition measures for the stochastic Burgers equation*, in Integrable Systems and Random Matrices, Contemp. Math. 458, AMS, Providence, RI, 2008, pp. 409–418.
[37] J. Newby and J. Keener, *An asymptotic analysis of the spatially inhomogeneous velocity-jump process*, Multiscale Model. Simul., 9 (2011), pp. 735–765.
[38] M. Reed, H.F. Nijhout, and J. Best, *Projecting biochemistry over long distances*, Math. Mode. Nat. Phenom., 9 (2014), pp. 130–138.
[39] S. M. Ross, *Stochastic Processes*, 2nd ed., Wiley, New York, 1996.
[40] B. Schmalfuss, *A random fixed point theorem based on Lyapunov exponents*, Random Comput. Dynam., 4 (1996), pp. 257–268.
[41] R. Schnaubelt and M. Veraar, *Stochastic Equations with Boundary Noise*, in Parabolic Problems, J. Escher, P. Guidotti, M. Hieber, P. Mucha, J. Pruss, Y. Shibata, G. Simonett, C. Walker, and W. Zajaczkowski, eds., Progr. Nonlinear Differential Equations Appl. 80, Springer, Berlin, 2011, pp. 609–629.
[42] W. Wang and J. Duan, *Reductions and deviations for stochastic partial differential equations under fast dynamical boundary conditions*, Stoch. Anal. Appl., 27 (2009), 431459.
[43] E. Weinan, K. Khanin, A. Mazel, and Y. Sinai, *Invariant measures for Burgers equation with stochastic forcing*, Ann. of Math. (2), 151 (2000), pp. 877–960.
[44] V. B. Wigglesworth, *The respiration of insects*, Bio. Rev., 6 (1931), 181220.
[45] IWolfram Research, *Mathematica*, 2012.
[46] G. Yin and C. Zhu, *Hybrid Switching Diffusions*, Springer, Berlin, 2010.
[47] C. Zhu and G. Yin, *On competitive Lotka Volterra model in random environments*, J. Math. Anal. Appl., 357 (2009), pp. 154–170.