5.2 The Pigeonhole Principle

$\boxed{\text{Thm 1 (Pigeonhole Principle) If } k \in \mathbb{Z}^+ \text{ and } k+1 \text{ or more objects are placed into } k \text{ boxes, then there is at least one box containing two or more of the objects.}}$

Corollary A fn f from a set w/ $k+1$ or more elements to a set w/ k elements is not one-to-one.

Ex 1 Show that if there are 28 people in a class, at least two of them have last names that start w/ same letter.

Thm 2 (Generalized Pigeonhole Principle) If N objects are placed into k boxes, then there is at least one box containing at least $\lceil \frac{N}{k} \rceil$ objects.

Pf (by Contradiction) Assume no box contains more than $\lceil \frac{N}{k} \rceil - 1$ objects.

5.2 (cont)

Ex2 How many cards must be selected from a deck of 52 cards to guarantee that at least 4 cards of same suit are chosen?

Ex3 What is the least number of area codes needed to guarantee 300,000,000 people in U.S. have distinct 10-digit phone numbers? (Assume an area code cannot start with 0.)

5.2 (cont)

Ex 4 How many numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these numbers add up to 16?

Ex 5 Let $n \in \mathbb{Z}^+$. Show that in any set of n consecutive integers $\exists$ exactly one divisible by n.

Thm 3 Every sequence of n^2+1 distinct $\mathbb{R}$ numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

5.2 (cont)

Ex 6 $3, 6, 4, -4, -1, 1, 7, 5, 0, 2$ is a sequence w/ 10 terms. $10 = 3^2+1 \Rightarrow n=3$ (from Thm 3)

Find all subsequences of length 4 that are strictly increasing or decreasing.

Ex 7 Let $a_1, a_2, \ldots, a_n$ be positive integers. Show that if $a_1+a_2+\cdots+a_n - n+1$ objects are placed into n boxes, then for some i $(i=1,2,\ldots,n)$, the i^{th} box contains at least a_i objects.

7.5 Inclusion-Exclusion

We know $|A \cup B| = |A| + |B| - |AB|$

<u>EX1</u> How many whole numbers not exceeding 2500 are divisible by 9 or 10?

Principle of Inclusion-Exclusion

<u>Thm</u> Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i A_j|$$

$$+ \sum_{1 \le i < j < k \le n} |A_i A_j A_k| - \cdots + (-1)^{n+1} |A_1 A_2 \cdots A_n|$$

<u>EX2</u> How many elements are in the union of 4 sets if each of the sets has 100 elements, each pair of the sets shares 50 elements, each three of the sets share 25 elements & 7 5 elements in all 4 sets?

7.5 (cont)

Ex 3 Students in the school of mathematics at a university major in one or more of the following four areas: applied math (A), pure math (P), operations research (R), and computer science (C). How many students are in this school if there are 23 students in A, 17 students in P, 44 in R, 63 in C, 5 in A and P, 8 in A and C, 4 in A and R, 6 in P and C, 5 in P and R, 14 in R and C, 2 in P, R & C, 2 in A, R & C, 1 in P & A & R, 1 in P & A & C and 1 in all four fields?

Ex 4 Find # of elements in $A_1 \cup A_2 \cup A_3$ if there are 300 elements in A_1, 500 elements in A_2 and 600 in A_3. if (a) sets are pairwise disjoint (b) $A_1 \subseteq A_2 \subseteq A_3$.

5.3 Permutations & Combinations

Thm 1 If $n \in \mathbb{Z}^+$, $r \in \mathbb{Z}^+ \ni 1 \le r \le n$, then $\exists$ $P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$ r-permutations of a set w/ n distinct elements

Corollary If $n, r \in \mathbb{W}$ $0 \le r \le n$, then $P(n,r) = \dfrac{n!}{(n-r)!}$

notation: it's sometimes denoted ${}_nP_r$.

Thm 2 The # of r-combinations of a set w/ n elements, where $n \in \mathbb{W}$ & $r \in \mathbb{Z}$, $0 \le r \le n$, is
$$C(n,r) = \frac{n!}{r!\,(n-r)!}$$

Corollary Let $n, r \in \mathbb{W}$ $r \le n$, then $C(n,r) = C(n, n-r)$

Ex 1 Find the number of 5-permutations of a set w/ 9 elements.

Ex 2 There are six different candidates for governor. In how many different orders can the names of the candidates be printed on a ballot?

5.3 (cont)

Ex 3 A coin is flipped 10 times. How many possible outcomes

(a) are there in total?

(b) contain exactly 2 Heads?

(c) contain at <u>most</u> 3 tails?

(d) contain same # of heads & tails?

5.3 (cont)

Ex 4 A group contains n men & n women. How many ways are there to arrange these people in a row if the men & women alternate?

Ex 5 How many ways are there for 10 women & 6 men to stand in a line so that no 2 men stand next to each other?

Ex 6 A club has 25 members.
(a) How many ways are there to choose 4 members to serve on a committee?

(b) How many ways are there to select 4 members to serve as President, V.P., Treasurer & Secretary?

Ex 7 Suppose a department has 10 men & 15 women. How many 6-member committees can be formed if each committee must have the same # of men & women?