6.2 (cont)

Defn 3 E, F are events $\ni p(F) > 0$. The conditional probability of E given F, $p(E|F)$, is defined as

$$p(E|F) = \frac{p(E \cap F)}{p(F)}.$$

Defn 4 Events E & F are independent iff

$$p(E \cap F) = p(E)p(F)$$

Ex 4 What is conditional probability that exactly 4 heads appear when a fair coin is flipped 7 times, given that the first flip was tails?

Ex 5 What is the probability that at least 4 Heads appear, out of 7 tosses, given that the first flip was a tail?

6.2 (cont)

Bernoulli Trials and the Binomial Distribution

Bernoulli trial $\Rightarrow$ has only 2 possible outcomes
(call two outcomes s = success, f = failure)

We say $p = P(s)$ and $q = P(f)$, and we know $p + q = 1$

$\Rightarrow q = 1 - p$

$$\boxed{\begin{array}{l}\text{\underline{Thm 2} The probability of \underline{exactly} } k \text{ successes in } n \\ \text{independent Bernoulli trials, with probability of success} \\ p \text{ and probability of failure } q = 1-p \text{, is} \\ \qquad \binom{n}{k} p^k q^{n-k}\end{array}}$$

__Ex 6__ Find the probability that a family w/ 5 children has __exactly__ two boys, if the sexes of the children are independent and the probability of a boy is 0.6.

__Ex 7__ Find probability that this family has __at least__ two boys.

6.2 (cont)

Ex 8 (The Birthday Problem) What is the minimum # of people who need to be in a room together so that the probability that at least two of them have the same birthday is greater than $\frac{1}{2}$?

Soln ① Assume bdays are independent for different people. ② Assume each bday is equally likely. ③ Assume 366 days of year (to include leap year bdays).

Pretend people walk in room one at a time.

n = # people in room	P_n = probability that n people all have different bdays (given first n-1 people have dif. bdays)
1	$\frac{366}{366} = 1$
2	$\frac{365}{366}$
3	$\frac{364}{366}$
4	$\frac{366-3}{366} = \frac{363}{366}$
⋮	⋮
j	$\frac{366-j+1}{366} = \frac{367-j}{366}$
⋮	⋮
n	$\frac{367-n}{366}$

Remember bdays of people in room are indep.

$\Rightarrow$ Prob. all have different bdays

$= \frac{366}{366} \cdot \frac{365}{366} \cdot \frac{364}{366} \cdots \frac{367-n}{366}$

$= p$

$\Rightarrow$ Probability that at least two people share a bday = 1 − probability no one shares a bday

$= 1 - \frac{365}{366} \cdot \frac{364}{366} \cdots \frac{367-n}{366} = 1 - p$

6.2 (cont)

After computation, we find for $n=23$,

$$1-p = 0.506$$

$\Rightarrow$ we need at least 23 people in the room to have a probability bigger than $\frac{1}{2}$ that at least 2 people share a bday.

Ex 9 Show that if E, F are events, then

$$p(E \cap F) \geq p(E) + p(F) - 1.$$

6.3 Bayes' Theorem

<u>Thm 1</u> (Bayes' Thm) E & F are events from a sample space $S \ni p(E) \neq 0$ & $p(F) \neq 0$. Then

$$p(F|E) = \frac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\bar{F})p(\bar{F})}$$

<u>pf</u> By defn, we know

$$p(F|E) = \frac{p(EF)}{p(E)} \quad \text{and} \quad p(E|F) = \frac{p(EF)}{p(F)}$$

$$\Leftrightarrow \quad p(EF) = p(F|E)\,p(E) = p(E|F)p(F)$$

(1) $$\Rightarrow \quad p(F|E) = \frac{p(E|F)p(F)}{p(E)}$$

and notice $E = (E \cap F) \cup (E \cap \bar{F})$ (since $F + \bar{F} = S$)

$$\Rightarrow p(E) = p(E \cap F) + p(E \cap \bar{F}) - \underbrace{p(EF \cap E\bar{F})}_{0}$$

but $p(E \cap F) = p(EF) = p(E|F)p(F)$

and likewise $p(E \cap \bar{F}) = p(E\bar{F}) = p(E|\bar{F})p(\bar{F})$

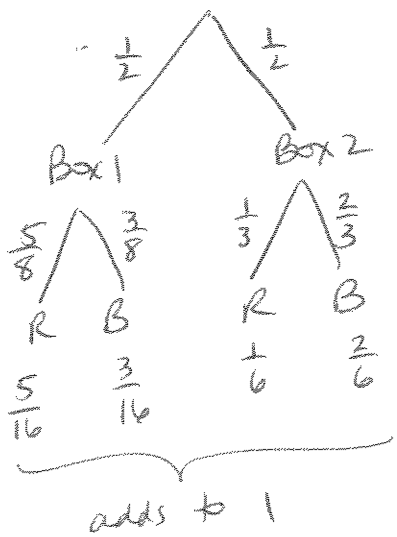
$$\Rightarrow (1) \text{ becomes } \quad p(F|E) = \frac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\bar{F})p(\bar{F})} \quad //$$

<u>Ex 1</u> We have 2 boxes. Box 1 has 3 blue balls and 2 red balls. Box 2 has 4 blue balls and 5 red balls. Karim selects a ball by first choosing the box and then choosing a ball (both at random). If Karim chooses a blue ball, what is probability he selected it from Box 2? i.e. P(Box 2 | Blue)

6.3 (cont)

Ex 1 (cont)

Probability Tree diagram



$\Downarrow$

$p(B|②) = \frac{2}{3}$

$p(B|①) = \frac{3}{8}$

$$P(\text{Box } 2 \mid \text{Blue}) = P(② \mid B)$$

$$= \frac{p(B|②)\,p(②)}{p(B|②)\,p(②) + p(B|①)\,p(①)}$$

Ex 2 Suppose that 8% of all bicycle racers use steroids, that a bicyclist who uses steroids tests positive for steroids 96% of the time, and that a bicyclist who does not use steroids tests positive for steroids 9% of the time. What is the probability that a randomly selected bicyclist who tests positive for steroids actually uses steroids?

6.3 (cont)

Ex 3 Suppose $E, F_1, F_2, \& F_3$ are events from a sample space S and that $F_1, F_2 \& F_3$ are mutually disjoint and $F_1 \cup F_2 \cup F_3 = S$. Find $p(F_2|E)$ if

$$p(E|F_1) = \frac{2}{7}, \quad p(E|F_2) = \frac{3}{8}, \quad p(E|F_3) = \frac{1}{2}$$

$$p(F_1) = \frac{1}{6}, \quad p(F_2) = \frac{1}{2}, \quad p(F_3) = \frac{1}{3}.$$

(Note: we need Generalized Bayes' Thm for this.

$$p(F_j|E) = \frac{p(E|F_j)p(F_j)}{\sum_{i=1}^{n} p(E|F_i)p(F_i)}$$

- $F_1, F_2, \ldots, F_n$ mutually exclusive events
- $\bigcup_{i=1}^{n} F_i = S$
- $p(E) \neq 0, p(F_i) \neq 0$ $\forall i = 1, \ldots, n$
