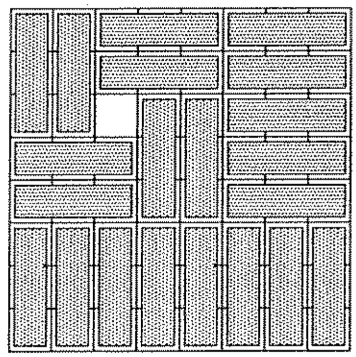
**46.** If we study Figure 7, we see that by rotating or reflecting the board, we can make any square we wish nonwhite, with the exception of the squares with coordinates $(3,3)$, $(3,6)$, $(6,3)$, and $(6,6)$. Therefore the same argument as was used in Example 22 shows that we cannot tile the board using straight triominoes if any one of those other 60 squares is removed. The following drawing (rotated as necessary) shows that we can tile the board using straight triominoes if one of those four squares is removed.



**48.** We will use a coloring of the $10 \times 10$ board with four colors as the basis for a proof by contradiction showing that no such tiling exists. Assume that 25 straight tetrominoes can cover the board. Some will be placed horizontally and some vertically. Because there is an odd number of tiles, the number placed horizontally and the number placed vertically cannot both be odd, so assume without loss of generality that an even number of tiles are placed horizontally. Color the squares in order using the colors red, blue, green, yellow in that order repeatedly, starting in the upper left corner and proceeding row by row, from left to right in each row. Then it is clear that every horizontally placed tile covers one square of each color and each vertically placed tile covers either zero or two squares of each color. It follows that in this tiling an even number of squares of each color are covered. But this contradicts the fact that there are 25 squares of each color. Therefore no such coloring exists.

## SUPPLEMENTARY EXERCISES FOR CHAPTER 1

**2.** The truth table is as follows.

| $p$ | $q$ | $r$ | $p \vee q$ | $p \wedge \neg r$ | $(p \vee q) \to (p \wedge \neg r)$ |
|---|---|---|---|---|---|
| T | T | T | T | F | F |
| T | T | F | T | T | T |
| T | F | T | T | F | F |
| T | F | F | T | T | T |
| F | T | T | T | F | F |
| F | T | F | T | F | F |
| F | F | T | F | F | T |
| F | F | F | F | F | T |

**4.** **a)** The converse is "If I drive to work today, then it will rain." The contrapositive is "If I do not drive to work today, then it will not rain." The inverse is "If it does not rain today, then I will not drive to work."
**b)** The converse is "If $x \geq 0$ then $|x| = x$." The contrapositive is "If $x < 0$ then $|x| \neq x$." The inverse is "If $|x| \neq x$, then $x < 0$."
**c)** The converse is "If $n^2$ is greater than 9, then $n$ is greater than 3." The contrapositive is "If $n^2$ is not greater than 9, then $n$ is not greater than 3." The inverse is "If $n$ is not greater than 3, then $n^2$ is not greater than 9."

**6.** The inverse of $p \to q$ is $\neg p \to \neg q$. Therefore the inverse of the inverse is $\neg\neg p \to \neg\neg q$, which is equivalent to $p \to q$ (the original proposition). The converse of $p \to q$ is $q \to p$. Therefore the inverse of the converse is $\neg q \to \neg p$, which is the contrapositive of the original proposition. The inverse of the contrapositive is $q \to p$, which is the same as the converse of the original statement.

**8.** Let $t$ be "Sergei takes the job offer"; let $b$ be "Sergei gets a signing bonus"; and let $h$ be "Sergei will receive a higher salary." The given statements are $t \to b$, $t \to h$, $b \to \neg h$, and $t$. By modus ponens we can conclude $b$ and $h$ from the first two conditional statements, and therefore we can conclude $\neg h$ from the third conditional statement. We now have the contradiction $h \wedge \neg h$, so these statements are inconsistent.

**10.** Since both knights and knaves claim that they are knights (the former truthfully and the latter deceivingly), we know that $A$ is a knave. But since $A$'s statement must be false, and the first part of the conjunction is true, the second part must be false, so we know that $B$ must be a knave as well. If $C$ were a knight, then $B$'s statement would be true, and knaves must lie, so $C$ must also be a knave. Thus all three are knaves.

**12.** If $S$ is a proposition, then it is either true or false. If $S$ is false, then the statement "If $S$ is true, then unicorns live" is vacuously true; but this statement *is* $S$, so we would have a contradiction. Therefore $S$ is true, so the statement "If $S$ is true, then unicorns live" is true and has a true hypothesis. Hence it has a true conclusion (modus ponens), and so unicorns live. But we know that unicorns do not live. It follows that $S$ cannot be a proposition.

**14.** **a)** The answer is $\exists x P(x)$ if we do not read any significance into the use of the plural, and $\exists x \exists y (P(x) \wedge P(y) \wedge x \neq y)$ if we do.
**b)** $\neg \forall x P(x)$, or, equivalently, $\exists x \neg P(x)$ **c)** $\forall y Q(y)$
**d)** $\forall x P(x)$ (the class has nothing to do with it) **e)** $\exists y \neg Q(y)$

**16.** The given statement tells us that there are exactly two elements in the domain. Therefore the statement will be true as long as we choose the domain to be anything with size 2, such as the United States presidents named Bush.

**18.** We want to say that for every $y$, there do not exist four different people each of whom is the grandmother of $y$. Thus we have $\forall x \neg \exists a \exists b \exists c \exists d (a \neq b \wedge a \neq c \wedge a \neq d \wedge b \neq c \wedge b \neq d \wedge c \neq d \wedge G(a,y) \wedge G(b,y) \wedge G(c,y) \wedge G(d,y))$.

**20.** **a)** Since there is no real number whose square is $-1$, it is true that there exist exactly 0 values of $x$ such that $x^2 = -1$.
**b)** This is true, because 0 is the one and only value of $x$ such that $|x| = 0$.
**c)** This is true, because $\sqrt{2}$ and $-\sqrt{2}$ are the only values of $x$ such that $x^2 = 2$.
**d)** This is false, because there are more than three values of $x$ such that $x = |x|$, namely all positive real numbers.

**22.** Let us assume the hypothesis. This means that there is some $x_0$ such that $P(x_0, y)$ holds for all $y$. Then it is certainly true that for all $y$ there exists an $x$ such that $P(x,y)$ is true, since in each case we can take $x = x_0$. Note that the converse is not always a tautology, since the $x$ in $\forall y \exists x P(x,y)$ can depend on $y$.

**24.** No. Here is an example. Let $P(x,y)$ be $x > y$, where we are talking about integers. Then for every $y$ there does exist an $x$ such that $x > y$; we could take $x = y + 1$, for example. However, there does not exist an $x$ such that for *every* $y$, $x > y$; in other words, there is no superlarge integer (if for no other reason than that no integer can be larger than itself).

**26.** a) It will snow today, but I will not go skiing tomorrow.
b) Some person in this class does not understand mathematical induction.
c) All students in this class like discrete mathematics.
d) There is some mathematics class in which all the students stay awake during lectures.

**28.** Let $W(r)$ means that room $r$ is painted white. Let $I(r,b)$ mean that room $r$ is in building $b$. Let $L(b,u)$ mean that building $b$ is on the campus of United States university $u$. Then the statement is that there is some university $u$ and some building on the campus of $u$ such that every room in $b$ is painted white. In symbols this is $\exists u \exists b(L(b,u) \wedge \forall r(I(r,b) \rightarrow W(r)))$.

**30.** To say that there are exactly two elements that make the statement true is to say that two elements exist that make the statement true, and that every element that makes the statement true is one of these two elements. More compactly, we can phrase the last part by saying that an element makes the statement true if and only if it is one of these two elements. In symbols this is $\exists x \exists y(x \neq y \wedge \forall z(P(z) \leftrightarrow (z = x \vee z = y)))$. In English we might express the rule as follows. The hypotheses are that $P(x)$ and $P(y)$ are both true, that $x \neq y$, and that every $z$ that satisfies $P(z)$ must be either $x$ or $y$. The conclusion is that there are exactly two elements that make $P$ true.

**32.** We give a proof by contraposition. If $x$ is rational, then $x = p/q$ for some integers $p$ and $q$ with $q \neq 0$. Then $x^3 = p^3/q^3$, and we have expressed $x^3$ as the quotient of two integers, the second of which is not zero. This by definition means that $x^3$ is rational, and that completes the proof of the contrapositive of the original statement.

**34.** Let $m$ be the square root of $n$, rounded down if it is not a whole number. (In the notation to be introduced in Section 2.3, we are letting $m = \lfloor \sqrt{n} \rfloor$.) We can see that this is the unique solution in a couple of ways. First, clearly the different choices of $m$ correspond to a partition of $\mathbf{N}$, namely into $\{0\}$, $\{1,2,3\}$, $\{4,5,6,7,8\}$, $\{9,10,11,12,13,14,15\}$, .... So every $n$ is in exactly one of these sets. Alternatively, take the square root of the given inequalities to give $m \leq \sqrt{n} < m+1$. That $m$ is then the floor of $\sqrt{n}$ (and that $m$ is unique) follows from statement (1a) of Table 1 in Section 2.3.

**36.** A constructive proof seems indicated. We can look for examples by hand or with a computer program. The smallest ones to be found are $50 = 5^2 + 5^2 = 1^2 + 7^2$ and $65 = 4^2 + 7^2 = 1^2 + 8^2$.

**38.** We claim that the number 7 is not the sum of at most two squares and a cube. The first two positive squares are 1 and 4, and the first positive cube is 1, and these are the only numbers that could be used in forming the sum. Clearly no sum of three or fewer of these is 7. This counterexample disproves the statement.

**40.** We give a proof by contradiction. If $\sqrt{2} + \sqrt{3}$ were rational, then so would be its square, which is $5 + 2\sqrt{6}$. Subtracting 5 and dividing by 2 then shows that $\sqrt{6}$ is rational, but this contradicts the theorem we are told to assume.

**42.** Exercise 77 in Section 2.3 gave a one-to-one correspondence between $\mathbf{Z}^+ \times \mathbf{Z}^+$ and $\mathbf{Z}^+$. Since $\mathbf{Z}^+$ is countable, so is $\mathbf{Z}^+ \times \mathbf{Z}^+$.

**44.** There are at most two real solutions of each quadratic equation, so the number of solutions is countable as long as the number of triples $(a, b, c)$, with $a$, $b$, and $c$ integers, is countable. But this follows from Exercise 41 in the following way. There are a countable number of pairs $(b, c)$, since for each $b$ (and there are countably many $b$'s) there are only a countable number of pairs with that $b$ as its first coordinate. Now for each $a$ (and there are countably many $a$'s) there are only a countable number of triples with that $a$ as its first coordinate (since we just showed that there are only a countable number of pairs $(b, c)$). Thus again by Exercise 41 there are only countably many triples.

**46.** We know from Example 21 that the set of real numbers between 0 and 1 is uncountable. Let us associate to each real number in this range (including 0 but excluding 1) a function from the set of positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ as follows: If $x$ is a real number whose decimal representation is $0.d_1d_2d_3\ldots$ (with ambiguity resolved by forbidding the decimal to end with an infinite string of 9's), then we associate to $x$ the function whose rule is given by $f(n) = d_n$. Clearly this is a one-to-one function from the set of real numbers between 0 and 1 and a subset of the set of all functions from the set of positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Two different real numbers must have different decimal representations, so the corresponding functions are different. (A few functions are left out, because of forbidding representations such as $0.239999\ldots$.) Since the set of real numbers between 0 and 1 is uncountable, the subset of functions we have associated with them must be uncountable. But the set of all such functions has at least this cardinality, so it, too, must be uncountable (by Exercise 37).

**48.** We follow the hint. Suppose that $f$ is a function from $S$ to $P(S)$. We must show that $f$ is not onto. Let $T = \{\, s \in S \mid s \notin f(s) \,\}$. We will show that $T$ is not in the range of $f$. If it were, then we would have $f(t) = T$ for some $t \in S$. Now suppose that $t \in T$. Then because $t \in f(t)$, it follows from the definition of $T$ that $t \notin T$; this is a contradiction. On the other hand, suppose that $t \notin T$. Then because $t \notin f(t)$, it follows from the definition of $T$ that $t \in T$; this is again a contradiction. This completes our proof by contradiction that $f$ is not onto.

## SUPPLEMENTARY EXERCISES FOR CHAPTER 2

**2.** We are given that $A \subseteq B$. We want to prove that the power set of $A$ is a subset of the power set of $B$, which means that if $C \subseteq A$ then $C \subseteq B$. But this follows directly from Exercise 15 in Section 2.1.

**4.** a) $\mathbf{Z}$ b) $\emptyset$ c) $\mathbf{O}$ d) $\mathbf{E}$

**6.** If $A \subseteq B$, then every element in $A$ is also in $B$, so clearly $A \cap B = A$. Conversely, if $A \cap B = A$, then every element of $A$ must also be in $A \cap B$, and hence in $B$. Therefore $A \subseteq B$.

**8.** This identity is true, so we must show that every element in the left-hand side is also an element in the right-hand side and conversely. Let $x \in (A - B) - C$. Then $x \in A - B$ but $x \notin C$. This means that $x \in A$, but $x \notin B$ and $x \notin C$. Therefore $x \in A - C$, and therefore $x \in (A - C) - B$. The converse is proved in exactly the same way.

**10.** The inequality follows from the obvious fact that $A \cap B \subseteq A \cup B$. Equality can hold only if there are no elements in either $A$ or $B$ that are not in both $A$ and $B$, and this can happen only if $A = B$.

**24.** We need to divide successively by 233, 144, 89, 55, 34, 21, 13, 8, 5, 3, 2, and 1, a total of 12 divisions.

**26.** **a)** The first statement is clear. For the second, if $a$ and $b$ are both even, then certainly 2 is a factor of their greatest common divisor, and the complementary factor must be the greatest common divisor of the numbers obtained by dividing out this 2. For the third statement, if $a$ is even and $b$ is odd, then the factor of 2 in $a$ will not appear in the greatest common divisor, so we can ignore it. Finally, the last statement follows from Lemma 1 in Section 3.5, taking $q = 1$ (despite the notation, nothing in Lemma 1 required $q$ to be the quotient).

**b)** All the steps involved in implementing part **(a)** as an algorithm require only comparisons, subtractions, and divisions of even numbers by 2. Since division by 2 is a shift of one bit to the right, only the operations mentioned here are used. (Note that the algorithm needs two more reductions: if $a$ is odd and $b$ is even, then $\gcd(a,b) = \gcd(a, b/2)$, and if $a < b$, then interchange $a$ and $b$.)

**c)** We show the operation of the algorithm as a string of equalities; each equation is one step.

$$\begin{aligned}\gcd(1202, 4848) &= \gcd(4848, 1202) = 2\gcd(2424, 601) = 2\gcd(1212, 601) = 2\gcd(606, 601)\\ &= 2\gcd(303, 601) = 2\gcd(601, 303) = 2\gcd(298, 303) = 2\gcd(303, 298)\\ &= 2\gcd(303, 149) = 2\gcd(154, 149) = 2\gcd(77, 149) = 2\gcd(149, 77)\\ &= 2\gcd(72, 77) = 2\gcd(77, 72) = 2\gcd(77, 36) = 2\gcd(77, 18)\\ &= 2\gcd(77, 9) = 2\gcd(68, 9) = 2\gcd(34, 9) = 2\gcd(17, 9)\\ &= 2\gcd(8, 9) = 2\gcd(9, 8) = 2\gcd(9, 4) = 2\gcd(9, 2)\\ &= 2\gcd(9, 1) = 2\gcd(8, 1) = 2\gcd(4, 1) = 2\gcd(2, 1)\\ &= 2\gcd(1, 1) = 2\end{aligned}$$

**28.** We can give a nice proof by contraposition here, by showing that if $n$ is not prime, then the sum of its divisors is not $n+1$. There are two cases. If $n = 1$, then the sum of the divisors is $1 \neq 1 + 1$. Otherwise $n$ is composite, so can be written as $n = ab$, where both $a$ and $b$ are divisors of $n$ different from 1 and from $n$ (although it might happen that $a = b$). Then $n$ has at least the three distinct divisors 1, $a$, and $n$, and their sum is clearly not equal to $n + 1$. This completes the proof by contraposition. One should also observe that the converse of this statement is also true: if $n$ is prime, then the sum of its divisors is $n + 1$ (since its only divisors are 1 and itself).

**30.** **a)** Each week consists of seven days. Therefore to find how many (whole) weeks there are in $n$ days, we need to see how many 7's there are in $n$. That is exactly what $n$ **div** 7 tells us.

**b)** Each day consists of 24 hours. Therefore to find how many (whole) days there are in $n$ hours, we need to see how many 24's there are in $n$. That is exactly what $n$ **div** 24 tells us.

**32.** We need to arrange that every pair of the four numbers has a factor in common. There are six such pairs, so let us use the first six prime numbers as the common factors. Call the numbers $a$, $b$, $c$, and $d$. We will give $a$ and $b$ a common factor of 2; $a$ and $c$ a common factor of 3; $a$ and $d$ a common factor of 5; $b$ and $c$ a common factor of 7; $b$ and $d$ a common factor of 11; and $c$ and $d$ a common factor of 13. The simplest way to accomplish this is to let $a = 2 \cdot 3 \cdot 5 = 30$; $b = 2 \cdot 7 \cdot 11 = 154$; $c = 3 \cdot 7 \cdot 13 = 273$; and $d = 5 \cdot 11 \cdot 13 = 715$. The numbers are mutually relatively prime, since no number is a factor of all of them (indeed, each prime is a factor of only two of them). Many other examples are possible, of course.

**34.** If $x \equiv 3 \pmod 9$, then $x = 3 + 9t$ for some integer $t$. In particular this equation tells us that $3 \mid x$. On the other hand the first congruence says that $x = 2 + 6s = 2 + 3 \cdot (2s)$ for some integer $s$, which implies that the remainder when $x$ is divided by 3 is 2. Obviously these two conclusions are inconsistent, so there is no simultaneous solution to the two congruences.

## SUPPLEMENTARY EXERCISES FOR CHAPTER 4

**2.** The proposition is true for $n = 1$, since $1^3 + 3^3 = 28 = 1(1+1)^2(2 \cdot 1^2 + 4 \cdot 1 + 1)$. Assume the inductive hypothesis. Then

$$\begin{aligned} 1^3 + 3^3 + \cdots + (2n+1)^3 + (2n+3)^3 &= (n+1)^2(2n^2+4n+1) + (2n+3)^3 \\ &= 2n^4 + 8n^3 + 11n^2 + 6n + 1 + 8n^3 + 36n^2 + 54n + 27 \\ &= 2n^4 + 16n^3 + 47n^2 + 60n + 28 \\ &= (n+2)^2(2n^2 + 8n + 7) \\ &= (n+2)^2(2(n+1)^2 + 4(n+1) + 1). \end{aligned}$$

**4.** Our proof is by induction, it being trivial for $n = 1$, since $1/3 = 1/3$. Under the inductive hypothesis

$$\begin{aligned} \frac{1}{1 \cdot 3} + \cdots + \frac{1}{(2n-1)(2n+1)} + \frac{1}{(2n+1)(2n+3)} &= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} \\ &= \frac{1}{2n+1}\left(n + \frac{1}{2n+3}\right) \\ &= \frac{1}{2n+1}\left(\frac{2n^2+3n+1}{2n+3}\right) \\ &= \frac{1}{2n+1}\left(\frac{(2n+1)(n+1)}{2n+3}\right) = \frac{n+1}{2n+3}, \end{aligned}$$

as desired.

**6.** We prove this statement by induction. The base case is $n = 5$, and indeed $5^2 + 5 = 30 < 32 = 2^5$. Assuming the inductive hypothesis, we have $(n+1)^2 + (n+1) = n^2 + 3n + 2 < n^2 + 4n < n^2 + n^2 = 2n^2 < 2(n^2+n)$, which is less than $2 \cdot 2^n$ by the inductive hypothesis, and this equals $2^{n+1}$, as desired.

**8.** We can let $N = 16$. We prove that $n^4 < 2^n$ for all $n > N$. The base case is $n = 17$, when $17^4 = 83521 < 131072 = 2^{17}$. Assuming the inductive hypothesis, we have $(n+1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1 < n^4 + 4n^3 + 6n^3 + 4n^3 + 2n^3 = n^4 + 16n^3 < n^4 + n^4 = 2n^4$, which is less than $2 \cdot 2^n$ by the inductive hypothesis, and this equals $2^{n+1}$, as desired.

**10.** If $n = 0$ (base case), then the expression equals $0 + 1 + 8 = 9$, which is divisible by $9$. Assume that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by $9$. We must show that $(n+1)^3 + (n+2)^3 + (n+3)^3$ is also divisible by $9$. The difference of these two expressions is $(n+3)^3 - n^3 = 9n^2 + 27n + 27 = 9(n^2 + 3n + 3)$, a multiple of $9$. Therefore since the first expression is divisible by $9$, so is the second.

**12.** The two parts are nearly identical, so we do only part **(a)**. Part **(b)** is proved in the same way, substituting multiplication for addition throughout. The basis step is the tautology that if $a_1 \equiv b_1 \pmod{m}$, then $a_1 \equiv b_1 \pmod{m}$. Assume the inductive hypothesis. This tells us that $\sum_{j=1}^{n} a_j \equiv \sum_{j=1}^{n} b_j \pmod{m}$. Combining this fact with the fact that $a_{n+1} \equiv b_{n+1} \pmod{m}$, we obtain the desired congruence, $\sum_{j=1}^{n+1} a_j \equiv \sum_{j=1}^{n+1} b_j \pmod{m}$ from Theorem 5 in Section 3.4.

**14.** After some computation we conjecture that $n + 6 < (n^2 - 8n)/16$ for all $n \geq 28$. (We find that it is not true for smaller values of $n$.) For the basis step we have $28 + 6 = 34$ and $(28^2 - 8 \cdot 28)/16 = 35$, so the statement is true. Assume that the statement is true for $n = k$. Then since $k > 27$ we have

$$\begin{aligned} \frac{(k+1)^2 - 8(k+1)}{16} = \frac{k^2 - 8k}{16} + \frac{2k-7}{16} &> k + 6 + \frac{2k-7}{16} \quad \text{by the inductive hypothesis} \\ &> k + 6 + \frac{2 \cdot 27 - 7}{16} > k + 6 + 2.9 > (k+1) + 6. \end{aligned}$$

**8.** Since the new permutation agrees with the old one in positions 1 to $j-1$, and since the new permutation has $a_k$ in position $j$, whereas the old one had $a_j$, with $a_k > a_j$, the new permutation succeeds the old one in lexicographic order. Furthermore the new permutation is the first one (in lexicographic order) with $a_1$, $a_2$, ..., $a_{j-1}$, $a_k$ in positions 1 to $j$, and the old permutation was the last one with $a_1$, $a_2$, ..., $a_{j-1}$, $a_j$ in those positions. Since $a_k$ was picked to be the smallest number greater than $a_j$ among $a_{j+1}$, $a_{j+2}$, ..., $a_n$, there can be no permutation between these two.

**10.** One algorithm would combine Algorithm 3 and Algorithm 1. Using Algorithm 3, we generate all the $r$-combinations of the set with $n$ elements. At each stage, after we have found each $r$-combination, we use Algorithm 1, with $n = r$ (and a different collection to be permuted than $\{1, 2, \ldots, n\}$), to generate all the permutations of the elements in this combination. See the solution to Exercise 11 for an example.

**12. a)** We find that $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $a_4 = 2$, and $a_5 = 3$. Therefore the number is $1 \cdot 1! + 1 \cdot 2! + 2 \cdot 3! + 2 \cdot 4! + 3 \cdot 5! = 1 + 2 + 12 + 48 + 360 = 423$.
**b)** Each $a_k = 0$, so the number is 0.
**c)** We find that $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$, and $a_5 = 5$. Therefore the number is $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4! + 5 \cdot 5! = 1 + 4 + 18 + 96 + 600 = 719 = 6! - 1$, as expected, since this is the last permutation.

**14. a)** We find the Cantor expansion of 3 to be $1 \cdot 1! + 1 \cdot 2!$. Therefore we know that $a_4 = 0$, $a_3 = 0$, $a_2 = 1$, and $a_1 = 1$. Following the algorithm given in the solution to Exercise 13, we put 5 in position $5 - 0 = 5$, put 4 in position $4 - 0 = 4$, put 3 in position $3 - 1 = 2$, and put 2 in the position that is 1 from the rightmost available position, namely position 1. Therefore the answer is 23145.
**b)** We find that $89 = 1 \cdot 1! + 2 \cdot 2! + 2 \cdot 3! + 3 \cdot 4!$. Therefore we insert 5, 4, 3, and 2, in order, skipping 3, 2, 2, and 1 positions from the right among the available positions, obtaining 35421.
**c)** We find that $111 = 1 \cdot 1! + 1 \cdot 2! + 2 \cdot 3! + 4 \cdot 4!$. Therefore we insert 5, 4, 3, and 2, in order, skipping 4, 2, 1, and 1 positions from the right among the available positions, obtaining 52431.

## SUPPLEMENTARY EXERCISES FOR CHAPTER 5

**2. a)** There are no ways to do this, since there are not enough items. **b)** $6^{10} = 60{,}466{,}176$
**c)** There are no ways to do this, since there are not enough items.
**d)** $C(6 + 10 - 1, 10) = C(15, 10) = C(15, 5) = 3003$

**4.** There are $2^7$ bit strings of length 10 that start 000, since each of the last 7 bits can be chosen in either of two ways. Similarly, there are $2^6$ bit strings of length 10 that end 1111, and there are $2^3$ bit strings of length 10 that both start 000 and end 1111 (since only the 3 middle bits can be freely chosen). Therefore by the inclusion–exclusion principle, the answer is $2^7 + 2^6 - 2^3 = 184$.

**6.** $9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 90{,}000$

**8. a)** All the integers from 100 to 999 have three decimal digits, and there are $999 - 100 + 1 = 900$ of these.
**b)** In addition to the 900 three-digit numbers, there are 9 one-digit positive integers, for a total of 909.
**c)** There is 1 one-digit number with a 9. Among the two-digit numbers, there are the 10 numbers from 90 to 99, together with the 8 numbers 19, 29, ..., 89, for a total of 18. Among the three-digit numbers, there are the 100 from 900 to 999; and there are, for each century from the 100's to the 800's, again $1 + 18 = 19$ numbers with at least one 9; this gives a total of $100 + 8 \cdot 19 = 252$. Thus our final answer is $1 + 18 + 252 = 271$. Alternately, we can compute this as $10^3 - 9^3 = 271$, since we want to subtract from the number of three-digit nonnegative numbers (with leading 0's allowed) the number of those that use only the nine digits 0 through 8.

d) Since we can use only even digits, there are $5^3 = 125$ ways to specify a three-digit number, allowing leading 0's. Since, however, the number $0 = 000$ is not in our set, we need to subtract 1, obtaining the answer 124.

e) The numbers in question are either of the form $d55$ or $55d$, with $d \neq 5$, or 555. Since $d$ can be any of nine digits, there are $9 + 9 + 1 = 19$ such numbers.

f) All 9 one-digit numbers are palindromes. The 9 two-digit numbers 11, 22, ..., 99 are palindromes. For three-digit numbers, the first digit (which must equal the third digit) can be any of the 9 nonzero digits, and the second digit can be any of the 10 digits, giving $9 \cdot 10 = 90$ possibilities. Therefore the answer is $9 + 9 + 90 = 108$.

**10.** Using the generalized pigeonhole principle, we see that we need $5 \times 12 + 1 = 61$ people.

**12.** There are $7 \times 12 = 84$ day-month combinations. Therefore we need 85 people to ensure that two of them were born on the same day of the week and in the same month.

**14.** We need at least 551 cards to ensure that at least two are identical. Since the cards come in packages of 20, we need $\lceil 551/20 \rceil = 28$ packages.

**16.** Partition the set of numbers from 1 to $2n$ into the $n$ pigeonholes $\{1,2\}$, $\{3,4\}$, ..., $\{2n-1, 2n\}$. If we have $n+1$ numbers from this set (the pigeons), then two of them must be in the same hole. This means that among our collection are two consecutive numbers. Clearly consecutive numbers are relatively prime (since every common divisor must divide their difference, 1).

**18.** Divide the interior of the square, with lines joining the midpoints of opposite sides, into four $1 \times 1$ squares. By the pigeonhole principle, at least two of the five points must be in the same small square. The furthest apart two points in a square could be is the length of the diagonal, which is $\sqrt{2}$ for a square 1 unit on a side.

**20.** If the worm never gets sent to the same computer twice, then it will infect 100 computers on the first round of forwarding, $100^2 = 10{,}000$ other computers on the second round of forwarding, and so on. Therefore the maximum number of different computers this one computer can infect is $100 + 100^2 + 100^3 + 100^4 + 100^5 = 10{,}101{,}010{,}100$. This figure of ten billion is probably comparable to the total number of computers in the world.

**22.** a) We want to solve $n(n-1) = 110$, or $n^2 - n - 110 = 0$. Simple algebra gives $n = 11$ (we ignore $n = -10$, since we need a positive integer for our answer).

b) We recall that $7! = 5040$, so the answer is 7.

c) We need to solve the equation $n(n-1)(n-2)(n-3) = 12n(n-1)$. Since we have $n \geq 4$ in order for $P(n,4)$ to be defined, this equation reduces to $(n-2)(n-3) = 12$, or $n^2 - 5n - 6 = 0$. Simple algebra gives $n = 6$ (we ignore the solution $n = -1$ since $n$ needs to be a positive integer).

**24.** An algebraic proof is straightforward. We will give a combinatorial proof of the equivalent identity $P(n+1,r)(n+1-r) = (n+1)P(n,r)$ (and in fact both of these equal $P(n+1,r+1)$). Consider the problem of writing down a permutation of $r+1$ objects from a collection of $n+1$ objects. We can first write down a permutation of $r$ of these objects ($P(n+1,r)$ ways to do so), and then write down one more object (and there are $n+1-r$ objects left to choose from), thereby obtaining the left-hand side; or we can first choose an object to write down first ($n+1$ to choose from), and then write down a permutation of length $r$ using the $n$ remaining objects ($P(n,r)$ ways to do so), thereby obtaining the right-hand side.

**26.** First note that Corollary 2 of Section 5.4 is equivalent to the assertion that the sum of the numbers $C(n,k)$ for even $k$ is equal to the sum of the numbers $C(n,k)$ for odd $k$. Since $C(n,k)$ counts the number of subsets of size $k$ of a set with $n$ elements, we need to show that a set has as many even-sized subsets as it has odd-sized subsets. Define a function $f$ from the set of all subsets of $A$ to itself (where $A$ is a set with $n$ elements, one of which is $a$), by setting $f(B) = B \cup \{a\}$ if $a \notin B$, and $f(B) = B - \{a\}$ if $a \in B$. It is clear that $f$ takes even-sized subsets to odd-sized subsets and vice versa, and that $f$ is one-to-one and onto (indeed, $f^{-1} = f$). Therefore $f$ restricted to the set of subsets of odd size gives a one-to-one correspondence between that set and the set of subsets of even size.

**28.** The base case is $n = 2$, in which case the identity simply states that $1 = 1$. Assume the inductive hypothesis, that $\sum_{j=2}^{n} C(j,2) = C(n+1,3)$. Then

$$\sum_{j=2}^{n+1} C(j,2) = \left(\sum_{j=2}^{n} C(j,2)\right) + C(n+1,2)$$
$$= C(n+1,3) + C(n+1,2) = C((n+1)+1,3),$$

as desired. The last equality made use of Pascal's identity.

**30.** a) For a fixed $k$, a triple is totally determined by picking $i$ and $j$; since each can be picked in $k$ ways (each can be any number from 0 to $k-1$, inclusive), there are $k^2$ ways to choose the triple. Adding over all possible values of $k$ gives the indicated sum.

b) A triple of this sort is totally determined by knowing the *set* of numbers $\{i,j,k\}$, since the order is fixed. Therefore the number of triples of each kind is just the number of sets of 3 elements chosen from the set $\{0,1,2,\ldots,n\}$, and that is clearly $C(n+1,3)$.

c) In order for $i$ to equal $j$ (with both less than $k$), we need to pick two elements from $\{0,1,2,\ldots,n\}$, using the larger one for $k$ and the smaller one for both $i$ and $j$. Therefore there are as many such choices as there are 2-element subsets of this set, namely $C(n+1,2)$.

d) This part is its own proof. The last equality follows from elementary algebra.

**32.** a) If we 2-color the $2d-1$ elements of $S$, then there must be at least $d$ elements of one color (if there were $d-1$ or fewer elements of both colors, then only $2d-2$ elements would be colored); this is just an application of the generalized pigeonhole principle. Thus there is a $d$-element subset that does not contain both colors, in violation of the condition for being 2-colorable.

b) We must show that every collection of fewer than three sets each containing two elements is 2-colorable, and that there is a collection of three sets each containing two elements that is not 2-colorable. The second statement follows from part (a), with $d = 2$ (the three sets are $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$). On the other hand, if we have two (or fewer) sets each with two elements, then we can color the two elements of the first set with different colors, and we cannot be prevented from properly coloring the second set, since it must contain an element not in the first set.

c) First we show that the given collection is not 2-colorable. Without loss of generality, assume that 1 is red. If 2 is red, then 6 must be blue (second set). Thus either 4 or 5 must be red (seventh set), which means that 3 must be blue (first or fourth set). This would force 7 to be red (sixth set), which would force both 4 and 5 to be blue (third and fifth sets), a contradiction. Thus 2 is blue. If 3 is red, then we can conclude that 5 is blue, 7 is red, 6 is blue, and 4 is blue, making the last set improperly colored. Thus 3 is blue. This implies that 4 is red, hence 7 is blue, hence 5 and 6 are red, another contradiction. So the given collection cannot be 2-colored. Next we must show that all collections of six sets with three elements each are 2-colorable. Since having more elements in $S$ at our disposable only makes it easier to 2-color the collection, we can assume that $S$ has only five elements; let $S = \{a,b,c,d,e\}$. Since there are 18 occurrences of elements in the collection, some element, say $a$, must occur at least four times (since $3 \cdot 5 < 18$). If $a$ occurs in six of the sets, then we can color $a$ red and the rest of the elements blue. If $a$ occurs in five of the sets, suppose without loss of generality that $b$ and $c$ occur in the sixth set. Then we can color $a$ and $b$ red and the remaining elements blue. Finally, if $a$ occurs in only four of the sets, then that leaves only four elements for the last two sets, and therefore a pair of elements must be shared by them, say $b$ and $c$. Again coloring $a$ and $b$ red and the remaining elements blue gives the desired coloring.

**34.** We might as well assume that the first person sits in the northernmost seat. Then there are $P(7,7)$ ways to seat the remaining people, since they form a permutation reading clockwise from the first person. Therefore the answer is $7! = 5040$.

**36.** We need to know the number of solutions to $d + m + g = 12$, where $d$, $m$, and $g$ are integers greater than or equal to 3. This is equivalent to the number of nonnegative integer solutions to $d' + m' + g' = 3$, where $d' = d-3$, $m' = m-3$, and $g' = g-3$. By Theorem 2 of Section 5.5, the answer is $C(3+3-1,3) = C(5,3) = 10$.

**38.** a) By Theorem 3 of Section 5.5, the answer is $10!/(3!2!2!) = 151{,}200$.
b) If we fix the start and the end, then the question concerns only 8 letters, and the answer is $8!/(2!2!) = 10{,}080$.
c) If we think of the three $P$'s as one letter, then the answer is seen to be $8!/(2!2!) = 10{,}080$.

**40.** There are 26 choices for the third letter. If the digit part of the plate consists of the digits 1, 2, and $d$, where $d$ is different from 1 or 2, then there are 8 choices for $d$ and $3! = 6$ choices for a permutation of these digits. If $d = 1$ or 2, then there are 2 choices for $d$ and 3 choices for a permutation. Therefore the answer is $26(8 \cdot 6 + 2 \cdot 3) = 1404$.

**42.** Let us look at the girls first. There are $P(8,8) = 8! = 40320$ ways to order them relative to each other. This much work produces 9 gaps between girls (including the ends), in each of which at most one boy may sit. We need to choose, in order without repetition, 6 of these gaps, and this can be done in $P(9,6) = 60480$ ways. Therefore the answer is, by the product rule, $40320 \cdot 60480 = 2{,}438{,}553{,}600$.

**44.** We are given no restrictions, so any number of the boxes can be occupied once we have distributed the objects.
a) This is a straightforward application of the product rule; there are $6^5 = 7776$ ways to do this, because there are 6 choices for each of the 5 objects.
b) This is similar to Exercise 50 in Section 5.5. We compute this using the formulae:

$$S(5,1) = \frac{1}{1!}\left(\binom{1}{0}1^5\right) = \frac{1}{1!}(1) = 1$$

$$S(5,2) = \frac{1}{2!}\left(\binom{2}{0}2^5 - \binom{2}{1}1^5\right) = \frac{1}{2!}(32-2) = 15$$

$$S(5,3) = \frac{1}{3!}\left(\binom{3}{0}3^5 - \binom{3}{1}2^5 + \binom{3}{2}1^5\right) = \frac{1}{3!}(243-96+3) = 25$$

$$S(5,4) = \frac{1}{4!}\left(\binom{4}{0}4^5 - \binom{4}{1}3^5 + \binom{4}{2}2^5 - \binom{4}{3}1^5\right) = \frac{1}{4!}(1024-972+192-4) = 10$$

$$S(5,5) = \frac{1}{5!}\left(\binom{5}{0}5^5 - \binom{5}{1}4^5 + \binom{5}{2}3^5 - \binom{5}{3}2^5 + \binom{5}{4}1^5\right) = \frac{1}{5!}(3125-5120+2430-320+5) = 1$$

$$\sum_{j=1}^{5} S(5,j) = 1 + 15 + 25 + 10 + 1 = 52$$

c) This is asking for the number of solutions to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 5$ in nonnegative integers. By Theorem 2 (see also Example 5) in Section 5.5, the answer is $C(6+5-1,5) = C(10,5) = 252$.

**42.** Let $X = X_1 + X_2 + \cdots + X_m$, where $X_i = 1$ if the $i^{\text{th}}$ ball falls into the first bin and $X_i = 0$ otherwise. Then $X$ is the number of balls that fall into the first bin, so we are being asked to compute $E(X)$. Clearly $E(X_i) = p(X_i = 1) = 1/n$. By linearity of expectation (Theorem 3), the expected number of balls that fall into the first bin is therefore $m/n$.

## SUPPLEMENTARY EXERCISES FOR CHAPTER 6

**2.** There are $C(52,13)$ possible hands. A hand with no pairs must contain exactly one card of each kind. The only choice involved, therefore, is the suit for each of the 13 cards. There are 4 ways to specify the suit, and there are 13 tasks to be performed. Therefore there are $4^{13}$ hands with no pairs. The probability of drawing such a hand is thus $4^{13}/C(52,13) = 67108864/635013559600 = 4194304/39688347475 \approx 0.000106$.

**4.** The denominator of each probability is the number of 7-card poker hands, namely $C(52,7) = 133784560$.

**a)** The number of such hands is $13 \cdot 12 \cdot 4$, since there are 13 ways to choose the kind for the four, then 12 ways to choose another kind for the three, then $C(4,3) = 4$ ways to choose which three cards of that second kind to use. Therefore the probability is $624/133784560 \approx 4.7 \times 10^{-6}$.

**b)** The number of such hands is $13 \cdot 4 \cdot 66 \cdot 6^2$, since there are 13 ways to choose the kind for the three, $C(4,3) = 4$ ways to choose which three cards of that kind to use, then $C(12,2) = 66$ ways to choose two more kinds for the pairs, then $C(4,2) = 6$ ways to choose which two cards of each of those kinds to use. Therefore the probability is $123552/133784560 \approx 9.2 \times 10^{-4}$.

**c)** The number of such hands is $286 \cdot 6^3 \cdot 10 \cdot 4$, since there are $C(13,3) = 286$ ways to choose the kinds for the pairs, $C(4,2) = 6$ ways to choose which two cards of each of those kinds to use, 10 ways to choose the kind for the singleton, and 4 ways to choose which card of that kind to use. Therefore the probability is $2471040/133784560 \approx 0.018$.

**d)** The number of such hands is $78 \cdot 6^2 \cdot 165 \cdot 4^3$, since there are $C(13,2) = 78$ ways to choose the kinds for the pairs, $C(4,2) = 6$ ways to choose which two cards of each of those kinds to use, $C(11,3) = 165$ ways to choose the kinds for the singletons, and 4 ways to choose which card of each of those kinds to use. Therefore the probability is $29652480/133784560 \approx 0.22$.

**e)** The number of such hands is $1716 \cdot 4^7$, since there are $C(13,7) = 1716$ ways to choose the kinds and 4 ways to choose which card of each of kind to use. Therefore the probability is $28114944/133784560 \approx 0.21$.

**f)** The number of such hands is $4 \cdot 1716$, since there are 4 ways to choose the suit for the flush and $C(13,7) = 1716$ ways to choose the kinds in that suit. Therefore the probability is $6864/133784560 \approx 5.1 \times 10^{-5}$.

**g)** The number of such hands is $8 \cdot 4^7$, since there are 8 ways to choose the kind for the straight to start at ($A$, 2, 3, 4, 5, 6, 7, or 8) and 4 ways to choose the suit for each kind. Therefore the probability is $131072/133784560 \approx 9.8 \times 10^{-4}$.

**h)** There are only $4 \cdot 8$ straight flushes, since the only choice is the suit and the starting kind (see part (g)). Therefore the probability is $32/133784560 \approx 2.4 \times 10^{-7}$.

**6. a)** Each of the outcomes 1 through 12 occurs with probability $1/12$, so the expectation is $(1/12)(1+2+3+\cdots+12) = 13/2$.

**b)** We compute $V(X) = E(X^2) - E(X)^2 = (1/12)(1^2+2^2+3^2+\cdots+12^2) - (13/2)^2 = (325/6) - (169/4) = 143/12$.

**8. a)** Since expected value is linear, the expected value of the sum is the sum of the expected values, each of which is $13/2$ by Exercise 6a. Therefore the answer is 13.

**b)** Since variance is linear for independent random variables, and clearly these variables are independent, the variance of the sum is the sum of the variances, each of which is $143/12$ by Exercise 6b. Therefore the answer is $143/6$.

10. a) Since expected value is linear, the expected value of the sum is the sum of the expected values, which are $9/2$ by Exercise 5a and $13/2$ by Exercise 6a. Therefore the answer is $(9/2) + (13/2) = 11$.
b) Since variance is linear for independent random variables, and clearly these variables are independent, the variance of the sum is the sum of the variances, which are $21/4$ by Exercise 5b and $143/12$ by Exercise 6b. Therefore the answer is $(21/4) + (143/12) = 103/6$.

12. We need to determine how many positive integers less than $n = pq$ are divisible by either $p$ or $q$. Certainly the numbers $p$, $2p$, $3p$, ..., $(q-1)p$ are all divisible by $p$. This gives $q-1$ numbers. Similarly, $p-1$ numbers are divisible by $q$. None of these numbers is divisible by both $p$ and $q$ since $lcm(p,q) = pq/\gcd(p,q) = pq/1 = pq = n$. Therefore $p+q-2$ numbers in this range are divisible by $p$ or $q$, so the remaining $pq - 1 - (p+q-2) = pq - p - q + 1 = (p-1)(q-1)$ are not. Therefore the probability that a randomly chosen integer in this range is not divisible by either $p$ or $q$ is $(p-1)(q-1)/(pq-1)$.

14. Technically a proof by mathematical induction is required, but we will give a somewhat less formal version. We just apply the definition of conditional probability to the right-hand side and observe that practically everything cancels (each denominator with the numerator of the previous term):

$$\begin{aligned} & p(E_1)p(E_2|E_1)p(E_3|E_1 \cap E_2) \cdots p(E_n|E_1 \cap E_2 \cap \cdots \cap E_{n-1}) \\ &= p(E_1) \cdot \frac{p(E_1 \cap E_2)}{p(E_1)} \cdot \frac{p(E_1 \cap E_2 \cap E_3)}{p(E_1 \cap E_2)} \cdots \frac{p(E_1 \cap E_2 \cap \cdots \cap E_n)}{p(E_1 \cap E_2 \cap \cdots \cap E_{n-1})} \\ &= p(E_1 \cap E_2 \cap \cdots \cap E_n) \end{aligned}$$

16. If $n$ is odd, then it is impossible, so the probability is 0. If $n$ is even, then there are $C(n, n/2)$ ways that an equal number of heads and tails can appear (choose the flips that will be heads), and $2^n$ outcomes in all, so the probability is $C(n, n/2)/2^n$.

18. There are $2^{11}$ bit strings. There are $2^6$ palindromic bit strings, since once the first six bits are specified arbitrarily, the remaining five bits are forced. If a bit string is picked at random, then, the probability that it is a palindrome is $2^6/2^{11} = 1/32$.

20. a) Since there are $b$ bins, each equally likely to receive the ball, the answer is $1/b$.
b) By linearity of expectation, the fact that $n$ balls are tossed, and the answer to part (a), the answer is $n/b$.
c) In order for this part to make sense, we ignore $n$, and assume that the ball supply is unlimited and we keep tossing until the bin contains a ball. The number of tosses then has a geometric distribution with $p = 1/b$ from part (a). The expectation is therefore $b$.
d) Again we have to assume that the ball supply is unlimited and we keep tossing until every bin contains at least one ball. The analysis is identical to that of Exercise 29 in this set, with $b$ here playing the role of $n$ there. By the solution given there, the answer is $b\sum_{j=1}^{b} 1/j$.

22. a) The intersection of two sets is a subset of each of them, so the largest $p(A \cap B)$ could be would occur when the smaller is a subset of the larger. In this case, that would mean that we want $B \subseteq A$, in which case $A \cap B = B$, so $p(A \cap B) = p(B) = 1/2$. To construct an example, we find a common denominator of the fractions involved, namely 6, and let the sample space consist of 6 equally likely outcomes, say numbered 1 through 6. We let $B = \{1, 2, 3\}$ and $A = \{1, 2, 3, 4\}$. The smallest intersection would occur when $A \cup B$ is as large as possible, since $p(A \cup B) = p(A) + p(B) - p(A \cap B)$. The largest $A \cup B$ could ever be is the entire sample space, whose probability is 1, and that certainly can occur here. So we have $1 = (2/3) + (1/2) - p(A \cap B)$, which gives $p(A \cap B) = 1/6$. To construct an example, again we find a common denominator of these fractions,