3.5 #14 (a) $6 = 1+2+3$ sum of its factors

$28 = 1+2+4+7+14$

(b) If 2^p-1 is prime, show $2^{p-1}(2^p-1)$ is a perfect #.

<u>pf</u> $n = 2^{p-1}(2^p-1)$ since (2^p-1) is prime, the factors of n are

$1, 2, 2^2, 2^3, \ldots, 2^{p-1}, (2^p-1), 2(2^p-1), 2^2(2^p-1), \ldots, 2^{p-2}(2^p-1)$

$\Rightarrow$ sum of those factors

$= 1+2+2^2+2^3+\cdots+2^{p-1}+(2^p-1)(1+2+2^2+\cdots+2^{p-2})$

$= \underbrace{\sum_{j=0}^{p-1} 2^j}_{\text{geometric series}} + (2^p-1)\sum_{j=0}^{p-2} 2^j = \frac{1(1-2^p)}{1-2} + (2^p-1)\left(\frac{1(1-2^{p-1})}{1-2}\right)$

$= (2^p-1) + (2^p-1)(2^{p-1}-1)$

$= (2^p-1)(1+2^{p-1}-1)$

$= 2^{p-1}(2^p-1)$ //

3.7 #4 Show 937 is inverse of 13 mod 2436

i.e. $937(13) \equiv 1 \pmod{2436}$

$\Leftrightarrow 937(13) - 1 = 2436m$ for some $m \in \mathbb{Z}^+$

$12180 = 2436m$

$m = 5$ //

3.7 #19

$x \equiv 1 \pmod 2$
$x \equiv 2 \pmod 3$
$x \equiv 3 \pmod 5$
$x \equiv 4 \pmod{11}$

2, 3, 5, 11 are pairwise relatively prime

$m_1 = 2 \quad m_2 = 3 \quad m_3 = 5 \quad m_4 = 11$

$m = 2(3)(5)(11) = 330$

$M_1 = \frac{330}{2} = 165 \qquad M_2 = 110$

$M_3 = 66 \qquad M_4 = 30$

$a_1 = 1,\ a_2 = 2,\ a_3 = 3,\ a_4 = 4$

$x = 165y_1 + 220y_2 + 198y_3 + 120y_4$

$y_1 M_1 = 1 \pmod{m_1} \iff 165y_1 \equiv 1 \pmod 2$

$165 = 82(2) + 1 \iff 1 = 165(1) - 82(2) \Rightarrow y_1 = 1$

$165 \bmod 2 = 1 \bmod 2$

$220y_2 \equiv 1 \pmod 3 \qquad 220 \equiv 1 \pmod 3 \Rightarrow y_2 = 1$

$198y_3 \equiv 1 \pmod 5$

$198 \equiv 3 \pmod 5$

$3 \cdot 2 \equiv 1 \bmod 5 \Rightarrow y_3 = 2$

$120\, y_4 \equiv 1 \pmod{11} \qquad 120 \equiv 10 \pmod{11}$

$10(10) \equiv 100 \equiv 1 \pmod{11} \rightarrow y_4 = 10$

$\Rightarrow x = 165(1) + 220(1) + 198(2) + 120(10)$

$x = 1981 \pmod{330} \equiv 1 \pmod{330}$

Chp 3 Supp. Ex #25)

gcd $(2n+1, 3n+2)$ $\quad n \in \mathbb{Z}^+$

$3n+2 = 1(2n+1) + (n+1)$

$2n+1 = 1(n+1) + n$

$n+1 = 1(n) + 1 \quad \Rightarrow \gcd = 1$

Chp 4 Supp. Ex. #2) $1^3 + 3^3 + 5^3 + \cdots + (2n+1)^3 = (n+1)^2(2n^2+4n+1)$.

$n \in \mathbb{Z}^+$. (Prove w/ induction.)

Claim $\sum_{i=0}^{n} (2i+1)^3 = (n+1)^2 (2n^2+4n+1) \quad n = 1, 2 \ldots$

pf ① check $n=1$ case.

$\sum_{i=0}^{1} (2i+1)^3 = 1^3 + 3^3 = 1+27 = 28$

$(1+1)^2(2+4+1) = 4(7) = 28$ ✓

② Assume true for n,

check $n+1$ case. We know (from induction assumptn)

$\sum_{i=0}^{n} (2i+1)^3 = (n+1)^2 (2n^2+4n+1)$

$\Leftrightarrow \sum_{i=0}^{n+1} (2i+1)^3 = (n+1)^2 (2n^2+4n+1) + (2(n+1)+1)^3$

(multiply all out & collect like terms)

$= 2n^4 + 16n^3 + 47n^2 + 60n + 28$

$= (n+2)^2 (2n^2 + 8n + 7)$ (I got by factoring hopefully.)

$= (n+2)^2 (2(n+1)^2 + 4(n+1) + 1)$ //

M2R-3

Chp 4 Supp Ex #4

claim $\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1} \quad n \in \mathbb{Z}^+$

pf ① check n=1 case.

$$\sum_{i=1}^{1} \frac{1}{(2(i)-1)(2(i)+1)} = \frac{1}{1(3)} = \frac{1}{3}$$

$$\frac{1}{2(1)+1} = \frac{1}{3} \checkmark$$

② Assume true ~~for~~ n, i.e. $\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$

check n+1 case. $\sum_{i=1}^{n+1} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1} + \frac{1}{(2(n+1)-1)(2(n+1)+1)}$

$$= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} = \frac{n(2n+3)+1}{(2n+1)(2n+3)}$$

$$= \frac{2n^2+3n+1}{(2n+1)(2n+3)}$$

$$= \frac{\cancel{(2n+1)}(n+1)}{\cancel{(2n+1)}(2n+3)} = \frac{n+1}{2n+3} \quad //$$

Chp 4 Supp Ex #8 Find $N \in \mathbb{Z} \ni 2^n > n^4$ whenever $n > N$. Prove your claim.

n	2^n	n^4
1	2	1
2	4	16
3	8	81
4	16	256
5	32	625
⋮	⋮	⋮
10	1024	10,000
15	32768	50625

n	2^n	n^4
20	1,048,576	160,000
16	65536	65536

claim

$2^n > n^4 \quad \forall n > 16$
$n \in \mathbb{Z}^+$

M2R-4

#8 (cont)

pf ① $n=17$, $2^{17}=131072$, $17^4=83521$ ✓

② Assume true for n, i.e. $2^n > n^4$.

(want $2^{n+1} > (n+1)^4$)

Check $n+1$ case.

$$(n+1)^4 = n^4+4n^3+6n^2+4n+1$$
$$< n^4+4n^3+6n^3+4n^3+2n^3$$
$$= n^4+16n^3$$
$$< n^4+n^4 \quad \text{(since } n>16\text{)}$$
$$=2n^4$$
$$<2(2^n) \quad \text{by induction assumptn}$$
$$=2^{n+1}$$

//

Chp 5 Supp Ex #2

choose 10 from 6 distinct items

(a) order matters, no repeats 0 (not enough items)

(b) order matters, repeats allowed 6^{10}

(c) unordered, no repeats 0 (not enough items)

(d) unordered, repeats allowed ${}_{6+10-1}C_{10} = {}_{15}C_{10}$

Chp 5 Supp Ex #10 12 signs how many people needed to guarantee at least 6 have same sign

(Pigeonhole principle) $5(12)+1=61$

Chp 5 Supp Ex # 22

Find n

(a) $_nP_2 = 110$ $\Leftrightarrow \dfrac{n!}{(n-2)!} = n(n-1) = 110$

$n^2 - n - 110 = 0$

$(n+10)(n-11) = 0$

$n = 11$

(b) $_nP_n = 5040$ $\dfrac{n!}{0!} = 5040$

$n! = 5040$ $n = 7$

(c) $_nP_4 = 12 \, _nP_2$ $\dfrac{n!}{(n-4)!} = 12 \dfrac{n!}{(n-2)!}$

$(n \neq 0)$

and $n \neq 1$

$\cancel{n(n-1)}(n-2)(n-3) = 12 \cancel{n(n-1)}$

$n^2 - 5n + 6 = 12$

$n^2 - 5n - 6 = 0$

$(n-6)(n+1) = 0$

$n = 6$