9.5 Permutations and Combinations

Permutation-->An ordered arrangement of objects.

Combination-->A collection of objects, in no particular order.

Factorial--> $n!=n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot3\cdot2\cdot1$ By definition, $1!=1$ and $0!=1$.

The number of permutations of n distinct objects, taken all together, is n!, i.e. “n factorial.”

Example 1: Simplify the following expressions.

(a) $\frac{10!}{3!} = \frac{10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = 10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4$

(b) $\frac{3!5!}{2!} = \frac{3\cdot\cancel{2}\cdot\cancel{1}\cdot5\cdot4\cdot3\cdot2\cdot1}{\cancel{2}\cdot\cancel{1}} = 360$

Let's make a table to record these next several results and try to generalize them to a formula.

n r

Permutation or Combination	*# things to choose from*	*# things we chose*	*# permutations or combinations*	*Formula guess*
P	26	4	$\frac{26!}{22!}$	$\boxed{\frac{n!}{(n-r)!} = {}_nP_r}$
P	3	3	$3!$	$\frac{3!}{0!}$ "n permute r"
P	6	6	$6!$	
P	15	3	$\frac{15!}{12!}$	
C	5	3	$10 = \frac{5!}{2!3!}$	$\boxed{{}_nC_r = \frac{n!}{(n-r)!\,r!}}$
C	10	4	$\frac{10!}{6!4!}$	
C	52	5	$\frac{52!}{47!5!}$	

Example 2: How many 4-letter "words" can we form (where we count any distinct grouping of letters as a word) with no repeating letters.

Permutation or Combination? P

$$\underline{26}\ \underline{25}\ \underline{24}\ \underline{23} = \frac{26!}{22!}$$

$$\frac{26!}{22!} = \frac{26\cdot 25\cdot 24\cdot 23\cdot \cancel{22!}}{\cancel{22!}}$$

Example 3: I want to arrange three marbles in a row. In how many ways can this be done?

Permutation or Combination? ___P___

$\underline{3} \cdot \underline{2} \cdot \underline{1} = 6 = 3!$

Example 4: If 6 horses are entered in a race and there can be no ties, how many different orders of finish are there?

Permutation or Combination? ___P___

$\underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 6!$

Example 5: In your sorority, you need to choose a president, treasurer and secretary out of a group of 15 women. How many ways can this occur?

Permutation or Combination? ___P___

$$\underset{P}{\underline{15}} \cdot \underset{T}{\underline{14}} \cdot \underset{S}{\underline{13}} = \frac{15!}{12!}$$

$$\frac{15!}{12!} = \frac{15 \cdot 14 \cdot 13 \cdot \cancel{12!}}{\cancel{12!}}$$

Example 6: My friend made 5 cakes and she has offered to let me take three of them home. How many different groups of cakes can I choose?

Permutation or Combination? ___C___

1, 2, 3, 4, 5

123	234
124	235
125	245
134	345
135	
145	

10 choices

(5 choose 3 = 10)

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$_5C_0$ $_5C_1$ $_5C_2$ $_5C_3$ $_5C_4$ $_5C_5$

Example 7: I have 10 marbles and I want to choose 4 to give to my best friend. How many different groupings of marbles can I give her?

Permutation or Combination? ___C___

"10 choose 4"

for this combo 1234, how many permutations are there?

1234
1243
1324
1342
1423
1432
⋮

$\underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1}$

$= 24$

$= 4!$

aside

${}_{10}P_4 = \frac{10!}{6!} = 10 \cdot 9 \cdot 8 \cdot 7$

$= 72(70)$

$= 5040$

$\frac{5040}{4!} = \#\text{ combos}$

$= \frac{10!}{6!\,4!}$

$\#\text{ permutations} = \#\text{ combos} * 4!$

Example 8: How many 5-card poker hands are there?

Permutation or Combination? ___C___

(52 choose 5)

$\frac{52!}{(52-5)!\,5!} = \frac{52!}{47!\,5!}$

<u>Number of Permutations</u>:

The number of permutations of r objects chosen from n objects, where $0 \le r \le n$, is

$$_nP_r = \frac{n!}{(n-r)!}$$ read "n permute r."

<u>Number of Combinations</u>:

The number of combinations of r objects chosen from n objects, where $0 \le r \le n$, is

$$\boxed{_nC_r = \frac{n!}{(n-r)!\,r!} = \binom{n}{\cancel{r}}}$$ read "n choose r." $\binom{n}{r}$

$$= \frac{number\ of\ permutations}{number\ of\ permutations\ per\ combination}$$

$$= \frac{number\ of\ permutations}{(number\ chosen)!}$$

Example 9: You have a class of 20 students. You need to select two students to be hall monitors. How many groups of two students can you choose from your class?

Permutation or Combination? C

$$_{20}C_2 = \frac{20!}{18!\,2!} = \frac{\overset{10}{\cancel{20}} \cdot 19 \cdot \cancel{18!}}{\cancel{18!} \cdot \cancel{2}} = 190$$

Example 10: I go to an ice cream store that has 30 flavors of ice cream.

(a) I want to get a bowl with three scoops to eat. How many different groupings of three flavors can I get?

Permutation or Combination? C

$$ {}_{30}C_3 = \frac{30!}{27!\,3!} = \frac{\cancel{30}^{5}\cdot 29\cdot 28\cdot \cancel{27!}}{\cancel{27!}\cdot \cancel{3\cdot 2\cdot 1}} = 5(29)(28) $$
$$ = 4060 $$

(b) I want to get my three scoops on a cone instead. How many different ways can I arrange three flavors on my cone?

Permutation or Combination? P

$$ {}_{30}P_3 = \frac{30!}{27!} $$
$$ = 30(29)(28) $$
$$ = 24{,}360 $$

Example 11: I want to choose 3 candies out of 5 different candies. How many choices do I have?

Permutation or Combination? ____C____

$${}_5C_3 = \frac{5!}{3!\,2!} = 10$$

Example 12: I have 4 different cookies. How many ways can I put them in a line?

Permutation or Combination? ____P____

$${}_4P_4 = \frac{4!}{0!} = 24$$

Example 13: How many 12-person juries can be chosen from 30 candidates?

Permutation or Combination? $\underline{\quad C \quad}$

$${}_{30}C_{12} = \frac{30!}{18!\,12!} = \frac{30\cdot 29\cdot 28\cdot 27\cdot 26\cdot 25\cdot 24\cdot 23\cdot 22\cdot 21\cdot 20\cdot 19}{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}$$

$$= 29(7)(13)(5)(23)(3)(19)$$

Example 14: On a 10-question True/False test, how many ways can 8 or more answers be correct?

Permutation or Combination? $\underline{\quad C \quad}$

8 qn correct $= {}_{10}C_8$

or 9 " " $= {}_{10}C_9$

or 10 " " $= {}_{10}C_{10} = 1$

$${}_{10}C_8 + {}_{10}C_9 + {}_{10}C_{10}$$

$$= \frac{10!}{2!\,8!} + \frac{10!}{9!\,1!} + 1$$

$$= \frac{10\cdot 9}{2} + 10 + 1$$

$$= 45 + 10 + 1 = 56$$

Example 15: At a pizza restaurant, you have 10 choices for toppings, 3 sauce choices, and 4 types of crust. They are running a special today for $6 you can get a 1-topping pizza with your choice of topping, sauce and crust. How many different pizzas can be ordered?

Permutation or Combination? C

$$\underset{\text{crust}}{\underline{{}_4C_1}} \cdot \underset{\text{sauce}}{\underline{{}_3C_1}} \cdot \underset{\text{topping}}{\underline{{}_{10}C_1}} = 4 \cdot 3 \cdot 10 = 120$$

Example 16: For a family of 9 children, with 4 girls, how many boy-girl arrangements can there be?

Permutation or Combination? C

(9th row of Pascal's Δ)

$${}_9C_4 = {}_9C_5$$

Example 17: You toss a coin 6 times and record the result—either H=heads or T=tails.

Permutation or Combination? ___C___

(a) How many ways can you get 2 heads?

$$_6C_2 = {_6C_4} = \frac{6!}{4!\,2!} = \frac{6\cdot 5}{2} = 15$$

(b) How many ways can you get 5 tails? $_6C_5 = {_6C_1} = 6$

Example 18: You have 3 red (triangular) flags, 2 green flags and 4 blue flags. How many different ways can you order the flags on your flagpole?

Permutation or Combination? ___C/P___

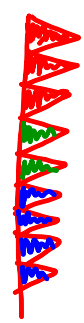
blue red green

$$_9C_4 \cdot {_5C_3} \cdot {_2C_2} = \frac{9!}{5!\,4!}\left(\frac{5!}{3!\,2!}\right)\left(\frac{2!}{2!\,0!}\right)$$

$$= \left(\frac{9\cdot\cancel{8}\cdot 7\cdot\overset{2}{\cancel{6}}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}}\right)\left(\frac{5\cdot\overset{2}{\cancel{4}}}{\cancel{2}}\right)(1)$$

$$= (9\cdot 7\cdot 2)(10)$$

$$= 126(10) = 1{,}260$$



9.5A #11)

$28 = 1+2+3+\dots+n$

formula:

$1+2+3+\dots+n$

$= (n+1)\frac{n}{2}$

$\frac{(n+1)n}{2} = 28$

~~guess~~ if $n=7$

$(n+1)\left(\frac{n}{2}\right) = \cancel{8}^{4}\left(\frac{7}{\cancel{2}}\right)$

$= 28$

$1+2+3+4+5+6+7 = 28$

$\Rightarrow$ 8 people

ex (in class)

$19+18+17+\dots+1+0 =$

$19+1 = 20$

$18+2 = 20$

$17+3 = 20$

$\vdots$

$11+9 = 20$

10

$9(20)+10$

$= 9.5(20)$

$= \boxed{\frac{19}{2}(20)}$

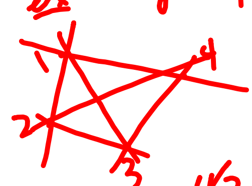
$= 190$

9.5A #9) 10 pts (no 3 are collinear)

\# lines = ?

((combinatn problem))

ex only 4 pts



${}_4C_2 = \frac{4!}{2!\,2!} = \frac{\cancel{4}\cdot 3\cdot 2}{\cancel{2}\cdot\cancel{2}}$

$= 6$

${}_{10}C_2 = \frac{10!}{8!\,2!}$

$= \frac{10\cdot 9\cdot \cancel{8!}}{\cancel{8!}\cdot 2} = 45$

9.5A#13)

$$P(\text{win}) = \frac{\#\text{ ways to win}}{\text{total }\#\text{ outcomes}} = \frac{1}{_{54}C_6}$$

9.5A#15)

choose committee of 8

20 Brits
21 Italians
4 Danes
total 45 people

(a) $P(2B, 4I, 2D)$

$$= \frac{_{20}C_2 \cdot {_{21}C_4}\, {_4C_2}}{_{45}C_8}$$

(b) $P(0B) = \frac{_{25}C_8}{_{45}C_8}$

(c) $P(\text{at least one Brit}) = 1 - P(\text{no Brits})$

$$= 1 - \frac{_{25}C_8}{_{45}C_8}$$

$P(A) = 1 - P(\bar{A})$

(d) $P(\text{all Brits}) = \frac{_{20}C_8}{_{45}C_8}$

9.5B#2)

(a) $\underline{2} \cdot \underline{26} \cdot \underline{26} = 2(26)^2$

(b) $\underline{2} \cdot \underline{26} \cdot \underline{26} \cdot \underline{26} = 2(26^3)$

9.5B#14)

3 person committee

7 A
5 F
3 E
15 total

(a) $P(\text{all A}) = \frac{_7C_3}{_{15}C_3}$

(b) $P(\text{no A}) = \frac{_8C_3}{_{15}C_3}$

9.5B#13)

die rolled 8 times

(a) $P(\text{all 1s}) = \frac{1}{6^8}$

(b) $P(\text{exactly 2 6s}) = \frac{_8C_2}{6^8}$

ex

6 _ _ 6 _ _ _ _

_ _ _ _ _ 6 _ 6

(c) $P(\text{at least one 6})$

$$= 1 - P(\text{no 6s})$$

$$= 1 - \frac{5^8}{6^8}$$

5 5 5 . . . . _ _ _

9.5 MC #8)

1

1 1

1 2 1

1 3 3 1

$_0C_0$

$_1C_0$ $_1C_1$

$_2C_0$ $_2C_1$ $_2C_2$

$_3C_0$ $_3C_1$ $_3C_2$ $_3C_3$

$_0C_0 = \frac{0!}{0!\cdot 0!} = 1$

$_1C_0 = \frac{1!}{0!\cdot 1!} = 1$

$\frac{1}{0} = ? \Leftrightarrow 1 = 0 \cdot ?$

B#19) 2 fair dice rolled 5 times

(a) $P(\text{sum of 7 on each of rolls}) = \frac{6^5}{36^5} = \frac{1}{6^5}$

$\underbrace{(6\cdot 6)}\underbrace{(6\cdot 6)}\underbrace{\_\ \_}\underbrace{\_\ \_}\underbrace{\_\ \_}$

5 rolls: 6 6 6 6 6

(b) $P(\text{sum of 7 exactly twice}) = \frac{_5C_2\,(6^2)}{36^5}$ (A)

S7 __ S7 __ __

__ S7 __ __ S7

9.5B#15)

32 people, choose 5 and then from 5 choose 1 prez.

$_{32}C_5 \cdot {_5C_1}$

9.5MC#4)

5 couples, put in a row, no couple is separated. how many choices?

ex

2 couples AB CD

AB CD AB DC BA CD BA DC $2!(2^2)$

CD AB . . . . .

5 couples

5 4 3 2 1

$\boxed{5!(2^5)}$