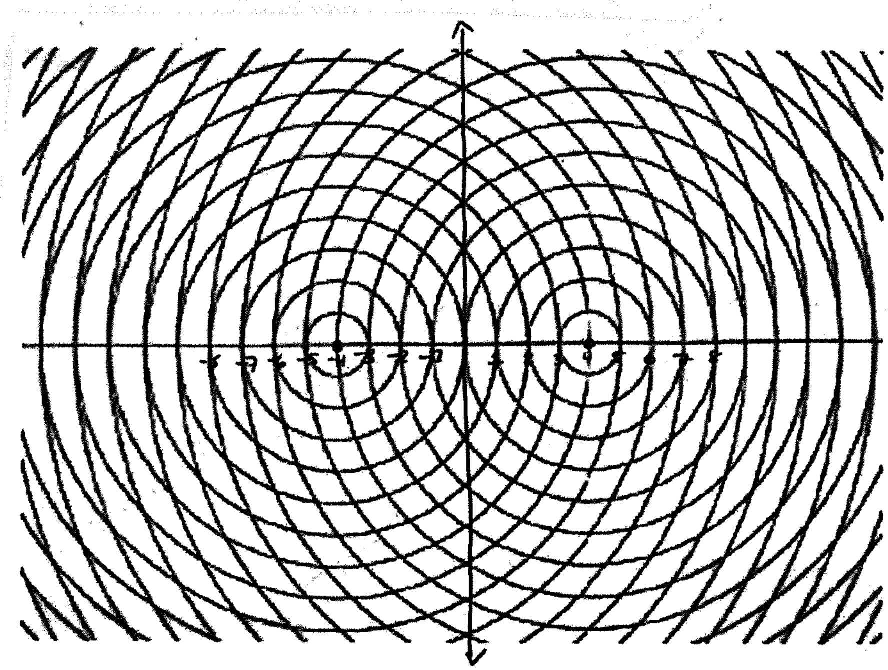
8.2 Ellipses

Ingredients: two points (foci) and a distance

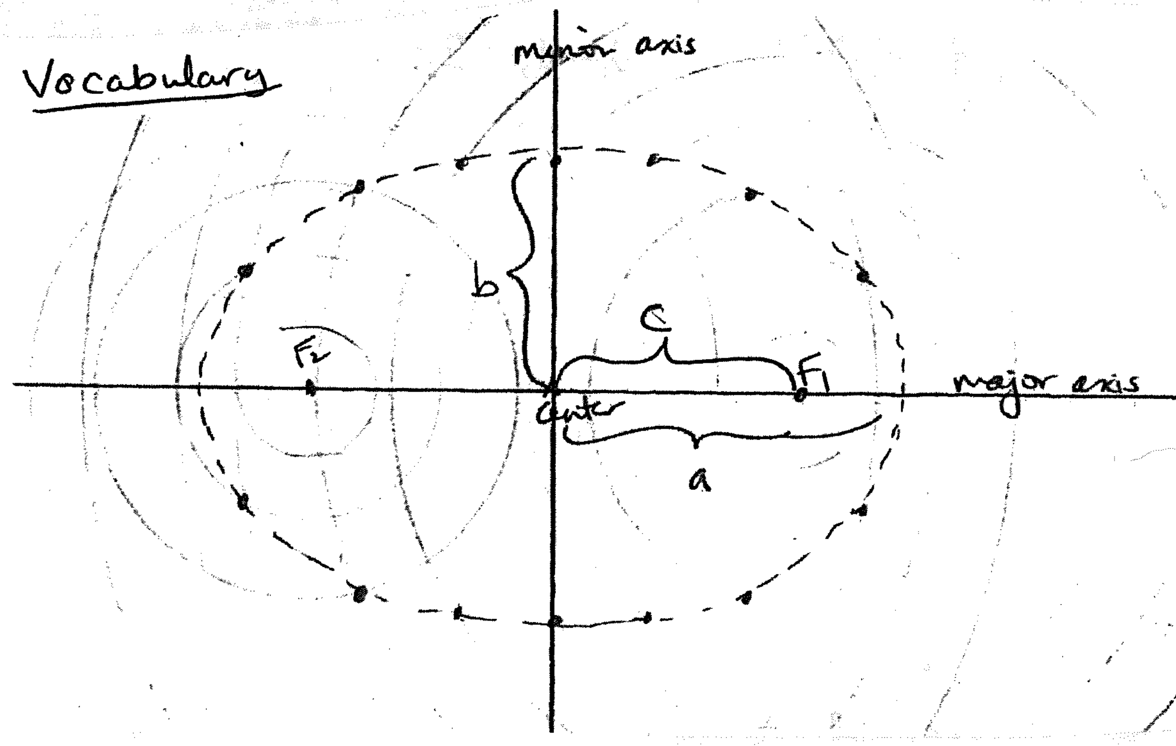
Defn of Ellipse: The set of all points in a plane such that for each pt on the ellipse, the sum of its distances from the two fixed pts (called foci) is constant.

Ex1 Given the pts $F_1(-4,0)$ and $F_2(4,0)$ (the two foci) and a constant distance of 12, draw the ellipse that these determine.



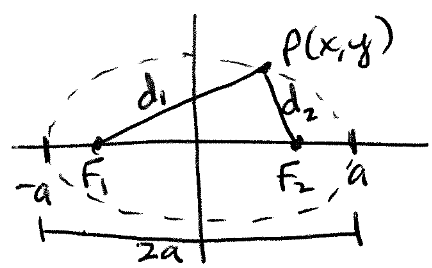
# 8.2 (cont)

Vocabulary




Standard form of ellipse w/ center at $(0,0)$:

by defn $d_1+d_2=2a$ (assume $a>b$)




$F_1(-c,0)$ $F_2(c,0)$

$$\sqrt{(x+c)^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2}=2a$$

$$\left(\sqrt{(x+c)^2+y^2}\right)^2=\left[2a-\sqrt{(x-c)^2+y^2}\right]^2$$

$$\cancel{x^2}+2cx+\cancel{c^2}+\cancel{y^2}=4a^2-4a\sqrt{(x-c)^2+y^2}+\cancel{x^2}-2cx+\cancel{c^2}+\cancel{y^2}$$

$$(4cx-4a^2)^2=\left(-4a\sqrt{(x-c)^2+y^2}\right)^2$$

$$16c^2x^2-32ca^2x+16a^4=16a^2(x^2-2cx+c^2+y^2)$$

$$16c^2x^2-\cancel{32ca^2x}+16a^4=16a^2x^2-\cancel{32ca^2x}+16a^2c^2+16a^2y^2$$

$$\frac{-1}{16}\left(16(c^2-a^2)x^2-16a^2y^2\right)=\left(16a^2c^2-16a^4\right)\frac{-1}{16}$$

$\boxed{\text{let } b^2=a^2-c^2}$

$$(a^2-c^2)x^2+a^2y^2=a^4-a^2c^2$$

$$b^2x^2+a^2y^2=a^2(a^2-c^2)$$

8.2 (cont)

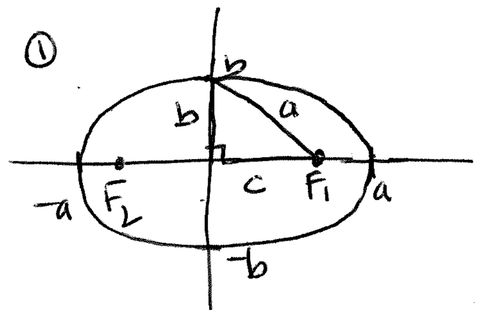
$$\frac{b^2x^2+a^2y^2}{a^2b^2}=\frac{a^2b^2}{a^2b^2}$$

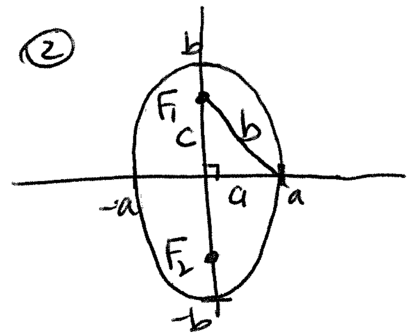
$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$ standard eqn for ellipse w/ center at $(0,0)$

goes through pts $(\pm a, 0)$ (x-intercepts) and $(0, \pm b)$ (y-intercepts)

---

Notice:

① 


or ② 


We have $b^2=a^2-c^2$ (from our proof)

here is geometric interpretation

in ①, $a>b$ $\quad b^2+c^2=a^2$
$\Leftrightarrow c^2=a^2-b^2$
(or $b^2=a^2-c^2$)

in ② $a<b$ $\quad a^2+c^2=b^2$
$c^2=b^2-a^2$
(or $a^2=b^2-c^2$)

(note: c is <u>not</u> the length of hypotenuse here)

---

length of horizontal axis $=2a$

length of vertical axis $=2b$

$\max(2a, 2b)=$ length of <u>major axis</u>

$\min(2a, 2b)=$ length of <u>minor axis</u>

<u>8.2 (cont)</u>

<u>Standard Ellipse</u>

center at $(0,0)$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

<u>Transformed Ellipse</u>

center at $(h,k)$

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$$c^2=|a^2-b^2|$$

$c$ = distance from center to each focus

| | $a<b$ | $a>b$ |
|---|---|---|
| vertices: | $(0,\pm b)$ | $(\pm a,0)$ |
| foci: | $(0,\pm c)$ | $(\pm c,0)$ |

| | $a<b$ | $a>b$ |
|---|---|---|
| vertices: | $(h,k\pm b)$ | $(h\pm a,k)$ |
| foci: | $(h,k\pm c)$ | $(h\pm c,k)$ |

---

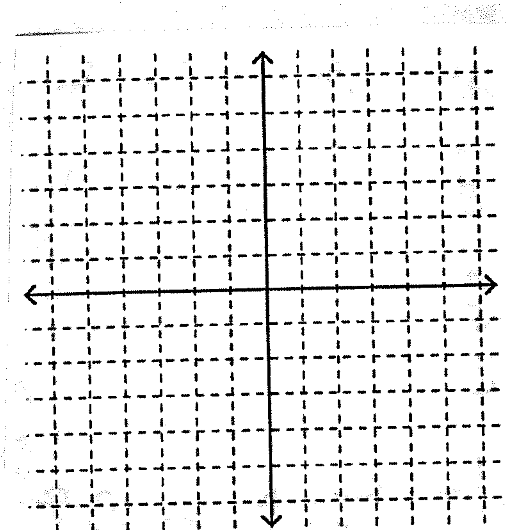
if $a=b$, then we have a <u>CIRCLE</u> with radius $r=a=b$ (and both foci in one pt at $(h,k)$)

---

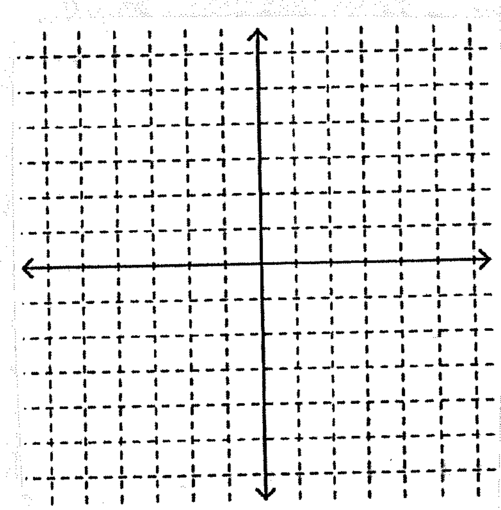
The eqn $(x-h)^2+(y-k)^2=0$ is ________________.

<u>Ex2</u> Sketch the curves:

(a) $4x^2+9y^2=36$

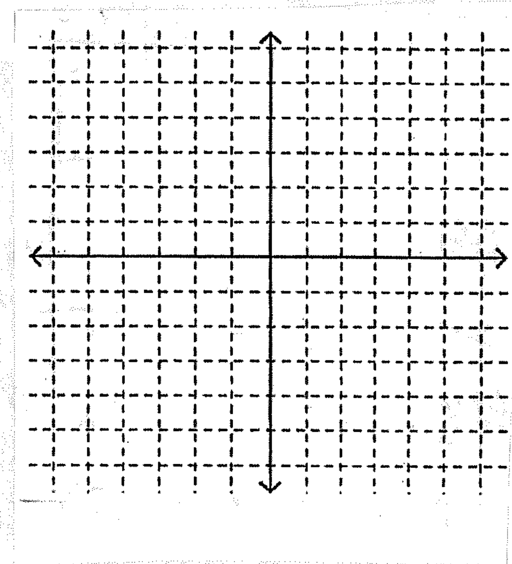


(b) $\frac{(x-3)^2}{9}+\frac{(y-2)^2}{16}=1$

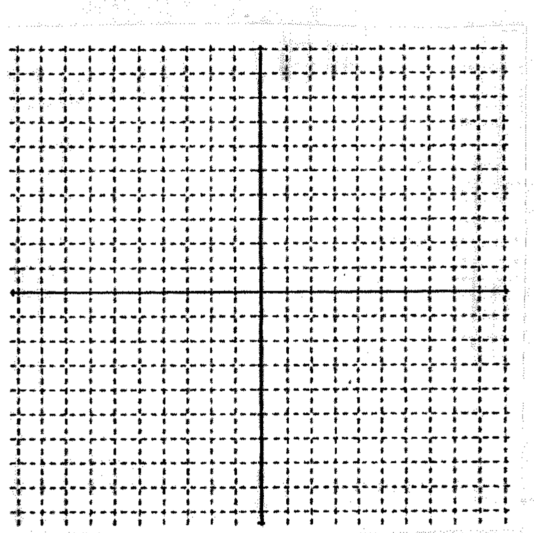


Ex 3 Graph the curve and find the eqn of the ellipse.

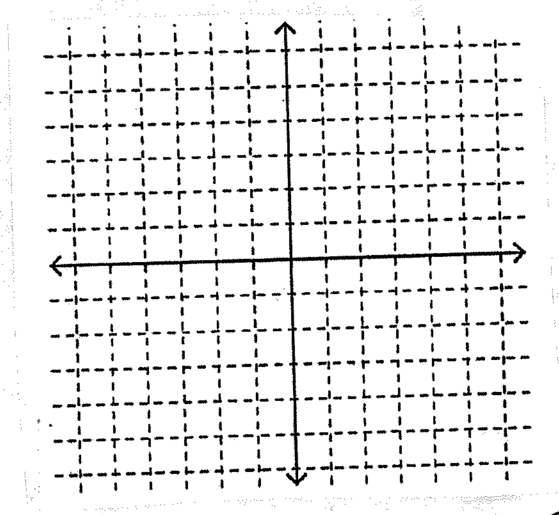
(a) set of pts 3 units away from (-2, 3)



(b) Set of pts such that sum of distances from (-4, 1) and (2, 1) is 10.

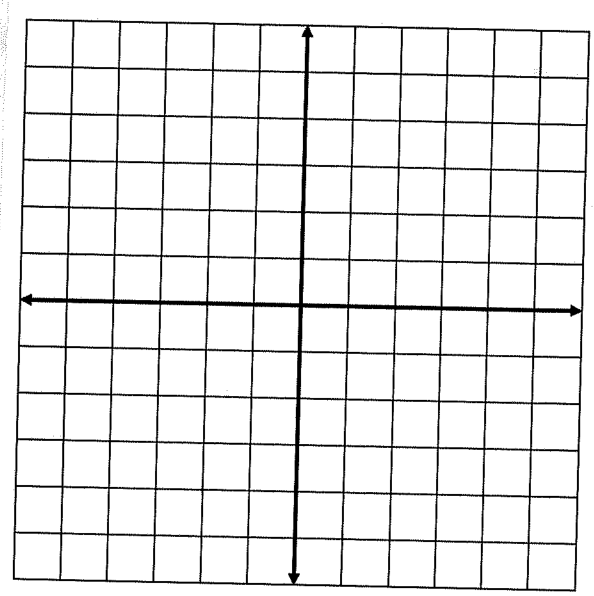


(c) Ellipse w/ vertices at (-6, 3) and (4, 3) and foci at (-4, 3) and (2, 3).



8.2 (cont)

Ex 4 Graph this curve. $x^2+9y^2-4x-18y-14=0$

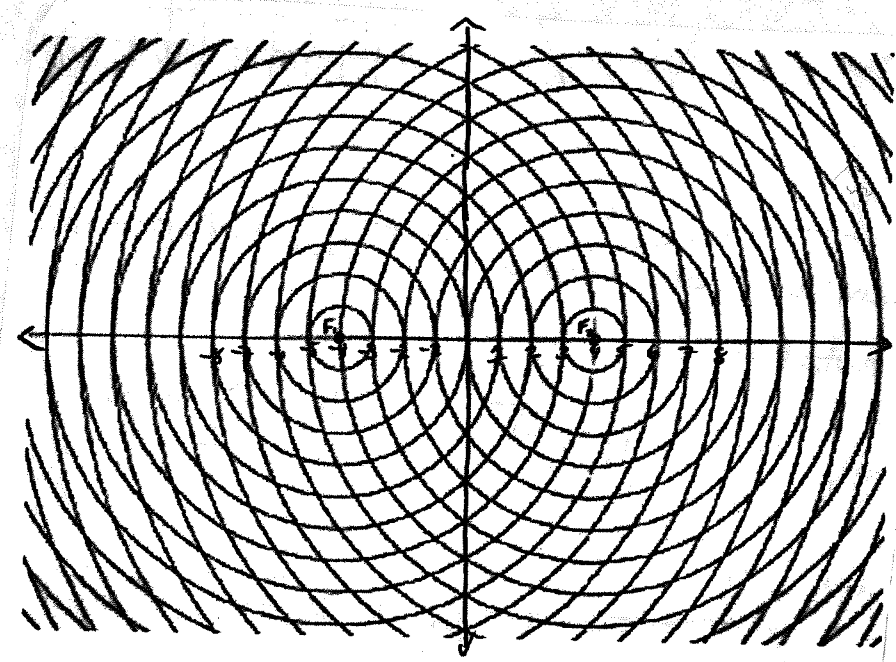


# 8.3 Hyperbolas

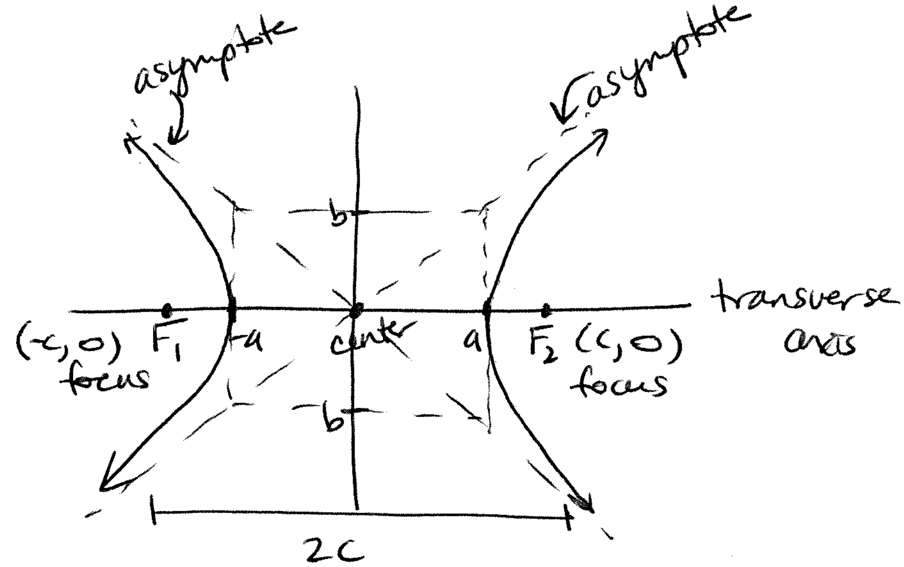
Ingredients: two points and a constant distance
(foci)

Defn of Hyperbola: The set of all points in a plane such that the difference of its distances from two fixed points is constant.

Ex 1 Given $F_1(-4,0)$ and $F_2(4,0)$ and the constant distance 1, draw the hyperbola that those determine.



Vocab:

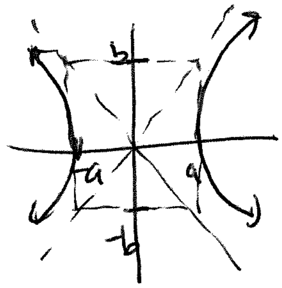



## Standard Form of Hyperbola w/ center at (0,0)

**horizontal**

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$a^2+b^2=c^2$$

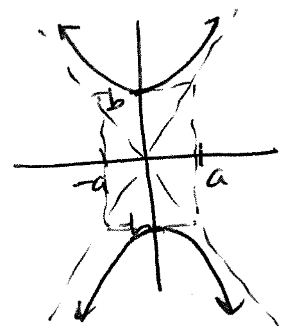



foci: $(\pm c,0)$
vertices: $(\pm a,0)$

**vertical**

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$$

$$a^2+b^2=c^2$$




foci: $(0,\pm c)$
vertices: $(0,\pm b)$

Eqn of Hyperbola (for horizontal case)

distance from $(x,y)$ on hyperbola to $(-c,0)$ and $(c,0)$ has constant difference of $2a$.

$$\Rightarrow \left|\underset{①}{\sqrt{(x+c)^2+(y-0)^2}}-\underset{②}{\sqrt{(x-c)^2+(y-0)^2}}\right|=2a$$

case 1: ① $\geq$ ② $\Rightarrow$ abs value does nothing.

$$\left(\sqrt{(x+c)^2+y^2}\right)^2 = \left(2a+\sqrt{(x-c)^2+y^2}\right)^2$$

$$(x+c)^2+\cancel{y^2} = 4a^2+4a\sqrt{(x-c)^2+y^2}+(x-c)^2+\cancel{y^2}$$

$$\cancel{x^2}+2cx+\cancel{c^2} = 4a^2+4a\sqrt{x^2-2cx+c^2+y^2}+\cancel{x^2}-2cx+\cancel{c^2}$$

$$\left(4cx-4a^2\right)^2 = \left(4a\sqrt{x^2-2cx+c^2+y^2}\right)^2$$

$$16c^2x^2-32ca^2x+16a^4 = 16a^2(x^2-2cx+c^2+y^2)$$

$$16c^2x^2-\cancel{32ca^2x}+16a^4 = 16a^2x^2-\cancel{32ca^2x}+16a^2c^2+16a^2y^2$$

$$\cancel{16}c^2x^2-\cancel{16}a^2x^2-\cancel{16}a^2y^2 = \cancel{16}a^2c^2-\cancel{16}a^4$$

$$(c^2-a^2)x^2-a^2y^2 = a^2(c^2-a^2)$$

but $b^2=c^2-a^2$

$$\frac{b^2x^2-a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2}$$

$$\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1}$$

(case 2 will yield same result)

---

Standard Hyperbola Eqns w/ center at $(h,k)$

<u>horizontal</u>

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

foci: $(h\pm c, k)$

vertices: $(h\pm a, k)$

<u>vertical</u>

$$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$$

foci: $(h, k\pm c)$

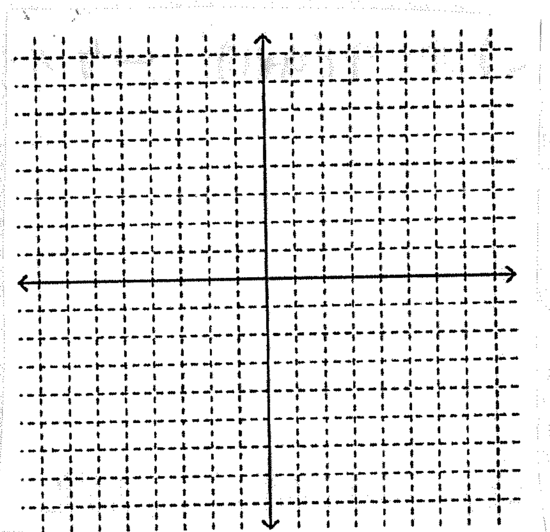
vertices: $(h, k\pm b)$

$a^2+b^2=c^2$

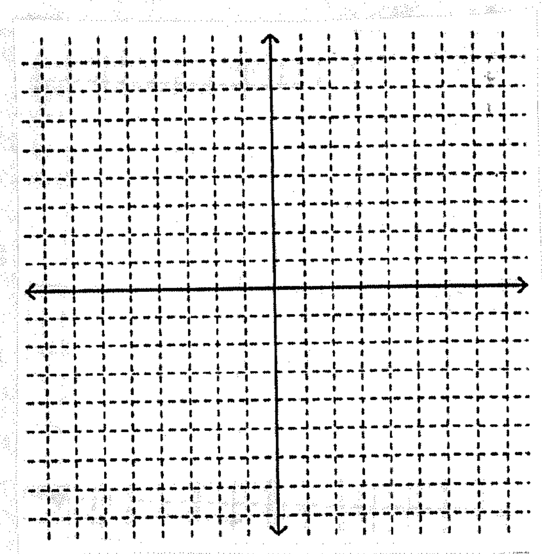
8.3 (cont)

EX 2 Graph.

(a) $\dfrac{x^2}{4} - \dfrac{y^2}{25} = 1$



(b) $9(y+1)^2 - 4(x-2)^2 = 1$



Given $Ax^2 + Cy^2 + Dx + Ey + F = 0$

(i) $A=0$ and $C=0$ ⇒ it's a line

(ii) $A=0$ OR $C=0$ ⇒ it's a parabola
(but not both 0)

(iii) A and C have same sign ⇒ it's an ellipse
(not both zero)

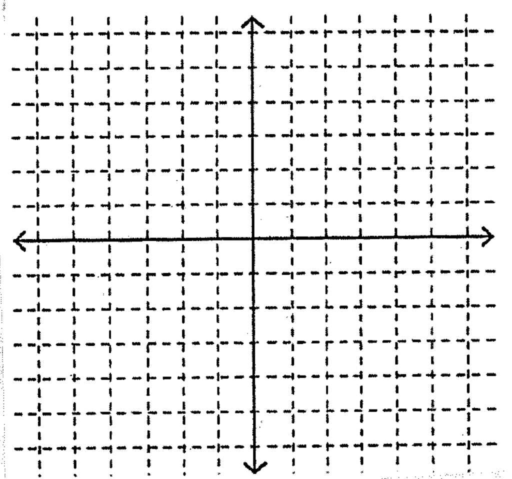
(iv) $A=C$ (not zero) ⇒ it's a circle

(v) A and C have opposite signs ⇒ it's a hyperbola

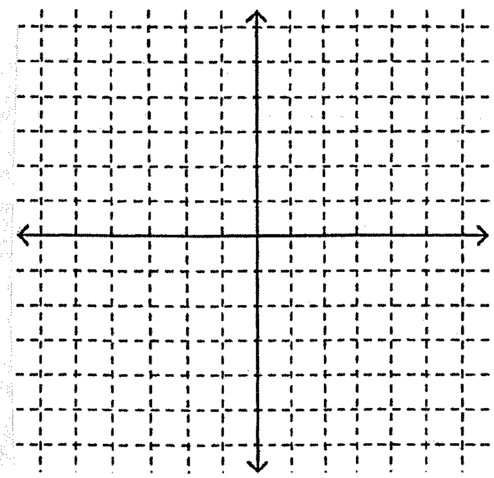
8.3 (Cont)

Ex 3 Identify each conic and graph.

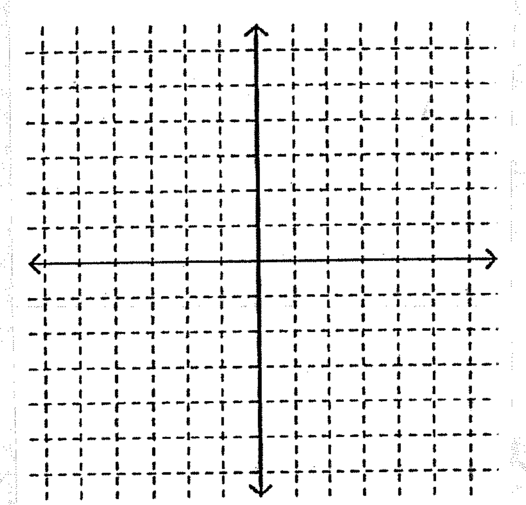
(a) $5(x-3)^2-(y+2)^2=15$



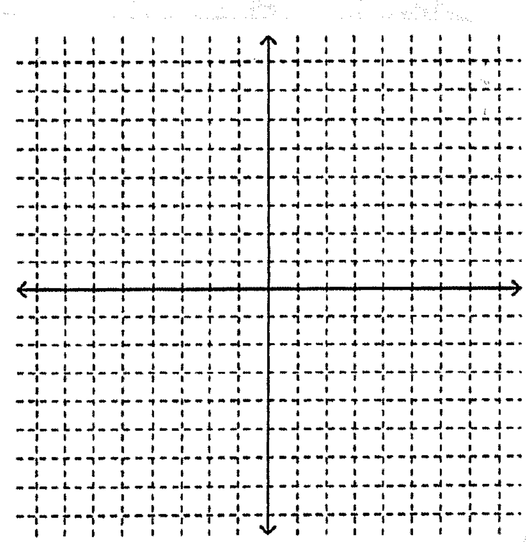
(b) $\frac{x-3}{9}+\frac{y-2}{4}=1$



(c) $9(x+3)^2+4(y-2)^2=0$



(d) $9(x+3)^2+18(y-2)=0$



# 8.3 (cont)

(e) $9x^2+16y^2-54x-128y+333=0$