MATH 1090 Final Exam 15 December 2016

Name: Solutions

UnID: Instructor:

Instructions:

•Please show all of your work as partial credit will be given where appropriate, and there may be no credit given for problems where there is no work shown.

•All answers should be completely simplified, unless otherwise stated.

•Make sure all cell phones (and all other electronics that can connect to the internet) are put away and out of sight. If you have a cell phone/other electronic device out at any point, for any reason, you will receive a zero on this exam.

•You may use a scientific calculator only. No graphing or programmable calculators allowed.

•You will be given an opportunity to ask clarifying questions about the instructions within the first fifteen minutes of the time the scheduled final exam is advertised by the University to begin. The questions will be answered for the entire class. After that, no further questions will be allowed, for any reason.

•You must show us your U of U student ID card when finished with the exam.

•You can ask the proctor/instructor for scratch paper. You may use **no** other scratch paper. Please transfer all finished work onto the proper page in the test for us to grade there. We will **not** grade the work on the scratch pages.

•You are allowed to use one 8.5 × 11 inch sheet of paper (front and back) for your reference during the exam.

DO NOT WRITE BELOW THIS LINE. THE TABLE IS FOR GRADING.

Question	Points	Score	Question	Points	Score
1	10		8	15	
2	12		9	18	
3	20		10	15	
4	15		11	15	
5	20		12	15	
6	10		13	15	
7	15		14	15	
			Total	210	

1. $\boxed{\text{10 points}}$ If three less than two times a number is equal to six more than the number, then what is the number?

Equation: $2N - 3 = N + 6$ (6)

Solve for N: $2N - 3 = N + 6$
$-N + 3 \quad -N + 3$ (3)
$N = 9$

Answer: $\boxed{N = 9}$ (1)

2. $\boxed{\text{12 points}}$ Find the equation of the line passing through the points $(0, 1)$ and $(1, 3)$.

- Slope-Intercept Eqn Form: $y = mx + b$ (3) (for writing down the form)
  need to find m & b
- Slope: $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3 - 1}{1 - 0} = 2$ (4)
- Intercept:
  method 1: Read from the point $(0, 1)$ (4)
  method 2: $y = 2x + b \Rightarrow 3 = 2(1) + b \Rightarrow b = 1$

Line: $\boxed{y = 2x + 1}$ (1)

(*put your answer in the form* $y = mx + b$)

2 points for each boundary line & total = 6 points.
2 points for each corner point & total = 6 points

3. $\boxed{\text{20 points}}$ Consider the following system of inequalities:

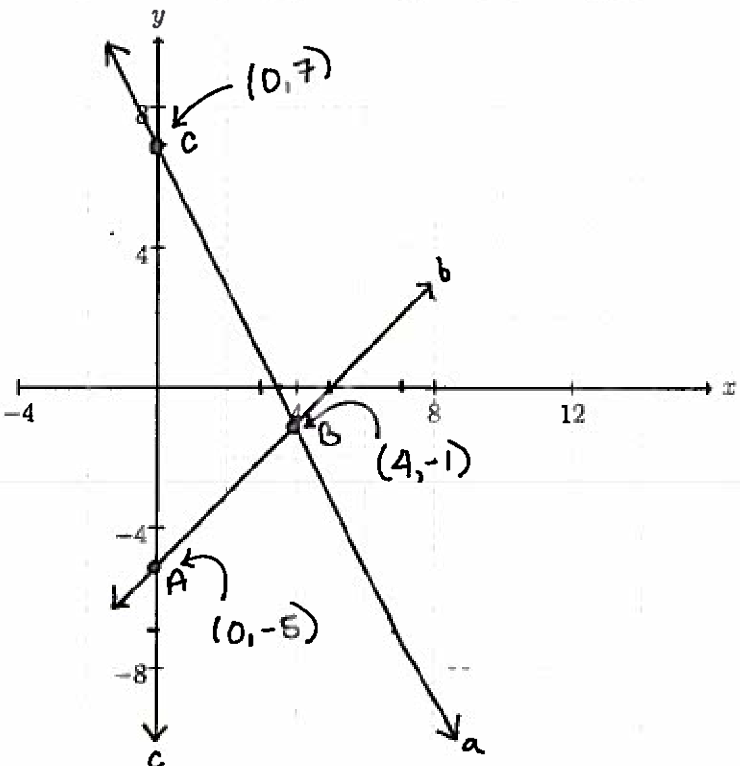
$$y \le -2x + 7 \quad (a)$$
$$y \ge x - 5 \quad (b)$$
$$x \ge 0 \quad (c)$$

(a) Graph the solution set and label the corner points with coordinates.

Point A:
b) $\begin{cases} y = x - 5 \\ x = 0 \end{cases}$ c)

Point B:
a) $\begin{cases} y = -2x + 7 \\ y = x - 5 \end{cases}$ b)

Point C:
a) $\begin{cases} y = -2x + 7 \\ x = 0 \end{cases}$ c)



Solve the system:
$\begin{cases} y = -2x + 7 \\ y = x - 5 \end{cases}$

Substitution:
$-2x + 7 = x - 5$
$-x \; -7 \quad -x \; -7$
$\dfrac{-3x}{-3} = \dfrac{-12}{-3}$
$x = 4$
Plug into (b)
$y = 4 - 5 = -1$

(b) Calculate the maximum and minimum of the objective function $P = 2x - 3y$ given the constraints above.

Given: $P(x,y) = 2x - 3y$

Point A: $P(0,-5) = 2(0) - 3(-5) = 15$ (2)
Point B: $P(4,-1) = 2(4) - 3(-1) = 11$ (2)
Point C: $P(0,7) = 2(0) - 3(7) = -21$ (2)

Maximum: 15 Located at: (0, -5) (1)

Minimum: -21 Located at: (0, 7) (1)

For substitution method:

(6) for putting the original system into a new system of linear eqn's in two ~~vari~~ variables. (for example $(x,y,z) \to (x,z)$)

4. 15 points Solve the following system of equations using any method.

(6) for solving the new system (for example (x,z))

(3) for solving for the last variable.

(1) $x + 2y + 3z = 3$

(2) $y + 4z = 11$

(3) $5x + 6y = -25$

method 1: Substitution

Eqn(2) $\Rightarrow y = 11 - 4z$

Plug into Eqns (1) & (3)

$$\begin{cases} x + 2(11-4z) + 3z = 3 \\ 5x + 6(11-4z) = -25 \end{cases}$$

$\Downarrow$

$$\begin{array}{ll} (4) & \begin{cases} x - 5z = -19 \\ 5x - 24z = -91 \end{cases} \\ (5) & \end{array}$$

Eqn(4) $\Rightarrow x = -19 + 5z$

Plug into Eqn(5):

$5(5z - 19) - 24z = -91$

$\Downarrow$

$\boxed{z = 4}$

Plug into Eqn(2) : $y + 4(4) = 11 \Rightarrow \boxed{y = -5}$

Plug into Eqn(3) : $5x + 6(-5) = -25$

$\Rightarrow 5x = 5 \Rightarrow \boxed{x = 1}$

method 2: Inverse method

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 - 5R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 + 4R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 = R_1 - 3R_3]{R_2 = R_2 - 4R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 16 & -12 & -3 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right]$$

Inverse matrix: $\begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$ (10)

Solution: $\begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 11 \\ -25 \end{bmatrix} =$

$= \begin{bmatrix} -72 + 198 - 125 \\ 60 - 165 + 100 \\ -15 + 44 - 25 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$

(5)

$(x, y, z) =$ $(1, -5, 4)$

For Gauss - Jordan Elimination method and finding the inverse ~~and~~ of the coefficient matrix in method 2, I think we can give students partial credits based on how far they go to put the coefficient matrix in identity

5. $\boxed{\text{20 points}}$ Consider the matrices:

$$A = \begin{bmatrix} 7 & 10 \\ 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 9 \\ 7 & -2 \\ -5 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

Calculate all of the following matrices. If a computation is impossible, state that and the reason why.

(a) Calculate B^T.

(4)

$B^T = \boxed{\begin{bmatrix} 4 & 7 & -5 \\ 9 & -2 & 6 \end{bmatrix}}$ (4)

(b) Calculate $3C - A$. (2) (for 3C)

(4)

$$3C-A = \begin{bmatrix} 3 & 6 \\ 0 & -3 \end{bmatrix} - \begin{bmatrix} 7 & 10 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -5 & -9 \end{bmatrix}$$

$3C - A = \boxed{\begin{bmatrix} -4 & -4 \\ -5 & -9 \end{bmatrix}}$ (2)

(c) Calculate A^{-1}.

(6)

(2) (for the correct formula)

(2) for setting up $\left(\begin{matrix} 7 & 10 & | & 1 & 0 \\ 5 & 6 & | & 0 & 1 \end{matrix}\right.$

method 1: Formula

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \left(\frac{1}{ad-bc}\right)\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{7(6)-5(10)} \begin{bmatrix} 6 & -10 \\ -5 & 7 \end{bmatrix}$$

(4) for computation

$A^{-1} = \boxed{-\frac{1}{8}\begin{bmatrix} 6 & -10 \\ -5 & 7 \end{bmatrix}}$

method 2: Gaussian Elimination

$$\left[\begin{array}{cc|cc} 7 & 10 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{array}\right] \xrightarrow{R_1 = 3R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & -2 & -2 & 3 \\ 5 & 6 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 = R_2 - 5R_1} \left[\begin{array}{cc|cc} 1 & -2 & -2 & 3 \\ 0 & 16 & 10 & -14 \end{array}\right]$$

(4) for ~~comput~~ computation

$$\xrightarrow{R_2 = R_2/16} \left[\begin{array}{cc|cc} 1 & -2 & -2 & 3 \\ 0 & 1 & 5/8 & -7/8 \end{array}\right] \xrightarrow{R_1 = R_1 + 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & -6/8 & -10/8 \\ 0 & 1 & 5/8 & -7/8 \end{array}\right]$$

(d) Calculate AC.

(6)

$$\begin{bmatrix} 7 & 10 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \overset{(1)}{7(1)+10(0)} & \overset{(1)}{7(2)+10(-1)} \\ \underset{(1)}{5(1)+6(0)} & \underset{(1)}{5(2)+6(-1)} \end{bmatrix}$$

$AC = \boxed{\begin{bmatrix} 7 & 4 \\ 5 & 4 \end{bmatrix}}$ (2)

6. $\boxed{\text{10 points}}$ The supply equation for a product is given by $p = 2q^2 - 24q + 4$ and the demand equation is given by $p = -8q + 100$.

(a) If the price increases, will consumers demand more or less product? Circle your answer.

(2)

More Less

(b) Find the market equilibrium.

Demand Eqn: $P = -8q + 100$

$\Rightarrow q = \dfrac{100 - P}{8}$

As p increases q decreases!

market equilibrium is the intersection of the supply & demand equations.

(4) for setting those eqn's equal & solving for (p, q)

$$2q^2 - 24q + 4 = -8q + 100$$

$$\Rightarrow 2q^2 - 16q - 96 = 0$$

$$\Rightarrow 2(q^2 - 8q - 48) = 0$$

$$\Rightarrow 2(q-12)(q+4) = 0$$

$$\Rightarrow q = 12 \text{ or } -4$$

$$q = 12 \Rightarrow p = -8(12) + 100 = 4$$

$$q = -4 \Rightarrow p = -8(-4) + 100 = 132$$

Two intersection Points:

(12, 4) & (-4, 132)

A negative quantity of items doesn't make sense so we choose (12, 4)

(2)

$p =$ $\boxed{4}$ (1) $q =$ $\boxed{\text{12 items}}$ (1)

7. $\boxed{\text{15 points}}$ An auto repair shop sells car repair services. Suppose that the shop has monthly fixed costs of \$100 dollars and variable costs of $x + 20$ dollars per service, where x is the total number of services sold. The revenue function is $R(x) = (60 - x)x$ in dollars.

(4) (a) Find the cost function.

Costs = fixed costs + variable costs (2)

Cost $C(x) = 100 + \underbrace{(x+20)}_{\text{cost per service}} \, x$ ← qty of services (2)

(2) (b) What kind of function is the revenue function? Circle your answer.

Linear **Quadratic**

(4) (c) How many services need to be sold to maximize revenue?

completing the square

$R(x) = (60-x)x = -x^2 + 60x = -(x^2 - 60x \quad)$ $\frac{60}{2}$

$= -(x^2 - 60x + 900) + 900 = -(x-30)^2 + 900 \Rightarrow$ Vertex @ $(30, 900)$

Services sold: $\boxed{\text{30 services}}$ they might do: x-intercepts: (0,0) & (60,0) $\Rightarrow$ axis of symmetry*: $x = \frac{0+60}{2} = 30$.

(3) (d) Find the profit function.

Profit = Revenue − Cost $= (-x^2 + 60x) - (x^2 + 20x + 100)$

(2)

Profit $P(x) = \boxed{-2x^2 + 40x - 100}$ (1)

(2) (e) Is it most valuable for a business owner to maximize revenue, or is there something better to maximize (or minimize)? Explain your thinking.

more valuable to maximize profit.

8. $\boxed{\text{15 points}}$ Consider the following functions:

$$f(x) = \frac{5-3x}{2}, \quad g(x) = x^3 + 1, \quad h(x) = \sqrt{x-4}$$

(2) (a) What is the domain of $h(x)$? We require $\underbrace{x-4 \geq 0}_{(1)} \Rightarrow \underbrace{x \geq 4}_{(1)}$

Domain: $\boxed{x \in \mathbb{R} \text{ s.t. } x \geq 4}$ - Alternatively - $\boxed{x \in [4, \infty)}$

(3) (b) Find $h(g(x))$.

I think it's fine if they leave the result as $\sqrt{(x^3+1)-4}$ without simplifying.

$h(g(x)) = \boxed{\sqrt{(x^3+1) - 4} = \sqrt{x^3 - 3}}$

(3) (c) Find $(f+h)(5)$.

$(f+h)(x) = \frac{5-3x}{2} + \sqrt{x-4} \Rightarrow (f+h)(5) = \frac{5-15}{2} + \sqrt{5-4} = -5+1$

$(f+h)(5) = \boxed{-4}$

(3) (d) Find $(gf)(1)$.

$(g \cdot f)(x) = \left(\frac{5-3x}{2}\right)(x^3+1) \Rightarrow (g \cdot f)(x) = 1(2) = 2$

$(gf)(1) = \boxed{2}$

(4) (e) Find $f^{-1}(x)$.

$y = \frac{5-3x}{2}$

Can be done in either order

Step 1: Solve for x : $2y = 5 - 3x \Rightarrow 3x = 5 - 2y \Rightarrow x = \frac{5-2y}{3}$ (3)

Step 2: Change Variables: $y = \frac{5-2x}{3}$ (1)

$\boxed{f^{-1}(x) = (5-2x)/3}$

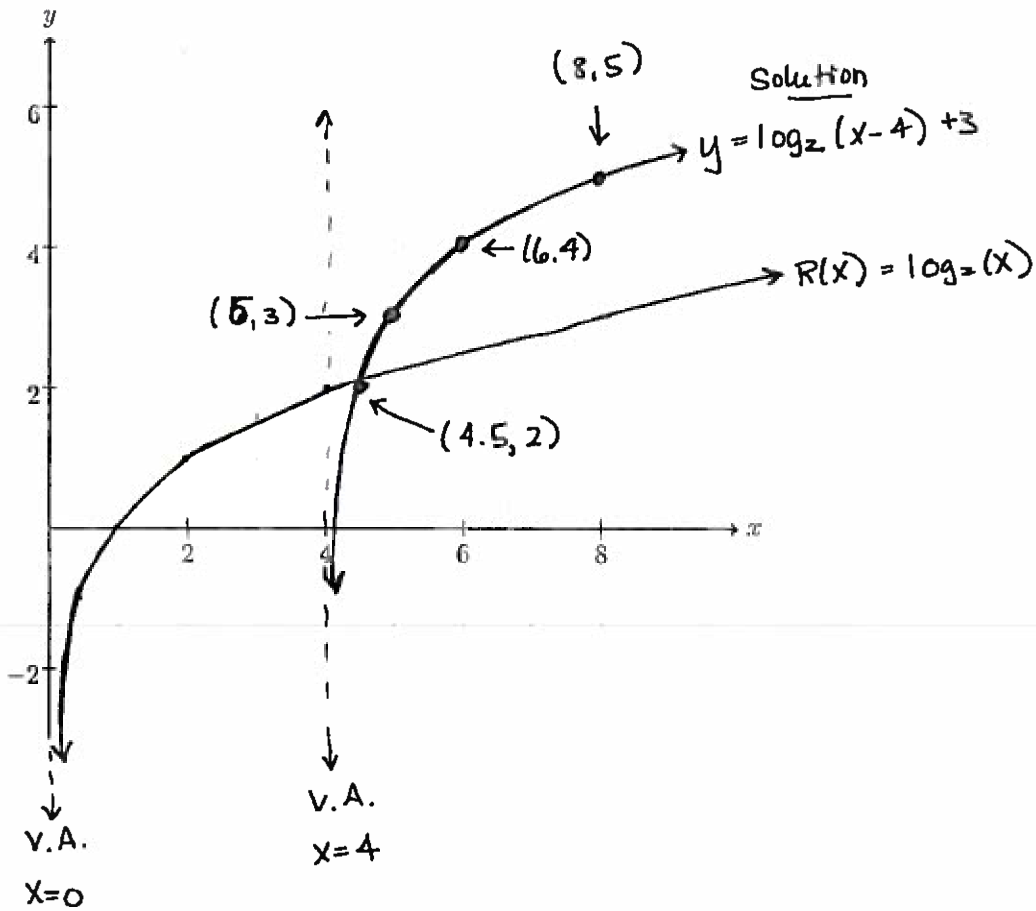
9. $\boxed{\text{18 points}}$ This question is about the graphs of various functions we studied in class.

(a) Graph $y = \log_2(x - 4) + 3$. Label at least two points on the graph with coordinates for full credit.

(6)

Reference function

$R(x) = \log_2(x)$

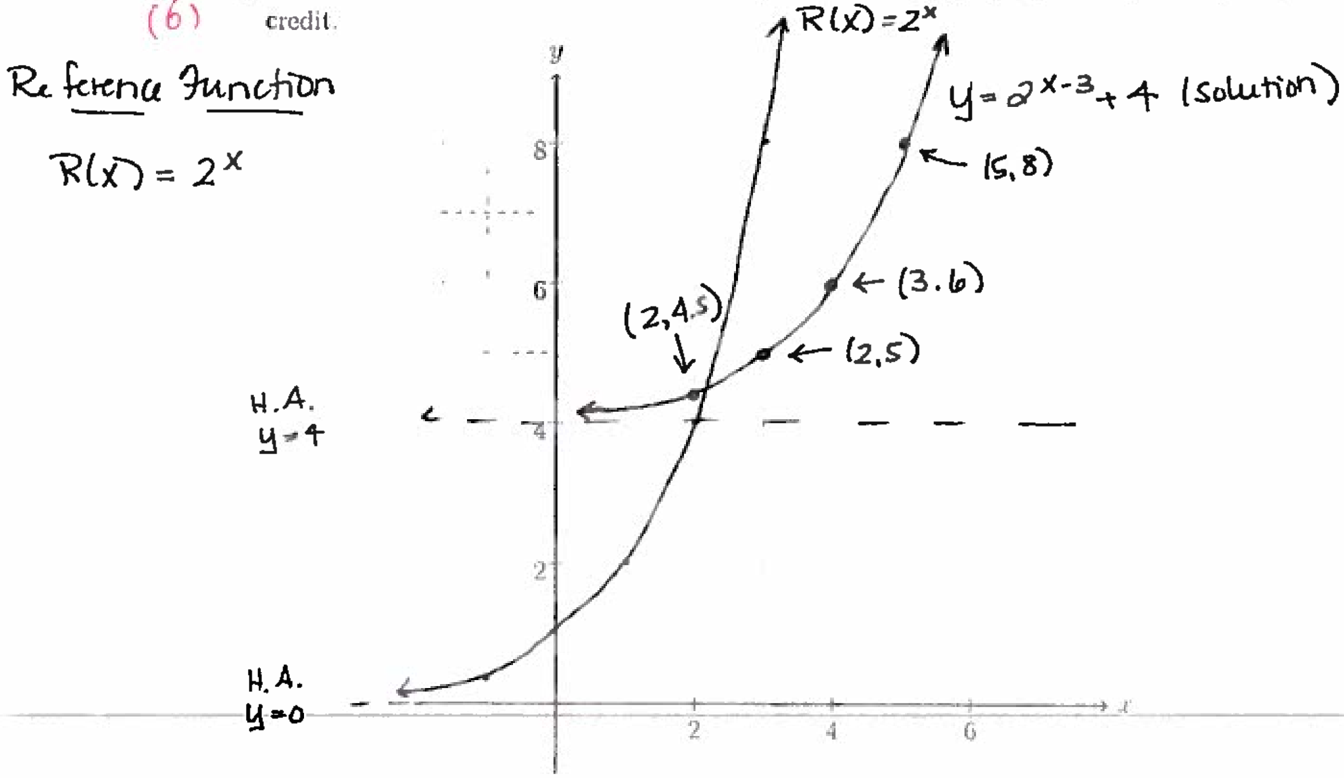


(4) for the correct graph (they might graph by plotting points)

(2) 1 point for each labeled point.

(b) Graph $y = 2^{x-3} + 4$. Label at least two points on the graph with coordinates for full credit.

(6)



(4) for the correct graph

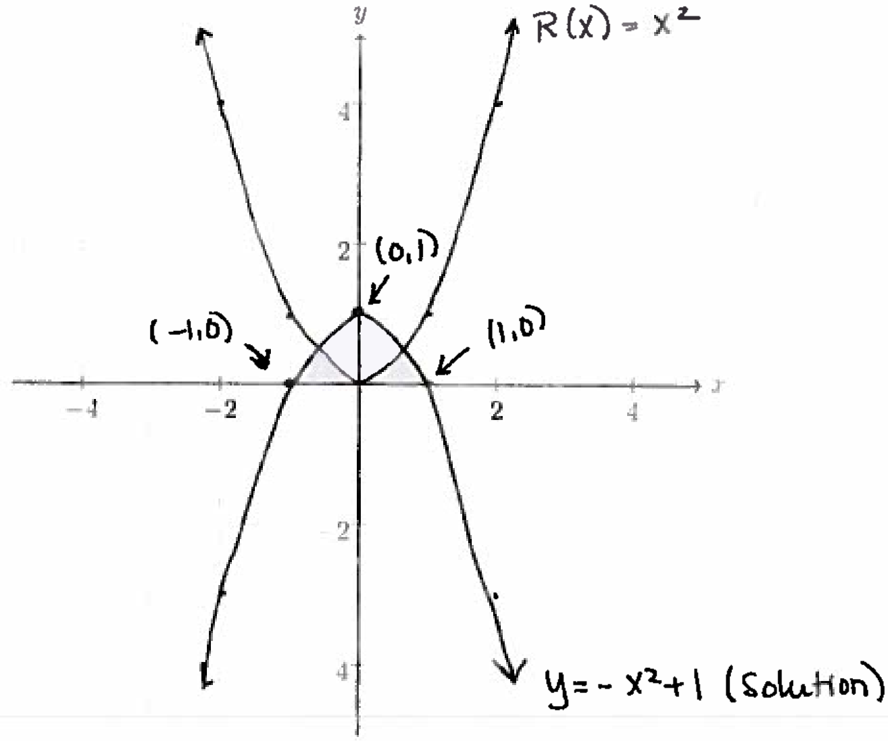
(2) for labeling (1 pt each)

(c) Graph $y = -x^2 + 1$. Label the vertex and x-intercepts with coordinates for full credit.

(6)

Reference Function:

$R(x) = x^2$



(3) for the correct graph.

Vertex: $\boxed{(0,1)}$ (1)

x-intercepts: $\boxed{(1,0) \;\&\; (-1,0)}$

(1) (1)

10. $\boxed{\text{15 points}}$ Solve the following equations.

(3) (a) $x = \log\left(100^{1011}\right)$. Your answer should be an integer. **Hint:** A calculator will not help you.

$x = \log(100^{1011}) \Rightarrow x = \underline{1011 \cdot \log(100)} = 1011\underline{(2)} = \underline{2022}$

(1) (1) (1)

Solution: $\boxed{2022}$

(6) (b) $2\left(e^{2x-5}\right) - 4 = 10$. Give the exact answer, with no calculator approximation.

$2(e^{2x-5}) - 4 = 10$

$+4 \quad +4$

$2(e^{2x-5}) = 14$

$\Rightarrow e^{2x-5} = 7$ (2)

$\rightarrow \ln(e^{2x-5}) = \ln(7)$

$\Rightarrow 2x - 5 = \ln(7)$ (2) (for getting rid of ~~ex~~ the ~~ex~~ exponential form for x

$\Rightarrow x = \dfrac{\ln(7)+5}{2}$

They might do. $e^{2x-5} = 7 \xrightarrow{\text{def of } \ln\text{"}} 2x-5 = \ln$

Solution: $\boxed{x = \dfrac{\ln(7)+5}{2}}$ (2)

(6) (c) $\log_2(x) + \log_2(x-1) = 1$. Check the domain for full credit.

Domain:

we require $x > 0$ & $x - 1 > 0 \Rightarrow x \in \mathbb{R}$ s.t $x > 1$ (2)

$\log_2(x) + \log_2(x-1) = 1 \xrightarrow{\text{Condense}} \log_2(x(x-1)) = 1$ (1)

(1)

$\Rightarrow \underline{x(x-1) = 2^1} \Rightarrow x^2 - x = 2 \Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x-2)(x+1) = 0 \Rightarrow \underline{x = 2, -1}$ (potential solutions)

$x = -1$ is not in the domain! (1)

Solution: $\boxed{x = 2}$ (1)

11. $\boxed{\text{15 points}}$ An investor has \$10,000 to invest. Suppose the money is invested at a rate of 7% compounded monthly. How many years will it take the investor to earn \$15,000?

Observations: lump sum in → lump sum out
interest compounded monthly

Formula needed: Compound Interest $S = P(1+\frac{r}{n})^{nt}$ (3)

Variables: $S = 15000$ (1) $r = .07$ (1) $t = \text{unknown}$

$P = 10000$ (1) $n = 12$ (1)

Calculations: $15000 = 10000\left(1+\frac{.07}{12}\right)^{12t}$

$$\Rightarrow \frac{15}{10} = \left(1+\frac{.07}{12}\right)^{12t}$$

$$\Rightarrow \ln(3/2) = \ln\left(\left(1+\frac{.07}{12}\right)^{12t}\right)$$ (5) (for putting "t" outside of "ln"

$$\Rightarrow \ln(3/2) = 12t\ln\left(1+\frac{.07}{12}\right)$$

$$\Rightarrow t = \frac{\ln(3/2)}{12\ln\left(1+\frac{.07}{12}\right)} = 5.81$$ (3)

Years: $\boxed{\text{5.81 years}}$

(round your answer to two decimal places)

12. $\boxed{\text{15 points}}$ Magnolia owes \$5,000 in student debt, to be paid off in regular payments over 4 years. If the interest rate is 3%, compounded monthly, what should her monthly payments be?

Observations: Loan!

Formula needed: Periodic Payment of an Amortized Loan (3)

$$R = S\left(\frac{r_c}{1-(1+r_c)^{-N}}\right)$$

Variables:

R = unknown $\quad r_c = r/n = .03/12$ (2)

S = 5000 $\quad N = n \cdot t = 12(4)$ (2)

(2)

Calculations: $R = 5000\left(\frac{(\frac{.03}{12})}{1-(1+\frac{.03}{12})^{-48}}\right)$

$= \$110.67$ (6)

Monthly payment: $\boxed{\$110.67}$

(round your answer to two decimal places)

13. $\boxed{\text{15 points}}$ Tony and Maria have a son. They decide to enroll in a savings plan on his second birthday to save for his college education. They want to have \$250,000 saved by the time their son turns 18. Assume their investment earns 6% every year, compounded quarterly, and that they will make deposits at the end of each quarter. How large must their quarterly deposits be to reach their investment goal?

Observations: multiple payments in → lumpsum out
deposits at the end of each quarter

(2)

Formula needed: Future value of an ordinary annuity

$$S = R\left(\frac{(1+r_c)^N - 1}{r_c}\right)$$

Variables:

$$\Rightarrow R = S\left(\frac{r_c}{(1+r_c)^N - 1}\right) \quad (3)$$

$S = 250000$ (1)

$R =$ unknown

$r_c = r/n = .06/4$ (2)

$N = n \cdot t = 4(16) = 64$ (2)

Calculations: $R = 250000\left(\frac{\left(\frac{.06}{4}\right)}{\left(1+\frac{.06}{4}\right)^{64} - 1}\right)$ (5)

$= 2353.84$

Quarterly deposit: $\boxed{\$2353.84}$

(round your answer to two decimal places)

14. $\boxed{\text{15 points}}$ Imagine Tony and Maria from the previous problem have reached their goal. Their son is starting college and their savings plan contains \$250,000. At the same annual interest rate of 6% compounded quarterly, how much money should their son withdraw from the account at the beginning of each quarter if he plans to spend down the entire account in 4 years?

Observations: lump sum in → multiple withdrawals out
withdrawals at the beginning of each quarter

Formula needed: Present value of an Annuity Due (2)

$$P_{due} = R \cdot (1+r_c)\left(\frac{1-(1+r_c)^{-N}}{r_c}\right) \quad (3)$$

Variables:

$P_{due} = 250000$ (1) $\quad r_c = r/n = .06/4$ (2)

$R = $ unknown $\quad N = n \cdot t = 4(4)$ (2)

Calculations:

$$P_{due} = R(1+r_c)\left(\frac{1-(1+r_c)^{-N}}{r_c}\right) \Rightarrow R = P_{due}\left(\frac{r_c}{(1+r_c)(1-(1+r_c)^{-N})}\right)$$

$$R = 250000\left(\frac{(\frac{.06}{4})}{(1+\frac{.06}{4})(1-(1+\frac{.06}{4})^{-16})}\right) \quad (5)$$

$= 17429.82$

Withdrawal amount: $\boxed{\$17429.82}$

(round your answer to two decimal places)