6.5 (cont)

Ex 1 Population of U.S. was 3.9 million in 1790 + 17.8 million in 1960. If the rate of growth is assumed proportional to the population, what estimate would you give for the population in 2000? (compare your answer w/ actual population of 275 million.)

let $t_0 = 0$ in 1790 then in 1960 $t_1 = 170$

Population is dependent on time $\Rightarrow P = P(t)$.

Basically, we have 2 pts our exponential curve goes thru. ① $(0, 3.9)$ and ② $(170, 17.8)$ where P is in millions

Fit this with $P = P_0 e^{rt}$

① $3.9 = P_0 e^0 \Rightarrow P_0 = 3.9$

② $17.8 = 3.9 e^{r(170)} \Rightarrow \frac{17.8}{3.9} = e^{170r}$

$\Rightarrow r = \frac{1}{170} \ln\left(\frac{17.8}{3.9}\right) \simeq 0.0224753$

$\Rightarrow P = 3.9 e^{0.0224753t}$ is function for population

in year 2000, $t = 2000 - 1790 = 210$

$P(210) = 3.9 e^{0.0224753(210)} \simeq 437.378$

$\Rightarrow$ about 437.378 million

6.5 (cont)

Ex 2 If a radio-active substance loses 15% of its radioactivity in 2 days, what is its half-life?

half-life means time it takes for half the substance to decay.

we have $y = y_0 e^{kt}$ & we know when $t = 2$ days $y = 0.85 y_0$

(i.e. if we lost 15% of y_0, we still have 85% of y_0 left)

$\Rightarrow 0.85 \cancel{y_0} = \cancel{y_0} e^{2k}$

$k = \frac{\ln 0.85}{2} \simeq -0.0812595$

half life: $(t, \frac{1}{2} y_0)$

$\Rightarrow y = y_0 e^{-0.0812595 t}$

$\frac{1}{2} \cancel{y_0} = \cancel{y_0} e^{-0.0812595 t}$

$\frac{\ln(\frac{1}{2})}{-0.0812595} = t \Rightarrow t \simeq 8.53$

$\boxed{\text{so about } 8.5 \text{ days}}$

Compound Interest

Put \$100 in a bank w/ 8% interest compounded quarterly.

$\left(\frac{0.08}{4} = 0.02 \text{ quarterly interest} \right)$

after qtr	value
0	100
1	$100(1+0.02)$
2	$[100(1+0.02)](1+0.02) = 100(1+0.02)^2$
3	$[100(1+0.02)(1+0.02)](1+0.02) = 100(1+0.02)^3$
⋮	
n	?

6.5 (cont)

Let's go back to ★ question. (continuous compounding)

$$A(t) = \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt}$$

$$= A_0 \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = A_0 \lim_{n\to\infty}\left[\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right]^{rt}$$

$$= A_0\left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{\frac{n}{r}}\right]^{rt}$$

as $n\to\infty$

$\frac{r}{n}\to 0$

let $h=\frac{r}{n}$

$$= A_0\left[\underbrace{\lim_{h\to 0}(1+h)^{1/h}}_{e}\right]^{rt}$$

$\boxed{A(t) = A_0 e^{rt}}$ continuous compounding

EX 3 If Methuselah's parents had put \$100 in the bank for him at birth and he left it there, what would Methuselah have had at his death (969 years later) if interest was 4% compounded annually?

$A_0 = \$100$

$r = 0.04$

$n = 1$

$t = 969$

$$A = A_0\left(1+\frac{r}{n}\right)^{nt}$$

$$A = 100(1+0.04)^{969} \simeq \$3{,}201{,}148{,}199{,}000{,}000{,}000$$