Ex (6) (pg 73 of the notes)

Find arc length for

$x = a\cos t + at\sin t$, $y = a\sin t - at\cos t$, $t \in [-1, 1]$

(a is a constant)

$$\frac{dx}{dt} = -a\sin t + a\sin t + at\cos t = at\cos t$$

$$\frac{dy}{dt} = a\cos t - a(\cos t - t\sin t) = at\sin t$$

$$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(at\cos t)^2 + (at\sin t)^2}$$

$$= \sqrt{a^2t^2[\cos^2 t + \sin^2 t]}$$

$$= \sqrt{a^2t^2} = |at|$$

$$\Rightarrow L = \int_{-1}^{1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_{-1}^{1} |at|\, dt = \int_{-1}^{0} |a|(-t)\, dt + \int_0^1 |a|\, t\, dt$$

$$= |a|\left(-\frac{t^2}{2}\right)\Big|_{-1}^{0} + |a|\left(\frac{t^2}{2}\right)\Big|_0^1$$

$$= \frac{|a|}{2}\big(0 - (-1)\big) + (1 - 0)\big)$$

$$= \frac{|a|}{2}(1+1) = \frac{2|a|}{2} = |a|$$

(you also could have recognized that

$$\int_{-1}^{1} |at|\, dt = 2\int_0^1 |at|\, dt$$

because $f(t) = |at|$ is even fn

$$= 2\int_0^1 at\, dt$$
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