4.1 & 4.2: Divisibility & Prime/Composite Numbers

Prime number--a natural number with exactly two factors, namely 1 and itself. ex 7

Composite number--a natural number with more than two factors. ex 4

Question: Is 1 prime or composite?

neither because it's multiplicative identity.

Divides (wording/notation)

a | b (read "a divides b")

other equivalent wording:

- a is a factor of b
- a is a divisor of b
- b is a multiple of a
- b is divisible by a
- 5 can be divided into 25

ex 5 | 25

- 5 divides 25
- 5 is a factor of 25
- 5 is a divisor of 25
- 25 is a multiple of 5
- 25 is divisible by 5
- 25 can be divided by 5

Ex. 5 | 25 but $3 \nmid 25$ (i.e. 5 divides 25 but 3 does not divide 25)

Tests for Divisibility

2 a # is divisible by 2 if it's even; i.e. if the 1's digit is divisible by 2.

5 a # ends in 0 or 5.

10 ends in 0.

3 add the digits; if that is divisible by 3, then the original # is div. by 3

ex 12348

9 add the digits; if that is divisible by 9, then the original # is div. by 9.

4 if the last 2 digits are divisible by 4

ex 7124 $7124 = 7100 + 24$

8 if the last 3 digits are divisible by 8

ex $9713888 = 9713000 + 888 = 9713(1000) + 888$

$1000 = 10^3 = (2 \cdot 5)^3 = 2^3 \cdot 5^3$

6 if it's divisible by both 3 and 2

ex 5034

Why does the divisibility test by 9 work? (And, can we extend such a rule to other bases?) For this argument, use a generic five-digit number.

ex 54369

$$= 5(10^4) + 4(10^3) + 3(10^2) + 6(10) + 9(1)$$

$+5-5 \quad +4-4 \quad +3-3 \quad +6-6$

$$= 5(10^4 - 1) + 4(10^3 - 1) + 3(10^2 - 1) + 6(10 - 1) + 9$$

$+5+4+3+6$

$$= \left(5(9999) + 4(999) + 3(99) + 6(9)\right) + (9 + 5 + 4 + 3 + 6)$$

$$= \underbrace{9\left(5(1111) + 4(111) + 3(11) + 6(1)\right)} + (\underbrace{9 + 5 + 4 + 3 + 6})$$

Examples:

(a) Is 7,465,$\underbrace{832}$ a multiple of 4? "7,465,832 divisible by 4?"

yes

(b) Is 8 a factor of 131,$\underbrace{888}$? yes

(c) Is 497 prime? $20 < \sqrt{497} < 25$ $20^2 = 400$

$497 \div 7 = 71 \Rightarrow$ no, it's composite

(d) Is this true or false? 3 | 6n for any natural number n

6 is divisible by 3 so any multiple of 6 is also divisible by 3

(e) Is this true or false? 0 | 0

(f) True or false: If a and b are both ~~whole~~ natural numbers, and $5 \nmid a$ and $5 \nmid b$, then $5 \nmid (a + b)$.

ex $5 \nmid 3$ and $5 \nmid 7$ but $5 \mid (3+7)$

(g) Is the number 57,729,364,583 divisible by 2, 3, 5, 6, 8, or 9?

no no no no no no

$5+7+7+2+9+3+6+4+5+8+3 = 59$

(h) Finish this number so that it is divisible by 9: 123456 7 8

(i) I know that 12 pizzas cost \$240.84. Can you fill in the missing digits? How much was each pizza? \$20.07

\$210.84 \$220.80 \$200.88

\$270.84 \$230.88

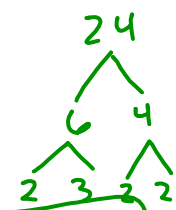
\$260.88

\$290.88

Fundamental Theorem of Arithmetic

There's only one prime factorization for a composite number.

ex



$\boxed{24 = 2^3 \cdot 3}$

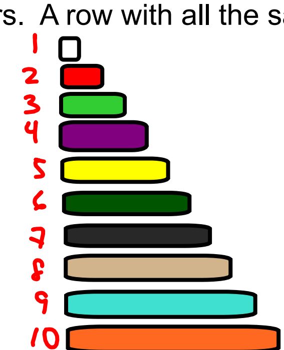
Prime factorization-- the list of prime factors for any composite #.

Examples:

(a) Factor 513 completely.

$$513 = 3(171) = 3^2(57)$$
$$= 3^2(3 \cdot 19)$$
$$= 3^3 \cdot 19$$

(b) Colored rods are used in many elementary classrooms. The rods vary in length from 1 to 10 cm (in whole number lengths). Various lengths have different colors. A row with all the same color rods is called a one-color train.



white is 1 unit

red is 2 units

light green is 3 units

purple is 4 units

yellow is 5 units

dark green is 6 units

black is 7 units

brown is 8 units

blue is 9 units

orange is 10 units

(i) What rods can be used to form one-color train for 18?

white, red, light green, dark green, blue

(ii) What one-color trains are possible for a length of 24?

white, red, light green, purple, dk green, brown

(iii) If a whole-number length can be represented by an all-red train, an all-green train, and an all-yellow train, what is the least number of factors it must have? What are they?

3 (they must be 2, 3 and 5)

Ex. Answer these questions about the factors of 97.

(a) If 2 is not a divisor of 97, can any other multiple of 2 be a divisor of 97? no

(b) If 3 is not a divisor of 97, can any multiple of 3 be a divisor of 97?
no

(c) If 5 is not a divisor of 97, what other numbers cannot be divisors of 97? 10, 15, 20, 25, . . .

(d) What numbers must you check to see if 97 has any factors before you decide it's prime?

$9 < \sqrt{97} < 10$

check: 2, 3, 4, . . ., 9

it's prime

(also not divisible by 7)

Ex. Find the prime factorizations for the following numbers.

(a) $36^{10}(49^{20})(6^{15})$ $= (2^2 \cdot 3^2)^{10} (7^2)^{20} (2 \cdot 3)^{15}$

$= 2^{20} \cdot 2^{15} \cdot 3^{20} \cdot 3^{15} \cdot 7^{40}$

$= 2^{35} \cdot 3^{35} \cdot 7^{40}$

(b) 2(3)(5)(7)(11) + 1

$= (21 \cdot 11)10 + 1$

$= 2310 + 1 = 2311$

check $48 < \sqrt{2311}$

~~11~~, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47

(c) $2(3^4)(5^{110})(7) + 4(3^4)(5^{110})$

$= 2(3^4)(5^{110})[7 + 2] = 2(3^4)(5^{110})(9) = 2(3^6)(5^{110})$

How many divisors does a composite number have?
(or factors)

Let's first try this with a few examples.

(a) 32 $= 2^5$

list all factors: $2^0, 2^1, 2^2, 2^3, 2^4, 2^5$

1, 2, 4, 8, 16, 32

\# of factors $= 6$

(b) 75 $= 3^1 \cdot 5^2$

list all factors: 1, 3, 5, 15, 25, 75

\# of factors $= 6$

exponents on 3: 0, 1
exp. on 5: 0, 1, 2

1	3	5	15	25	75
$3^0 \cdot 5^0$	$3^1 \cdot 5^0$	$3^0 \cdot 5^1$	$3^1 \cdot 5^1$	$3^0 \cdot 5^2$	$3^1 \cdot 5^2$

\# factors $= (1+1)(2+1) = 6$

(c) $2^3(5^2)(3)$
$= 600$
$= \underbrace{2 \cdot 2} \cdot 2 \cdot 5 \cdot 5 \cdot 3$

factors: 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 25, 30, 40, 50, 60, 75, 100, 120, 150, 200, 300, 600

\# factors $= 24$

1	2	3	4	5	6	8
$2^0 3^0 5^0$	$2^1 3^0 5^0$	$2^0 3^1 5^0$	$2^2 3^0 5^0$	$2^0 3^0 5^1$	$2^1 3^1 5^0$	$2^3 3^0 5^0$
10	12	15	20	24	25	30
$2^1 3^0 5^1$	$2^2 3^1 5^0$	$2^0 3^1 5^1$	$2^2 3^0 5^1$	$2^3 3^1 5^0$	$2^0 3^0 5^2$	$2^1 3^1 5^1$

\# of factors for $2^3 3^1 5^2 = (4)(2)(3)$
$= 24$

(d) Is this enough to see a pattern emerging? If so, predict the total number of factors for $2^4(3^5)(5)(7^6)$

\# of factors $= (5)(6)(2)(7) = 420$

Formula: If a composite number has prime factorization of $p_1^{n}(p_2^{m})(p_3^{r})$, then it has a total of this many factors:

\# factors $= (n+1)(m+1)(r+1)$

ex $2^2 \cdot 3 \cdot 5^3 \cdot 11^9$ \# factors $= 3 \cdot 2 \cdot 4 \cdot 10 = 240$

Quiz 7

1) (a) $15^5 \cdot \underbrace{3^4 \cdot 5^4}_{15^4} = 15^9$

(b) $8^5 \cdot 2^5 \cdot 16^2 = 16^5 \cdot 16^2 = 16^7$

$(2^3)^5 \cdot 2^5 \cdot (2^4)^2 = 2^{15} \cdot 2^5 \cdot 2^8 = 2^{28}$ (or 4^{14})

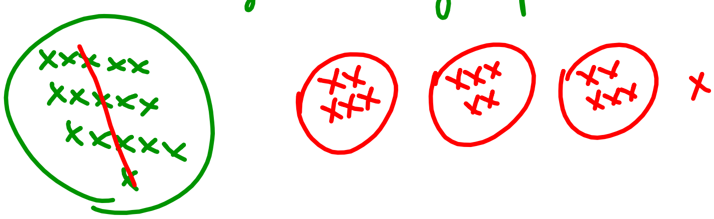
(c) 0^0 undefined

$2^0 = 1$	$0^2 = 0$
$1^0 = 1$	$0^1 = 0$
$0^0 = 1?$	$0^0 = ?$

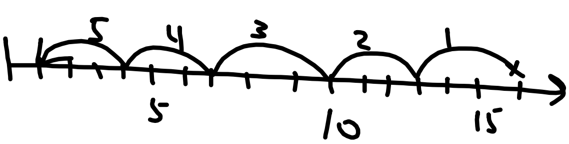
because two arguments give different possible values, 0^0 cannot be defined

2) $16 \div 3 = ?$

(a) I have 16 students to split into 3 groups. How many in each group?



(b) We have 16 students, with 3 students at each table, w/ any remaining students in the lunch line. How many tables?



① * $16 \div 3 = 5$

② $16 \div 3 = 5\ R1$

③ $16 \div 3 = 5\frac{1}{3}$

④ $16 \div 3 = 6$

3) $\underbrace{\{3, \cancel{5}, 6, \cancel{9}, \cancel{12}, 20\}}_{A} - \underbrace{\{4, 5, 9, 12, 15\}}_{B} = \{3, 6, 20\}$

$A - B$

4) (a) $45(61) + 49(45) = 45(61 + 49) = 45(110)$
$= 4500 + 450 = 4950$

(b) $6284(5) = \frac{6284}{2}(10) = 31420$

(c) $8200 \div 25 = \frac{8200}{25}\left(\frac{4}{4}\right) = 82(4) = 328$

(d) $9539 + 701 = 9540 + 700 = 10{,}240$

(e) $8706 - 87 = 8706 - 86 - 1 = 8620 - 1 = 8619$

$8706 - 87 = 8706 - 87 + 1 - 1$
$= 8707 - 87 - 1$

5) (a) $4671(304) \simeq 5000(300) = 1{,}500{,}000$

(b) $98034 - 5689 \simeq 100{,}000 - 6000$
$= 94{,}000$

4.1 MC

#10) (a) base 10,

in base 4, ex 3202_4

in base 6, ex 5514_6

even bases

(b) base 3, ex 212_3

base 5,

odd bases

because in base b (where b is odd), we can test for divisibility by (b-1) by adding up the digits (like the 9 test in base 10) & since b is odd, (b-1) is even, then if the # is divisible by (b-1) (an even #), then it's also divisible by 2.

4.1 A

8) (a) T

sum of #s (where each # is divisible by 3) is also divisible by 3

(b) false, ex 12

(c) false

A 14) (b) ex 822

$$= 800+20+2+8-8+2-2$$

$$= 800-8+20-2+(2+8+2)$$

$$= 8(100-1)+2(10-1)+(8+2+2)$$

$$= \underbrace{8(99)+2(9)}+(8+2+2) = \underbrace{9(8(11)+2(1))}_{\text{divisible by 3}} + \underbrace{(8+2+2)}_{\text{leftover}}$$

likewise

$$823 = \underbrace{9(8(11)+2(1))}_{\text{divisible by 3}} + \underbrace{(8+2+3)}_{\text{some groups of 3 + 1 remainder}}$$

4.2 HW

B8) divisible by $\cancel{1},\cancel{2},\cancel{3},\cancel{4},\cancel{5},\cancel{6},\cancel{7},\cancel{8},\cancel{9},\cancel{10},\cancel{11},\cancel{12}$

$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$

B10) $97^2 = \underline{9}\ \underline{4}\ \underline{0}\ \underline{9}$ has exactly 3 pos. factors

3 total factors, let's call the # n, | biggest 2-digit prime # is 97

list of factors: 1, , n

ex factors of 25: 1, 5, 25

ex ~~1, 10, 100~~ ex 1, 19, 361

$19^2 = (20-1)(20-1)$

$= 20^2 - 40 + 1 = 361$

B7) 435 total; each committee has 2, 3, 4, ..., or 29 members

each committee is same size

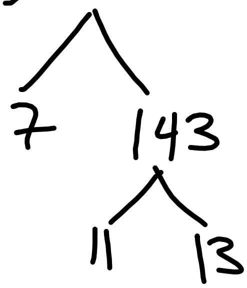
(a) possible sizes: 3, 5, 15, 29

(b) # committees: 145 87 29 15

B1) smallest # divisible by 4 primes $= 2 \cdot 3 \cdot 5 \cdot 7 = 210$

4.2B #6)

(a) 1001



$1001 = 7 \cdot 11 \cdot 13$

(d) $111^{10} - 111^{9}$

$= 111^{9}(111 - 1)$

$= 111^{9}(110)$

$= (3 \cdot 37)^{9}(10 \cdot 11)$

$= \boxed{3^{9} \cdot 37^{9} \cdot 2 \cdot 5 \cdot 11}$

$= 2 \cdot 3^{9} \cdot 5 \cdot 11 \cdot 37^{9}$

$5^3 - 5^2 \cancel{\overset{?}{=}} 5$

$= 125 - 25$

$= 100$

$5^3 - 5^2 = 125 - 25$

$= 25(5-1)$

$5^2(5-1)$

4.2B

5) is 503 prime?

$\sqrt{503} \simeq 22.43$

$\Rightarrow$ you only check for factors up to 22.

$\Rightarrow$ so biggest prime factor to check is 19.

4.2 A 10) least __ __ __ $= n$ w/ exactly 5 factors

factors: $1, a, c, b, n$ $(ab = n)$

$c^2 = n$

$\Rightarrow c$ must be 2-digit #

guess:

~~1, 2, 11, ,~~

~~1, , 13, ,~~

~~1, 2, 15,~~

~~1, 2, 16,~~

~~1, 3, 5, 15, n~~

$\boxed{1, 5, 25, 125, 625}$

WS "Prime concerns"

$P_{41} = 41^2 - 41 + 41 = 41^2$ composite

5) $p_n = n^2 - n + 41$

any multiple of 41 gives P_n composite

$n = 82$ $\quad p_{82} = 82^2 - 82 + 41 = (41\cdot 2)(41\cdot 2) - 41(2) + 41(1)$

$= 41\left[41(4) - 2 + 1\right]$

at least two factors $\Rightarrow P_{82}$ is composite

try $P_{42} = 42^2 - 42 + 41 = 42^2 - 1 = 1763$

Can you... $17 = 1 \,\underline{+}\, 2 \,\underline{-}\, 3 \,\underline{-}\, 5 \,\underline{+}\, 7 \,\underline{-}\, 11 \,\underline{+}\, 13 \,\underline{+}\, 13$

Twin Primes

products	
$15 = 3\cdot 5$	$15 + 1 = 16 = 4^2 = 2^2\cdot 2^2$
$35 = 5\cdot 7$	$35 + 1 = 36 = 6^2 = 2^2\cdot 3^2$
$143 = 11\cdot 13$	$143 + 1 = 144 = 12^2 = 2^2\cdot 6^2 = 2^4\cdot 3^2$
$323 = 17\cdot 19$	$323 + 1 = 324 = 18^2 = 2^2\cdot 3^4$
$899 =$	$899 + 1 = 900 = 30^2 = 3^2\cdot 2^2\cdot 5^2$