10.4 Parametric Eqns

Ex 1 Eliminate the parameter t. Graph the curve and tell if it's simple & closed.

$x = t^2+1, \quad y = t-1, \quad -2 \le t \le 2$

Vocab

parametric eqn

closed curve

simple curve

If $x = f(t)$, $y = g(t)$ and $f'(t), g'(t)$ exist and are continuous (and $f'(t) \neq 0$ on $\alpha < t < \beta$), then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$$

10.4 (cont)

Ex2 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, given $x = \cot t - 3$

(do not eliminate parameter) $y = -2\csc t$

$t \in (0, \pi)$

Ex3 Find the length of the curve given by $x = t^3$, $y = 6t^2$, $t \in [1, 4]$

Length of a curve

$$L = \int_a^b \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}_{\text{a little bit of arc length}}$$

This is "calc" version of what famous theorem?

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11.1 Cartesian Coordinates in 3-space

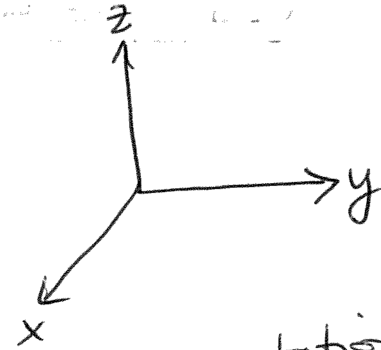
Ex1 Plot:

(a) the ________ $(1, 3, -4)$

(b) the ________ $3x - 2y + z = 6$

(c) the ________ $x = 4$

(d) the ________ w/ center $(2, 4, 7)$ and radius 1.



(default orientation of 3d axes in your book)

(x_1, y_1, z_1) and (x_2, y_2, z_2) are two 3-d pts

① distance between them

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

② midpt:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

③ Sphere w/ center (h, k, l) and radius r:

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

11.1 (cont)

Ex 2 Describe the graph $xz=0$ in 3-space.

Ex 3 Calculus hero is at $(5,4,-2)$ and there is a treasure at $(1,-3,-1)$. Calculus hero's lasso is 10 units long. Can he/she use the lasso to reach the treasure?

Qn: In 2-d, I can draw axes and a pt and estimate what pt it is. Can I do this in 3-d also?

Ex 4 Calculus hero's nemesis lands halfway between Calculus hero and the treasure. At what pt is the nemesis?

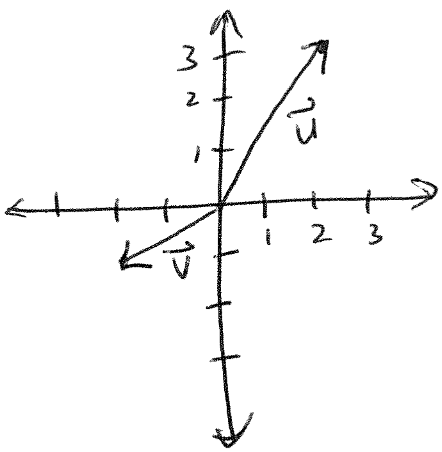
Ex 5 Find the center and radius of the sphere.

$$x^2+y^2+z^2-2x+18z=-57$$

11.2 Vectors

Ex 1 vectors $\vec{u}$ and $\vec{v}$ are shown.

(a) Describe $\vec{u}$ and $\vec{v}$ algebraically.



(b) Find $\|\vec{u}\|$ and $\|\vec{v}\|$ and $\hat{u}$ and $\hat{v}$.

(c) Calculate (i) $2\vec{u}+\vec{v}$ algebraically and geometrically.
and (ii) $\vec{u}-\vec{v}$

Ex2 The water from a fire hose exerts a force of 150 lbs on the person holding the hose. The nozzle weighs 10 lbs. What is the magnitude and direction of the force exerted by the person holding the hose?

Ex 3 An airplane flies 400 mph in still air. How should the airplane be headed and how fast will it be flying (wrt the ground) if it flies against a 20 mph wind blowing N40°W? Assume the plane wants to travel due north.

11.3 The Dot Product

Ex 1 let $\vec{u} = \langle 3, -1, 1 \rangle$ and
$\vec{v} = \langle 2, 1, 0 \rangle$.

Find

(a) $\|\vec{u}\|(\vec{u} \cdot \vec{v})$

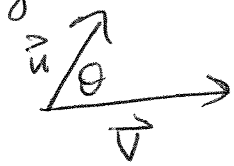
(b) angle between $\vec{u}$ and $\vec{v}$

(c) Is angle between $\hat{u}$ and $\hat{v}$ different or same as angle between $\vec{u}$ and $\vec{v}$? why?

$\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

angle between them $= \theta$



① $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

② $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

$= [u_1\ u_2\ u_3] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

<u>note</u>: ① $\Rightarrow$

③ $\hat{u} \cdot \hat{v} = \cos\theta$

and notice

④ $\|\vec{u}\|^2 = \vec{u} \cdot \vec{u}$

11.3 (cont)

Ex 2 Let $\vec{u}$ and $\vec{v}$ be vectors and θ be angle between $\vec{u}$ and $\vec{v}$. Write three more statements that are different but equivalent to first one.

(i) $\vec{u}$ and $\vec{v}$ are orthogonal

(ii)

(iii)

(iv)

Qn:
what is a normal vector?

Ex 3 Show $\langle -1, 2, 0 \rangle$ is parallel to plane $6x + 3y + z = 2$

11.3 (cont)

Ex 4 The planes

$2x+3y-5z=2$ and

$2x+3y-5z=9$ are parallel.

What is distance between them?

④ shortest distance from pt (x_0, y_0, z_0) to a plane $Ax+By+Cz=D$

$$L = \frac{|Ax_0+By_0+Cz_0-D|}{\sqrt{A^2+B^2+C^2}}$$

Important Facts/Formulas

① angle between $\vec{u}$ and $\vec{v}$

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)$$

$$= \cos^{-1}(\hat{u}\cdot\hat{v})$$

② projection of $\vec{u}$ onto $\vec{v}$:

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$$

$$= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$$

$$= (\vec{u}\cdot\hat{v})\hat{v}$$

③ eqn of a plane

(a) $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$

$\Longleftrightarrow \langle A,B,C\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0$

where (x_0,y_0,z_0) is a pt on the plane and $\langle A,B,C\rangle$ is normal vector

(b) can also be written as $Ax+By+Cz=D$ where $D=Ax_0+By_0+Cz_0$

Ex 5 Find the eqn of the plane through $(-1, 2, -3)$ and parallel to $2x + 4y - z = 6$.

11.4 The Cross Product

Ex 1 $\vec{a} = \langle 3, 4, 1 \rangle$
$\vec{b} = \langle -2, 0, 5 \rangle$
$\vec{c} = \langle 2, -1, 3 \rangle$

Find

(a) $(\vec{a} + \vec{b}) \times \vec{c}$

(b) $(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{a})$

Cross Product Facts

$\vec{u} = \langle u_1, u_2, u_3 \rangle$
$\vec{v} = \langle v_1, v_2, v_3 \rangle$

① $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

produces a ____________

② $\vec{u} \times \vec{v} = \vec{0} \iff$

____

③ $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta$

$\Rightarrow \|\hat{u} \times \hat{v}\| = \sin\theta$

Compare to Dot Product:

① $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

produces a ____________

② $\vec{u} \cdot \vec{v} = 0 \iff$

____

③ $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

$\hat{u} \cdot \hat{v} = \cos\theta$

11.4 (cont)

Ex 2 Find all vectors $\perp$ to both $\vec{a} = \langle -2, 5, -2 \rangle$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 4\hat{k}$

$\vec{w}, \vec{u}, \vec{v} \in \mathbb{R}^3$, $c \in \mathbb{R}$

(4) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

(5) $\vec{u} \times (\vec{v} + \vec{w})$
$= (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$

(6) $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v}$
$= \vec{u} \times (c\vec{v})$

(7) $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$

(8) $(\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w})$

(9) $\vec{u} \times (\vec{v} \times \vec{w})$
$= (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

(10) area of parallelogram spanned by $\vec{u}$ and $\vec{v}$:
$A = \|\vec{u} \times \vec{v}\|$

11.4 (cont)

Ex 3 Find area of Δ with vertices $(1,2,3)$, $(3,1,5)$ and $(4,5,6)$.

Ex 4 Find eqn of plane through $(2,-1,4)$ that is $\perp$ to both the planes $x-3y+2z=7$ and $2x-2y-z=-3$.

Ex 5 Given pts $A(1,0,5)$, $B(2,1,3)$ and $C(4,2,9)$, find the eqn of the plane through them.

Strategy

① create two vectors (say $\vec{AB}$ and $\vec{BC}$)

② take cross product of these 2 vectors to create a normal vector to the plane

③ use one of the pts and the normal vector to create the plane eqn.

11.5 Vector-valued Functions and Curvilinear Motion

Ex1 Find limit, if it exists.

$$\lim_{t \to -2} \left[\frac{2t^2 - 10t - 28}{t+2} \hat{\imath} - \frac{7t^3}{t-3} \hat{\jmath} \right]$$

f, g, h are $\mathbb{R}$-valued fns (i.e. input = $\mathbb{R}$ number and output = $\mathbb{R}$ number)

vector-valued fn F is

$$\vec{F}(t) = f(t)\,\hat{x} + g(t)\,\hat{y} + h(t)\,\hat{z}$$
$$= f(t)\,\hat{\imath} + g(t)\,\hat{\jmath} + h(t)\,\hat{k}$$
$$= \langle f(t), g(t), h(t) \rangle$$

input for F: ____________

output for F: ____________

Ex2 what is domain of $\vec{r}(t) = \ln(t-1)\,\hat{\imath} + \sqrt{20-t}\,\hat{\jmath}$

$\vec{r}(t)$ is ____________

$\vec{v}(t)$ is ____________

$\vec{a}(t)$ is ____________

Qn: what is difference between speed and velocity?

11.5 (cont)

Ex 3 Given position vector $\vec{r}(t) = t^6\hat{\imath} + (6t^2-5)^6\hat{\jmath} + t\hat{k}$

(a) find $\vec{v}(t)$

(b) find $\vec{a}(t)$

(c) what is speed when $t=1$?

Ex 4 Find the length of the curve w/ given vector eqn $\vec{r}(t) = t\cos t\,\hat{\imath} + t\sin t\,\hat{\jmath} + \sqrt{2t}\,\hat{k}$, $t \in [0, 2]$.

11.6 Lines and Tangent lines

Ex 1 Find the eqns of the line through $(-1,3,6)$ and parallel to $\langle 2,0,5\rangle$.

Symmetric eqns of a line:

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

(eliminates the parameter)

In 2-d:

point $(0,b)$ (y-intercept) and a slope (direction information)

$\Rightarrow$ line: $y=mx+b$

In 3-d:

point (x_0,y_0,z_0)

and a direction vector $\vec{v}=\langle a,b,c\rangle$

$\Rightarrow$ line: $\vec{r}=\vec{r}_0+\vec{v}t$

where $\vec{r}_0$ = vector from origin to pt on line $=\langle x_0,y_0,z_0\rangle$

t is a parameter

i.e. $\langle x,y,z\rangle=\langle x_0,y_0,z_0\rangle+\langle a,b,c\rangle t$

$$\Leftrightarrow \left.\begin{array}{l} x=x_0+at \\ y=y_0+bt \\ z=z_0+ct \end{array}\right\} \text{parametric eqns of a 3-d line}$$

Ex 2 Find the line eqns for the line through $(2,-4,5)$ that is parallel to the plane $3x+y-2z=5$ and $\perp$ to line $\frac{x+8}{2}=\frac{y-5}{3}=\frac{z-1}{-1}$

To find the line

① Given a pt and direction vector $\vec{v}$

$\vec{r}=\vec{r}_0+\vec{v}t$

($\vec{r}_0$ = vector form of given point

OR

② Given 2 pts:

(a) find $\vec{v}$ = vector from one pt to the other

(b) $\vec{r}=\vec{r}_0+\vec{v}t$ where $\vec{r}_0$ is vector form of one of the given pts

Ex 3 Find the eqn of the plane containing the line $x=3,\ y=1+t,\ z=2t$ and perpendicular to the intersection of the planes $2x-y+z=0$ and $y+z+1=0$

Ex 4 Find the line of intersection between the 2 planes $x-3y+z=5$ and $6x-5y+4z=3$.

11.6 (cont)

Ex 5 Find the line that is tangent to the curve $x = 2t^2$, $y = 4t$, $z = t^3$, at $t = 2$.

11.8 Surfaces in 3-space

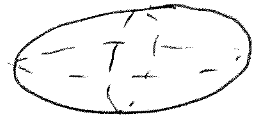
Ex1 Name and sketch the graph.

(a) $z^2+x^2=9$

(b) $y^2+z^2-4x^2+4=0$

① ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



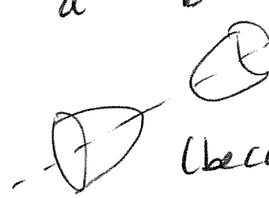
② hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

(major axis: z-axis because it follows −)

③ hyperboloid of 2 sheets

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



major axis: x-axis (because it's the only one not subtracted)

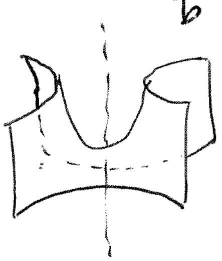
④ elliptic paraboloid

$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

(major axis: z-axis, because it's the variable not squared)

⑤ hyperbolic paraboloid

$$z=\frac{y^2}{b^2}-\frac{x^2}{a^2}$$

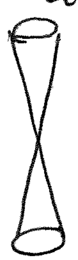


(looks like a pringles chip)

major axis: z-axis (because z is not squared)

⑥ elliptic cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$



major axis: z-axis because it's only one being subtracted

⑦ cylinder: one of the variables is missing

11.8 (cont)

Ex 1 (cont) (c) $9x^2+25y^2+9z^2=225$

(d) $y^2-x^2+z=0$

(e) $2x^2-6z^2=0$

11.8 (cont)

Ex 2 What surface results when the curve $z=2y$ in the yz-plane is revolved around the z-axis? Sketch it and find the eqn for it.

11.9 Cylindrical and Spherical Coordinates

Ex 1 Change $\left(4, \frac{\pi}{3}, \frac{3\pi}{4}\right)$ from S to R coords.

Ex 2 Convert $(1, -3, 2)$ to spherical coords.

C = Cylindrical
R = rectangular
S = Spherical

Cylindrical

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\} \text{from C to R}$$

$$\left.\begin{aligned} r &= \sqrt{x^2+y^2} \\ \tan\theta &= \frac{y}{x} \\ z &= z \end{aligned}\right\} \text{from R to C}$$

notice: $r \geq 0$, $\theta \in [0, 2\pi)$

Spherical

$$\left.\begin{aligned} x &= \rho\sin\varphi\cos\theta \\ y &= \rho\sin\varphi\sin\theta \\ z &= \rho\cos\varphi \end{aligned}\right\} \text{S to R}$$

$$\left.\begin{aligned} \rho &= \sqrt{x^2+y^2+z^2} \\ \tan\theta &= \frac{y}{x} \\ \cos\varphi &= \frac{z}{\sqrt{x^2+y^2+z^2}} \end{aligned}\right\} \text{R to S}$$

note: $\theta \in [0, 2\pi)$, $r \geq 0$
$\varphi \in [0, \pi]$

11.9 (cont)

Ex 3 Sketch graph of given cylindrical or spherical eqn.

Qn: can you tell by looking at a pt if it's in spherical, cylindrical or rectangular coords?

(a) $r = 2\sin(2\theta)$

(b) $\rho = \sec \varphi$

(c) $\theta = \pi/3$

11.9 (cont)

Ex 4 Change the eqn to the indicated coord. system

(a) $\rho \sin\varphi = 2$ to cylindrical

(b) $r^2 + 6z^2 = 7$ to spherical