12.1 Functions (Fns) of Two or More Variables

Ex1 let $f(x,y,z) = \sqrt{x\cos y} + z^2$.

(a) what is the domain?

(b) what dimension space does the graph of this fn live in?

(c) Find $f(2, \pi/3, -1)$.

Ex2 Find domain of $f(x,y,z) = z\ln(xy)$

Ex 3 Sketch the level curves ($z=k$ where $k=0,1,2,3$) for $z=f(x,y)=2-x-y^2$

Ex 4 Sketch the graph of $z=f(x,y)=x^2+y^2-4$.

12.2 Partial Derivatives

(wrt = with respect to)

Ex 1 Find f_x and f_y given $f(x,y) = \ln(x^2 - y^2)$

Given $z = f(x,y)$

The partial derivative of z wrt x is

$$f_x(x,y) = z_x = \frac{\partial z}{\partial x} = \frac{\partial f(x,y)}{\partial x}$$

(likewise for partial wrt y)

$$f_y(x,y) = z_y = \frac{\partial z}{\partial y} = \frac{\partial f(x,y)}{\partial y}$$

Ex 2 Find the four second order partial derivatives for $f(x,y) = 2x^3\cos(4y)$

For f_{xyy}, work "inside to outside" for this notation.

$$f_x \xrightarrow{\text{then}} (f_x)_y = f_{xy}$$

$$\xrightarrow{\text{then}} (f_{xy})_y = f_{xyy}$$

Note:

$$f_{xyy} = \frac{\partial^3 f}{\partial x \partial^2 y}$$

(in this notation, it's from right to left in the denominator)

12.2 (cont)

Ex 3 Imagine you are on the surface $z = \frac{5\sqrt{16-x^2}}{4}$ at pt $(2, 3, \frac{5\sqrt{3}}{2})$. Find the slope of the tangent line to this pt that lies in the $x=2$ plane (i.e. the tangent to the intersection curve of the surface w/ the $x=2$ plane). Repeat for the tangent line to the pt that lies in the $y=3$ plane.

(i) what type of surface is this? Sketch it.

(ii) slope at pt in $x=2$ plane

(iii) slope at pt in $y=3$ plane

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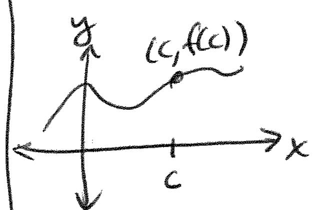
12.3 Limits & Continuity

Ex 1 Find the limit or justify that it DNE.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$

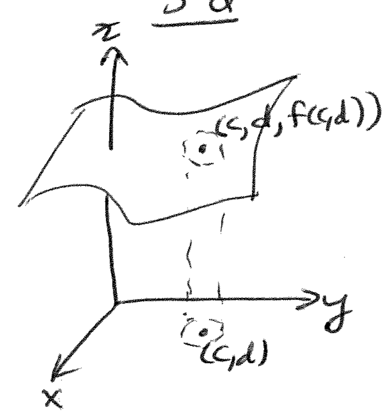
(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

2-d



Does $\displaystyle\lim_{x\to c} f(x)$ exist?
approach from left and right to see.

3-d



Does $\displaystyle\lim_{(x,y)\to(c,d)} f(x,y)$ exist?
approach from ∞ directions in disk around (c,d)

Strategies to show limit exists:

① plug in numbers, everything is fine

② algebraic manipulation
- factoring / dividing out
- use trig identities

③ change to polar coords
if $(x,y)\to(0,0) \Leftrightarrow r\to 0$.

Strategies to show limit DNE:

① show limit is different if approach from different paths

② switch to polar coords and limit DNE

12.3 (cont)

Ex 1 (cont)

(c) $\lim_{(x,y)\to(0,0)} xy\left(\frac{x^2-y^2}{x^2+y^2}\right)$

A fn $z=f(x,y)$ is continuous at (a,b) if

$f(a,b) = \lim_{(x,y)\to(a,b)} f(x,y)$

i.e. ① limit exists

② fn value is defined

③ they are the same value

Ex 2 Describe the largest set S on which f is continuous.

(a) $f(x,y) = \frac{1}{\sqrt{1+x+y}}$

(b) $f(x,y,z) = \ln(4-x^2-y^2-z^2)$

12.4 Differentiability

Ex 1 Find the gradient ∇f of $f(x,y) = 4xe^{9xy}$.

Ex 2 Find the gradient of $f(x,y) = \frac{x^2}{y}$ at input pt $(2,-1)$. Then find eqn of tangent plane at this pt.

Let $z=f(x,y)$ be a fn and (a,b) be a pt in the domain.

Gradient of f at pt (a,b):

$$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle$$

(a vector)

Tangent plane to $z=f(x,y)$ at pt (a,b):

$$z = f(a,b) - \nabla f(a,b) \cdot \langle x-a, y-b\rangle$$

Qns

① Does the gradient always exist?

② Does the tangent plane at a pt on the surface always exist?

③ How is the gradient related to differentiability?

12.4 (cont)

Ex 3 Find the eqn of the tangent "hyperplane" to $w = f(x,y,z) = 2y\cos(2\pi x) + 4x\cos(\pi y) + xz$ at input pt $(1, \frac{1}{2}, 3)$

12.5 Directional Derivatives

Let $z = f(x,y)$ be a fn, (a,b) a pt in the domain (i.e. an input pt) and $\hat{u}$ a unit vector (2-d).

3-d

Thm If f is __________ then f has a directional derivative at (a,b) in the direction of $\hat{u} = u_1\hat{\imath} + u_2\hat{\jmath}$.

$$D_{\vec{u}} f(a,b) = \hat{u} \cdot \nabla f(a,b)$$

$$= \langle u_1, u_2 \rangle \cdot \langle f_x(a,b), f_y(a,b) \rangle$$

$$= u_1 f_x(a,b) + u_2 f_y(a,b)$$

(this returns a number/scalar)

4-d version:

$$D_{\vec{u}} f(a,b,c) = \hat{u} \cdot \nabla f(a,b,c)$$

$$= \langle u_1, u_2, u_3 \rangle \cdot \langle f_x(a,b,c), f_y(a,b,c), f_z(a,b,c) \rangle$$

Ex 1 Find the directional derivative of $f(x,y) = e^{-xy}$ at the input pt $(1,-1)$ in the direction of $\vec{u} = -\hat{\imath} + \sqrt{3}\hat{\jmath}$.

(Note: Is $\vec{u}$ a unit vector?)

Ex 2 Find a unit vector in the direction in which $f(x,y,z) = 4xyz^2$ decreases most rapidly at pt $(2,-1,1)$. What is the rate of change in this direction?

Thm

At input pt (a,b), the fn $z = f(x,y)$ increases most rapidly in the direction

____

at a rate ________________

and decreases most rapidly in the direction

____

at a rate ________________.

Ex 3 Find the directional derivative of $f(x,y) = e^{-x}\cos y$ at $(0, \pi/3)$ in the direction toward the origin.

12.6 Chain Rule(s)

Ex 1 Find $\frac{dw}{dt}$ using a chain rule.

$w = xy + yz + xz$

$x = t^2, \; y = 1 - t^2, \; z = 1 - t$

Ex 2 Find $\frac{\partial w}{\partial t}$ using a chain rule.

$w = x^2 - y \ln x$

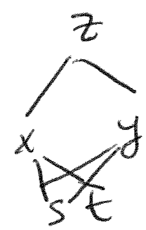
$x = \frac{s}{t}, \; y^2 = s^2 t$

$z = f(x, y)$

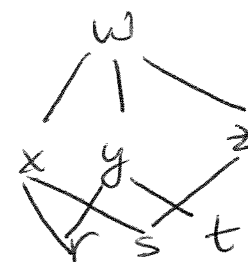
let $x = x(s,t)$, $y = y(s,t)$ w/ first partial derivatives at $(x(s,t), y(s,t))$. Then z has first partial derivatives

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \_\_\_ + \frac{\partial z}{\partial y} \_\_\_$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \_\_\_ + \frac{\partial z}{\partial y} \_\_\_$$



If the flow chart is



Find: $\frac{\partial w}{\partial r} =$

$\frac{\partial w}{\partial s} =$

$\frac{\partial w}{\partial t} =$

12.6 (cont)

Ex3 If $w = x^2 y + z^2$, $\quad x = \rho \cos\theta \sin\varphi$

$y = \rho \sin\theta \sin\varphi$

$z = \rho \cos\varphi$

find $\left.\dfrac{\partial w}{\partial \theta}\right|_{\rho=2,\ \theta=\pi,\ \varphi=\pi/2}$

Ex4. Airplanes A & B depart from point P at the same time. Plane A flies due east & plane B flies N50°E. At a certain time, plane A is 200 miles from P flying 450 mph and plane B is 150 miles from P flying 400 mph. How fast are they separating at that instant?

12.7 Tangent Planes

Ex1 Find eqn of tangent plane to $x^2+y^2-z^2=4$ at $(2,1,1)$.

Ex2 Find a pt on the surface $z=2x^2+3y^2$ where the tangent plane is parallel to the plane $8x-3y-z=0$.

Defn

Let $F(x,y,z)=k$ be a surface.
$P_0=(x_0,y_0,z_0)$ be a pt on F.
If ____________________

and ____________________

then the tangent plane to F at P_0 exists.
It is the plane $\perp$ to

____

and passes through

____

Eqn of Tangent Plane

$$\nabla F(x_0,y_0,z_0)\cdot\langle x-x_0,\, y-y_0,\, z-z_0\rangle=0$$

12.7 (cont)

Ex 3 Use differentials to approximate the change in $z=\tan^{-1}(xy)$ from $P(2,-0.5)$ to $Q(-2.03,-0.51)$. Then find Δz (the actual change).

Ex 4 An object's weight in air is 36 lbs $=A$ and its weight in water is $W=20$ lbs, with a possible error in each measurement of 0.02 lbs. Find (by approximating) the maximum possible error in calculating its specific gravity from

$$S(A,W)=\frac{A}{A-W}.$$

Let $z=f(x,y)$ be a differentiable fn.

Total differential of f $=dz$

$$dz=\nabla f\cdot\langle dx,dy\rangle$$

(this dz is the approximate change in z)

the actual change in z is

$$\Delta z=z_2-z_1$$

(i.e. the difference in the z-values)

12.8 Maxima and Minima

Ex 1 What is the boundary of
$\{(x,y): 1 < x \leq 4\}$

Ex 2 For $f(x,y) = x^2 + a^2 - 2ax\cos y$
$y \in [-\pi, \pi]$, find all critical pts. Determine if they are min, max or saddle pts. (Assume a is fixed.) (more space on next page)

Vocab For fn $z = f(x,y)$

① global max:

② global min:

③ local max:

④ local min:

Critical pts:
① stationary pts
② singular pts
③ bndry pts

12.8 (cont)

Ex 2 (cont)

$z = f(x,y)$

Strategy to find max/min:

① Find all (x,y) pts such that $\nabla f(x,y) = \vec{0}$.

② let $D = f_{xx}(x,y) f_{yy}(x,y) - f_{xy}^2(x,y)$

If (a) $D > 0$ and $f_{xx}(x,y) < 0$, $f(x,y)$ is local max. value

(b) $D > 0$ and $f_{xx}(x,y) < 0$, $f(x,y)$ is local min. value

(c) $D < 0$, $(x,y, f(x,y))$ is a saddle pt

(d) $D = 0$, test is in conclusive.

③ Determine if any boundary pt gives min or max. Typically, we have to parameterize boundary & then reduce to a Calc I type of min/max problem to solve.

12.8 (cont)

Ex 3 Find the min + max values of

$z = y^2 - x^2$ (hyperbolic paraboloid - like a Pringles chip)

on the closed triangle w/ vertices $(0,0)$, $(1,2)$

and $(2,-2)$.

12.8 (cont)

Ex 4 (Setup this problem & take home to solve.)

Find the pt on the plane $x+2y+3z=12$ that is closest to the origin. What is the minimum distance?

12.9 Lagrange Multipliers

Ex 1 Find the minimum of $f(x,y) = x^2+4xy+y^2$ subject to constraint $x-y-6=0$.

Given fn $z=f(x,y)$

WANT pts that produce global max/min and satisfy an additional constraint / boundary

$g(x,y)=0$.

To find those max/min pts: solve

① $\nabla f = \lambda \nabla g$

and ② $g(x,y)=0$

simultaneously.

Note: we can easily extend this to fns of more variables than two.

Ex 2 Use Lagrange method to solve:

(a) find min & max values for $f(x,y) = x^2 - y^2 - 1$ on
$S = \{(x,y) \mid x^2 + y^2 \le 1\}$

(Qn: can we use only Lagrange method here? Why or why not?)

12.9 (cont)

Ex 2 (cont)

(b) Find 3d vector of length 9 with the largest possible sum of its components.

12.9 (cont)

Ex 3 Find the pt on the plane $x+2y+3z=12$ that is closest to the origin. (This is same problem as 12.8 Ex 4 notes, but this time, see how much faster it is to solve using Lagrange method. :))