13.1 & 13.2 Double Integrals over Rectangles and Iterated Integration

Ex 1 $\iint_R (y-x+4)\,dA$

$R = \{(x,y) : 0 \le x \le 4,\ 0 \le y \le 4\}$

(a) sketch solid whose volume is given by the integral.

(b) Calculate approximate volume.

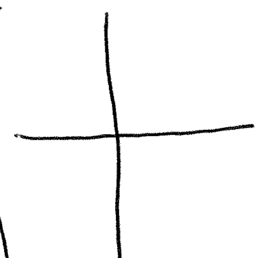
(c) calculate exact volume.

Calculating Signed Volume

Q1: what is signed volume?

Let $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$

R



(draw R in 2-d space)

For $f(x,y)$ continuous over R:

signed volume between $z=f(x,y)$ and xy-plane (over R) is

$V = \iint_R f(x,y)\,dA$

$= \underline{\qquad\qquad\qquad}$

Ex 2 Evaluate

$$\int_0^{\ln 3}\int_0^{\ln 2} e^{x+y}\,dy\,dx$$

Ex 3 Evaluate $\int_0^1\int_0^2 \frac{y}{1+x^2}\,dy\,dx$

Qn: can we separate integral? i.e. is

$$\int_0^1\int_0^2 \frac{y}{1+x^2}\,dy\,dx = \left(\int_0^1 \frac{1}{1+x^2}\,dx\right)\left(\int_0^2 y\,dy\right) ?$$

13.1 & 13.2 (cont)

Ex 4: $\int_0^1 \int_0^1 x e^{xy} \, dy \, dx$

Ex 5: Evaluate $\iint_R f(x,y) \, dA$, where $f(x,y) = 1 + x^2 + y^2$,
$R = \{(x,y) : 0 \le x \le 4, 0 \le y \le 3\}$

13.3 Double Integrals Over Nonrectangular Regions

Ex 1 Evaluate

$$\int_1^2\int_0^{x^2} \frac{y^2}{x}\,dy\,dx$$

Ex 2 Evaluate

$$\int_1^5\int_0^x \frac{3}{x^2+y^2}\,dy\,dx$$

$$\iint_S f(x,y)\,dA = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx$$

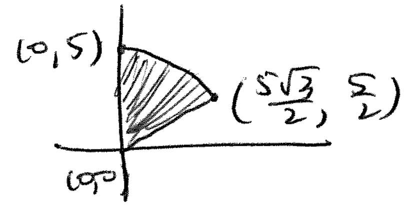
OR

$$\iint_S f(x,y)\,dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy$$

- S is a simple closed curve (not necessarily a rectangle)
- $dA = dx\,dy$ or $dy\,dx$
- the limits of inner integration can be fns of x (or y) depending on which variable hasn't been integrated yet
- Most likely, we can NOT separate the integrals into a product
- NOTE: to switch the order of integration requires some geometric thinking (it's not trivial)

13.3 (cont)

Ex 3 let $f(x,y) = x+y$ S be the region below



(a) determine limits of integration for

$$V = \iint_S f(x,y)\,dA$$

(note: you must first choose if you want $dA = dx\,dy$ or $dy\,dx$)

(b) Calculate V.

13.3 (cont)

Ex 4 Sketch the solid in the first octant bounded by the coordinate planes, $2x+y-4=0$ and $8x+y-4z=0$. Then calculate its volume using iterated integration.

Ex 5 For the triangular region in the xy-plane bounded by the vertices $(1,7)$, $(4,1)$ and $(-2,1)$, set up $\iint_S f(x,y)\,dx\,dy$ and $\iint_S f(x,y)\,dy\,dx$. (S is the inside of the triangle)

13.4 Double Integrals in Polar Coordinates

Ex 1 Evaluate (one of these is "illegal"... which one? and why?)

(a) $\int_0^{\pi/2}\int_0^{\sin\theta} r\,dr\,d\theta$

(b) $\int_0^{\sin\theta}\int_0^{\pi/2} r\,d\theta\,dr$

Volume of a solid between $z=f(x,y)$ and the xy-plane, over a simple closed region S:

$$V=\iint_S f(r,\theta)\,r\,dr\,d\theta$$

- $dA = r\,dr\,d\theta$ or $r\,d\theta\,dr$

13.4 (cont)

Ex 2 Sketch the region, convert to polar coords and evaluate.

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2y^2+y^4)\,dy\,dx$$

13.4 (cont)

Ex 3 Consider the solid inside the paraboloid $z = 4 - x^2 - y^2$, outside the cylinder $x^2 + y^2 = 1$ and above the xy-plane. Sketch the solid + calulate the volume.

13.6 Surface Area

Ex1 Make a sketch and find the area of the part of the surface $z=\sqrt{4-y^2}$ in Octant 1 that is directly above the circle $x^2+y^2=4$ in the xy-plane.

$f(x,y)=z$ continuous over S (a closed region in 2-d domain)

Then surface area of $z=f(x,y)$ over S is

$$SA=\iint_S \sqrt{f_x^2+f_y^2+1}\,dA$$

13.6 (cont)

Ex 2 Find area of surface $z = \frac{x^2}{4} + 4$ that is cut off by the planes $x=0$ $x=1$, $y=0$ and $y=2$.

Ex 3 Find area of surface that is part of the cylinder $x^2+y^2=ay$ inside the sphere $x^2+y^2+z^2=a^2$, where $a>0$ (fixed).

13.7 Triple Integrals in Cartesian Coordinates

Ex1 Write iterated integral

$\iiint_S xyz \, dV$ where

$S = \{(x,y,z) : 0 \le x \le 5,\ z^2 \le y \le 9,\ 0 \le z \le 3\}$

(sketch the solid S.)

$\iiint_S f(x,y,z)\,dV$

$= \int_{a_1}^{a_2} \int_{\varphi_1(x)}^{\varphi_2(x)} \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)\,dz\,dy\,dx$

☆ dV can be exchanged for $dx\,dy\,dz$ in any order, but you must then choose your limits of integration according to that order

13.7 (Cont)

Ex2 Find the volume of the solid bounded by the cylinder $y = x^2 + 2$ and the planes $y = 4$, $z = 0$, $3y - 4z = 0$.

Ex3 Rewrite the integral $\int_0^2 \int_0^{4-2y} \int_0^{4-2y-z} f(x,y,z)\, dx\, dz\, dy$ with order $dy\, dx\, dz$.

13.8 Triple Integrals in Cylindrical & Spherical Coords

Ex 1 Sketch region of integration and evaluate integral.

$$\int_0^{\pi}\int_0^{\sin\theta}\int_0^{2} r\,dz\,dr\,d\theta$$

$dx\,dy\,dz$
$= r\,dr\,dz\,d\theta$

and

$dx\,dy\,dz$
$= \rho^2 \sin\varphi\,d\rho\,d\theta\,d\varphi$

13.8 (cont)

Ex 2 Sketch the region bounded above by the plane $z = y + 4$, below by the xy-plane, and laterally by the right circular cylinder having radius 4 & whose axis is the z-axis.

Find its volume.

13.8 (cont)

Ex 3 Change to spherical coords and evaluate.

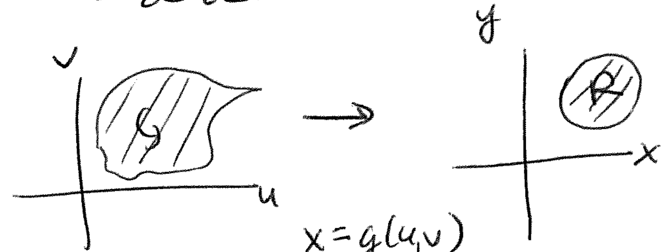
$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} (x^2+y^2+z^2)\,dz\,dy\,dx$$

13.9 Change of Variables (Jacobi Method)

Ex 1 Find image of the rectangle w/ (u,v) corners $(0,0)$ $(3,0)$ $(3,1)$ $(0,1)$ under the transformation $x = 2u + 3v$ and $y = u - v$.

Then find $J(u,v)$.

Idea:



$x = g(u,v)$
$y = h(u,v)$

$$\iint_G f(g(u,v), h(u,v))\, |J(u,v)|\, du\, dv = \iint_R f(x,y)\, dx\, dy$$

absolute value

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

determinant

13.9 (cont)

Ex 2 Use a transformation to evaluate

$$\iint_R (2x-y)\cos(y-2x)\,dA$$ over R where R is

the triangle with vertices $(1,0)$, $(4,0)$ and $(4,3)$.