Math 1210 Midterm Review

(Sections 1.4, 1.5, 1.6, 2.1, 2.2, 2.3)

Spring 2016 Instructor: Kelly MacArthur

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Instructions: Please show all of your work. All answers should be completely simplified, unless otherwise stated. No calculators or electronics of any kind are allowed.

1. Calculate the following limits. If they are infinite or do not exist, state this.

   (a) $\lim_{x\to -2} \frac{x^3+8}{x^2-2x+4}$

   Solution: If $x = -2$ is substituted into this expression, the denominator is nonzero. We conclude the function is continuous at $x = -2$, and therefore

   $$\lim_{x\to -2} \frac{x^3+8}{x^2-2x+4} = \frac{-8+8}{4} = 0.$$

   (b) $\lim_{\theta\to 0} \frac{\cot(\pi\theta)\cos\theta}{2\sec\theta}$

   Solution: This function looks like it might have a discontinuity at $\theta = 0$. We can simplify it by applying trig identities $\sec\theta = \frac{1}{\cos\theta}$ and $\cot(\theta) = \frac{\cos\theta}{\sin\theta}$, such that

   $$\lim_{\theta\to 0} \frac{\cot(\pi\theta)\cos\theta}{2\sec\theta} = \lim_{\theta\to 0} \frac{\cos(\pi\theta)\cos^2\theta}{2\sin(\pi\theta)}.$$

   As θ approaches 0, the denominator $2\sin(\pi\theta)$ approaches zero and the numerator stays nonzero. We conclude this limit may not exist, or may go off to infinity. To test this, we take left and right limits

   $$\lim_{\theta\to 0^-} \frac{\cos(\pi\theta)\cos^2\theta}{2\sin(\pi\theta)} = -\infty \qquad \text{and} \qquad \lim_{\theta\to 0^+} \frac{\cos(\pi\theta)\cos^2\theta}{2\sin(\pi\theta)} = \infty;$$

   implying no limit exists at $\theta = 0$.

   (c) $\lim_{t\to 0} \frac{\sin^2(3t)}{4t}$

   Solution: This problem can be approached using our knowledge that $\lim_{t\to 0} \frac{\sin(t)}{t} = 1$. First observe the limit can be broken up multiplicatively (as long as both limits exist) to

   $$\lim_{t\to 0} \frac{\sin^2(3t)}{4t} = \frac{3}{4}\left(\lim_{t\to 0} \frac{\sin(3t)}{3t}\right)\left(\lim_{t\to 0} \sin(3t)\right),$$

noticing how the constants have been shifted around slightly in the first limit. The first limit goes to 1 and the second to 0, so the original limit is

$$\lim_{t\to 0} \frac{\sin^2(3t)}{4t} = \frac{3}{4} \cdot 1 \cdot 0 = 0.$$

(d) $\lim_{x\to\infty} \sqrt{\frac{x^2+x+3}{(x-1)(x+2)}}$

Solution: An infinite limit such as this one can be calculated by considering the highest-order powers of the numerator and the denominator; this one reduces to

$$\lim_{x\to\infty} \sqrt{\frac{x^2+x+3}{(x-1)(x+2)}} = \lim_{x\to\infty} \sqrt{\frac{x^2}{x^2}} = \sqrt{1} = 1.$$

Therefore, our answer is 1.

(e) $\lim_{x\to\infty} \sqrt[3]{\frac{\pi x^3+7x}{\sqrt{2}x^3+3x^2}}$

Solution: We take the same approach here of only considering the highest-order powers of the numerator and denominator, such that

$$\lim_{x\to\infty} \sqrt[3]{\frac{\pi x^3+7x}{\sqrt{2}x^3+3x^2}} = \lim_{x\to\infty} \sqrt[3]{\frac{\pi x^3}{\sqrt{2}x^3}} = \frac{\sqrt[3]{\pi}}{\sqrt[6]{2}}.$$

(f) $\lim_{x\to-\infty} \frac{3\sqrt{-x^3}+4x}{\sqrt{-8x^3}}$

Solution: We take the same approach here of only considering the highest-order powers of the numerator and denominator, such that

$$\lim_{x\to-\infty} \frac{3\sqrt{-x^3}+4x}{\sqrt{-8x^3}} = \lim_{x\to-\infty} \frac{3\sqrt{-x^3}}{\sqrt{-8x^3}} = \lim_{x\to-\infty} \frac{3\sqrt{-x^3}}{2\sqrt{-2x^3}} = \frac{3\sqrt{2}}{4}.$$

Equivalently, you can obtain $\frac{3}{2\sqrt{2}}$ by not rationalizing the denominator.

(g) $\lim_{x\to 5^-} \frac{\sin|x-5|}{x-5}$

Solution: First observe that we are approaching 5 from a smaller number, so $x-5$ will always be negative, and we can replace $|x-5|$ with $5-x$ (that is, $-(x-5)$) in this case. Next observe that sine is an odd function, so $\sin(5-x) =$

$-\sin(x-5)$. Given these two facts together, and our knowledge of the limit of $\frac{\sin x}{x}$ as $x \to 0$, we conclude

$$\lim_{x\to 5^-} \frac{\sin|x-5|}{x-5} = \lim_{x\to 5^-} \frac{-\sin(x-5)}{x-5} = -\lim_{x\to 5^-} \frac{\sin(x-5)}{x-5} = -1.$$

(h) $\lim\limits_{x\to 5^+} \frac{\sin|x-5|}{\tan(x-5)}$

Solution: Here we use the same trick to get rid of the absolute value: because the quantity in the sine term is always positive for this limit, $\sin|x-5| = \sin(x-5)$. The definition of $\tan x$ allows us to cut down this expression even further, so that

$$\lim_{x\to 5^+} \frac{\sin|x-5|}{\tan(x-5)} = \lim_{x\to 5^+} \frac{\sin(x-5)\cos(x-5)}{\sin(x-5)} = \lim_{x\to 5^+} \cos(x-5) = 1.$$

(i) $\lim\limits_{x\to\infty} x^{-1/2}\sin x$

Solution: This limit can be calculated using the Squeeze Theorem; note $-1 \leq \sin x \leq 1$, then multiplying all parts of this inequality by $x^{-1/2}$, which is always positive, we obtain

$$\frac{-1}{x^{1/2}} \leq \frac{\sin x}{x^{1/2}} \leq \frac{1}{x^{1/2}}.$$

Since $\lim\limits_{x\to\infty} \frac{-1}{x^{1/2}} = 0$ and $\lim\limits_{x\to\infty} \frac{1}{x^{1/2}} = 0$, the desired limit must also be 0 as well.

(j) $\lim\limits_{x\to -\infty} \sin\left(x + \frac{1}{x}\right)$

Solution: The trick to this problem is using an additive trig identity to simplify it and then trying to apply the Squeeze Theorem. First note

$$\lim_{x\to -\infty} \sin\left(x + \frac{1}{x}\right) = \lim_{x\to -\infty} \left[\sin x \cos\left(\frac{1}{x}\right) + \cos x \sin\left(\frac{1}{x}\right)\right]$$

by the corresponding additive trig identity. Next, we can bound $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$, so that

$$-\cos\left(\frac{1}{x}\right) \leq \sin x \cos\left(\frac{1}{x}\right) \leq \cos\left(\frac{1}{x}\right)$$

and

$$-\sin\left(\frac{1}{x}\right) \leq \cos x \sin\left(\frac{1}{x}\right) \leq \sin\left(\frac{1}{x}\right)$$

for values where $\cos\left(\frac{1}{x}\right)$ or $\sin\left(\frac{1}{x}\right)$ are positive and

$$\cos\left(\frac{1}{x}\right) \leq \sin x \cos\left(\frac{1}{x}\right) \leq -\cos\left(\frac{1}{x}\right)$$

and

$$\sin\left(\frac{1}{x}\right) \leq \cos x \sin\left(\frac{1}{x}\right) \leq -\sin\left(\frac{1}{x}\right)$$

where they are negative. In either case, the limits on either side for $\sin\left(\frac{1}{x}\right)$ both go to 0, but there's no way to deal with $\cos\left(\frac{1}{x}\right)$ (one is -1 and one is 1 for both sets of inequalities). We conclude no limit exists.

(k) $\lim\limits_{x\to-\infty}\left[\sin\left(x+\frac{1}{x}\right)-\sin x\right]$

Solution: The same thing happens as before, except we can factor out an additional $-\sin x$ at the beginning:

$$\lim_{x\to-\infty}\left[\sin\left(x+\frac{1}{x}\right)-\sin x\right] = \lim_{x\to-\infty}\left[\sin x\left(\cos\left(\frac{1}{x}\right)-1\right)+\cos x\sin\left(\frac{1}{x}\right)\right].$$

We again bound sine and cosine as above, so that

$$-\left[\cos\left(\frac{1}{x}\right)-1\right] \leq \sin x\left[\cos\left(\frac{1}{x}\right)-1\right] \leq \left[\cos\left(\frac{1}{x}\right)-1\right]$$

and

$$-\sin\left(\frac{1}{x}\right) \leq \cos x \sin\left(\frac{1}{x}\right) \leq \sin\left(\frac{1}{x}\right)$$

for values where $\cos\left(\frac{1}{x}\right)$ or $\sin\left(\frac{1}{x}\right)$ are positive and

$$\left[\cos\left(\frac{1}{x}\right)-1\right] \leq \sin x\left[\cos\left(\frac{1}{x}\right)-1\right] \leq -\left[\cos\left(\frac{1}{x}\right)-1\right]$$

and

$$\sin\left(\frac{1}{x}\right) \leq \cos x \sin\left(\frac{1}{x}\right) \leq -\sin\left(\frac{1}{x}\right)$$

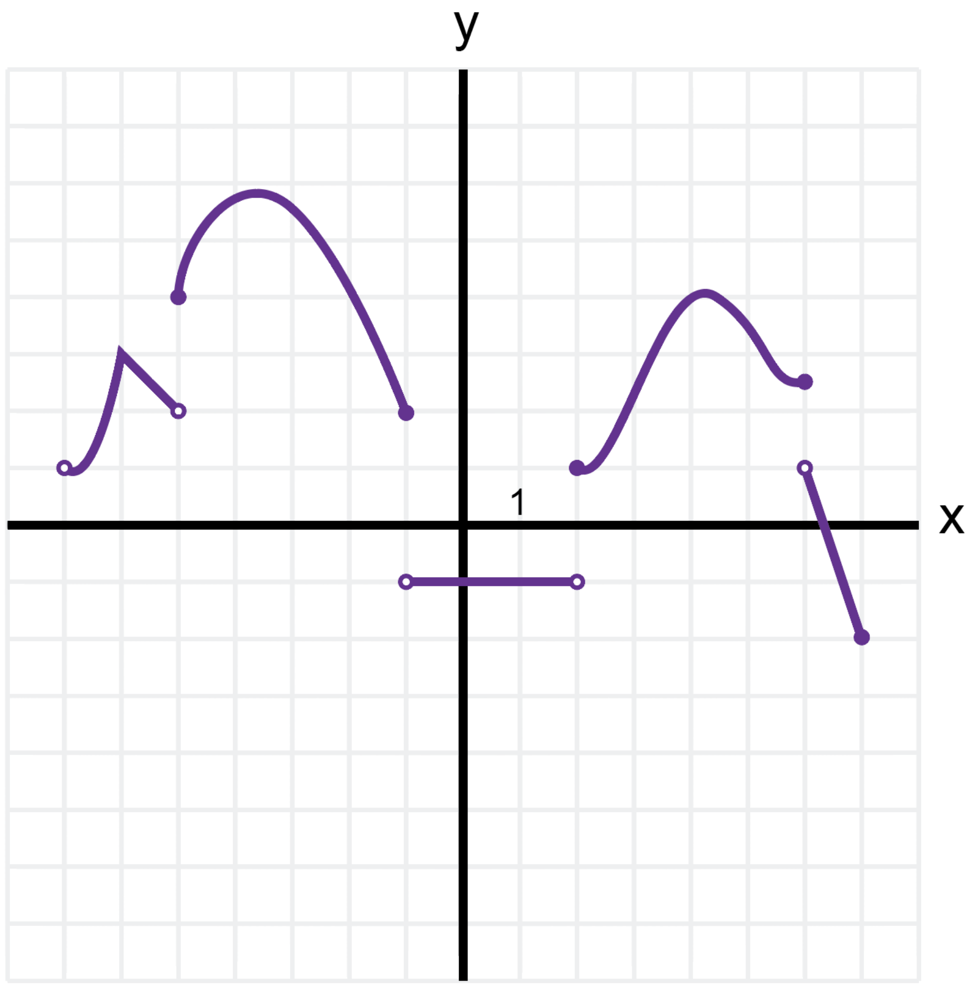
where they are negative. However, now the left- and right-hand sides of each inequality all do go to 0 when x approaches $-\infty$! This implies

$$\lim_{x\to-\infty}\left[\sin\left(x+\frac{1}{x}\right)-\sin x\right] = 0-0=0$$

by the Squeeze Theorem.

2. **Continuity (Graph-Based)**

   (a) From the graph of h given below, indicate the intervals on which h is continuous. Indicate left or right continuity when possible.



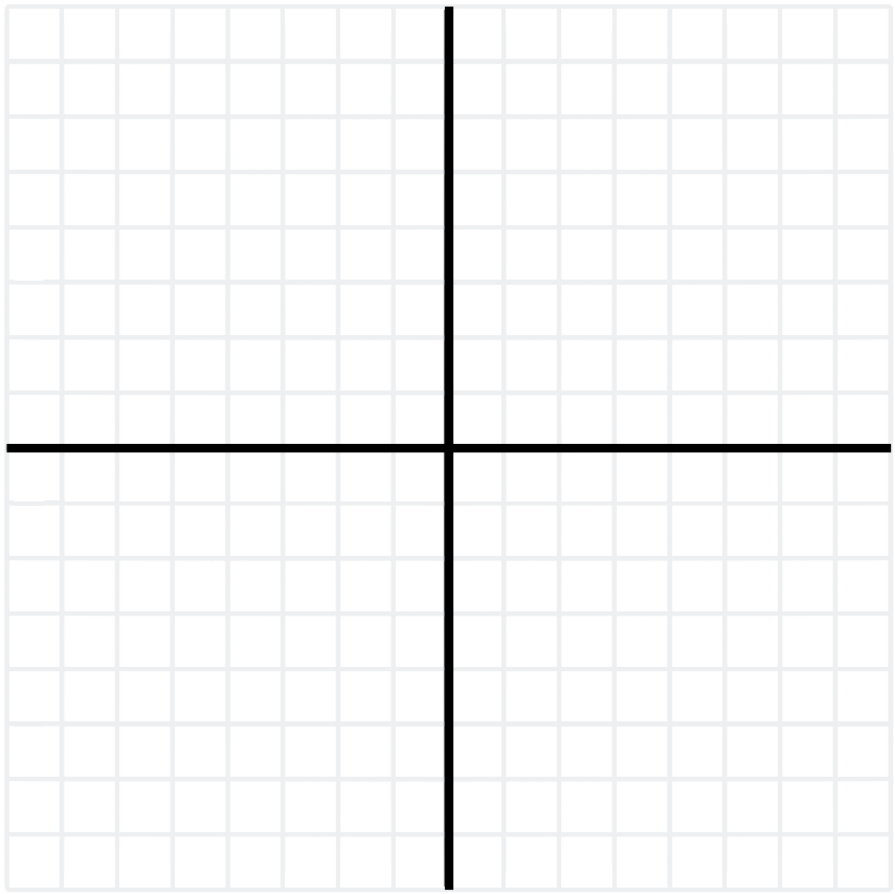
Solution: This function is discontinuous at -7, -5, -1, 2, and 6. It is right continuous at -5, left continuous at -1, right continuous at 2, and left continuous at 6. Therefore, the intervals on which this function is continuous are

$$(-7,\ -5) \cup [-5,\ -1] \cup (-1,\ 2) \cup [2, 6] \cup (6, 7].$$

(b) Sketch the graph of a function $g(x)$ that satisfies the following conditions:

- Its domain is [-2,2].
- It satisfies $f(-2) = f(-1) = f(1) = f(2) = 1$.
- It is discontinuous at -1 and 1.
- It is right continuous at -1 and left continuous at 1.

You need not provide an algebraic definition for $g(x)$.



Solution: This problem has infinitely many solutions. In particular, the points (-1, 1) and (1, 1) can be connected by one line.

3. **Constructing Examples**

(a) Give an example of a function such that $f(x)$ is not continuous everywhere, but $|f(x)|$ is.

Solution: One example would be $f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$; that is, sgn x.

(b) Give an example of a function with discontinuities at infinitely many points.

Solution: One example would be $f(\theta) = \tan\theta$.

(c) Give an example of a real-valued function that is not continuous on the left half plane.

Solution: One example would be $f(x) = \sqrt{x}$.

4. Find the values of a and b such that the function

$$f(x) = \begin{cases} x+1 & x < 1 \\ ax+b & 1 \leq x < 2 \\ 3x & x \geq 2 \end{cases}$$

is continuous everywhere.

Solution: Observe that, for continuity to be preserved, we require $1a+b = 1+1 = 2$ and $2a+b = 3(2) = 6$. This is a linear system and can be solved by substitution, for example. We obtain $a = 4$ and $b = -2$.

5. **Tangent Lines**
   Using the definition of the derivative, not shortcuts, find the slope of the tangent line to the curve at $x = -2$:

   (a) $y = x^2 - 3$

   > **Solution:** We substitute $f(x) = x^2 - 3$ into the formula $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ to obtain $y'(x) = 2x$ for any point x, so that the slope of the tangent line at -2 is$f'(-2) = -4$.

   (b) $y = -x^3 + x$

   > **Solution:** We substitute $f(x) = -x^3 + x$ into the formula to obtain $y'(x) = -3x^2 + 1$ for any point x, making the slope of the tangent line $f'(-2) = -11$.

   (c) $y = 2x^3 - x^2 - 6$

   > **Solution:** We substitute $f(x) = 2x^3 - x^2 - 6$ into the formula to obtain $y'(x) = 6x^2 - 2x$ for any point x, making the slope of the tangent line $f'(-2) = 28$.

6. **Falling Bodies**
A falling body will fall approximately $10t^2$ meters in t seconds under the influence of gravity.

(a) How far will an object fall between $t = 1$ and $t = 3$ (assuming it doesn't hit the ground)?

Solution: At $t = 1$, the object will have fallen 10 meters, and at $t = 3$, the object will have fallen 90 meters. Therefore, the object will fall 80 meters in this timespan.

(b) What is its average velocity on the interval $0 \leq t \leq 3$?

Solution: At $t = 0$, the object will have fallen 0 meters, and at $t = 3$, the object will have fallen 90 meters. We find the average velocity by dividing this difference by the difference in time (*i.e.*, finding the slope of the secant line), such that

$$\frac{90-0}{3-0} = 30$$

is the average velocity on this interval in m/s.

(c) What is its average velocity on the interval $1 \leq t \leq 3$?

Solution: Taking the same approach as the previous part, we note the time the object has fallen to find

$$\frac{10(3)^2 - 10(1)^2}{3-1} = \frac{90-10}{2} = 40$$

is the average velocity on this interval in m/s.

(d) Find its instantaneous velocity at $t = 3$.

Solution: We use the instantaneous velocity formula $v = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ to find

$$\begin{aligned} v(3) &= \lim_{h\to 0} \frac{10(3+h)^2 - 10(3)^2}{h} \\ &= \lim_{h\to 0} \frac{90 + 60h + 10h^2 - 90}{h} \\ &= \lim_{h\to 0} (60 + 10h) = 60. \end{aligned}$$

This implies the instantaneous velocity at $t = 3$ is 60 m/s.

7. Use "shortcuts" to find $f'(x)$ for given functions.

(a) $f(x) = \pi x^4 + 2x^2 - 5x + 100$

Solution: Using the derivative shortcut for polynomials, the derivative of this function is $f'(x) = 4\pi x^3 + 4x - 5$.

(b) $f(x) = \frac{3x^2 - x + 10}{2x+1}$

Solution: Using the Quotient Rule, the derivative of this function is

$$f'(x) = \frac{(6x-1)(2x+1) - 2(3x^2 - x + 10)}{(2x+1)^2} = \frac{6x^2 + 6x - 21}{(2x+1)^2}.$$

(c) $f(x) = \frac{2\pi}{x^3} - x^{-4} + \frac{9}{x^7}$

Solution: Using the derivative rule for powers of x, we get

$$f'(x) = -6\pi x^{-4} + 4x^{-5} - 63x^{-8}$$

(d) $f(x) = (5x^3 + 1)(x^4 - 2x^2 - \frac{1}{2}x)$

Solution: Using the product rule and the shortcuts for polynomials,

$$\begin{aligned} f'(x) &= (5x^3 + 1)\left(4x^3 - 4x - \frac{1}{2}\right) + 15x^2\left(x^4 - 2x^2 - \frac{1}{2}x\right) \\ &= 35x^6 - 50x^4 - 6x^3 - 4x - \frac{1}{2}. \end{aligned}$$

8. (a) Find the equation of the tangent line to $y = x^3 - x + 2$ at $x = -1$.

Solution: To find the slope of the tangent line at this point, take the derivative $y'(x) = 3x^2 - 1$ and substitute in -1 for x, such that $y'(-1) = 2$. Next, use the given function to find the y-value at the point on the function the tangent line touches, such that $y(1) = 2$. We can then use the point-slope formula to derive

$$\begin{aligned} y - 2 &= 2(x+1) \\ y &= 2x + 4. \end{aligned}$$

(b) Find all points on the graph of $y = \frac{1}{3}x^3 - 16x + 2$ where the tangent line is horizontal.

Solution: The tangent line will be horizontal when the derivative of this function is zero. To this end, we differentiate y and set it equal to zero, allowing us to solve for

$$\begin{aligned} 0 = y'(x) &= x^2 - 16 \\ &= (x-4)(x+4). \end{aligned}$$

Therefore, the tangent line is horizontal when $x = 4$ and $x = -4$, corresponding to the points $\left(4, \frac{-122}{3}\right)$ and $\left(-4, \frac{134}{3}\right)$.