II. Problems

Problem 1. Given that $f(3)=5$, $f'(3)=1$, $g(3)=2$, $g'(3)=-2$ find the value of $((2f+3g)^4)'(3)$.

$$((2f+3g)^4)'(3)$$
$$= ((2f(x)+3g(x))^4)'\Big|_{x=3}$$
$$= 4(2f(x)+3g(x))^3(2f'(x)+3g'(x))\Big|_{x=3}$$
$$= 4(2f(3)+3g(3))^3(2f'(3)+3g'(3))$$
$$= 4(10+6)^3(2+(-6)) = \boxed{-16^4 = -65536}$$

Problem 2. Given that $f(e)=e$ and $f'(e)=\sqrt[4]{5}$. Find the derivative of $f(f(f(f(x))))$ at $x=e$.

$$[f(f(f(f(x))))]'\Big|_{x=e} = f'(f(f(f(x))))\cdot f'(f(f(x)))\cdot f'(f(x))\cdot f'(x)\Big|_{x=e} \quad \cdots (*)$$

Because $f(e)=e$, $(*) = (f'(e))^4 = (\sqrt[4]{5})^4 = \boxed{5}$

Problem 3. Find the equation of the tangent line to $y=(x^2+1)^3(x^4+1)^2$ at $x=1$.

① slope $= y'\big|_{x=1} = 3(x^2+1)^2(2x)(x^4+1)^2 + (x^2+1)^3(2(x^4+1)(4x^3))\Big|_{x=1}$

$$= 3(2)^2(2)(2)^2 + (2)^3(2(2)\cdot(4)) = 224$$

② y coordinate $= (1+1)^3(1+1)^2 = 32$

$\therefore\ y-32 = 224(x-1) \Rightarrow \boxed{y=224x-192}$

Problem 4. $\frac{d^n}{dx^n}(\cos x)$

$n=0 \quad \cos x$

$n=1 \quad -\sin x$

$n=2 \quad -\cos x$

$n=3 \quad \sin x$

$n=4 \quad \cos x$

So

$$\boxed{\frac{d^n(\cos x)}{dx^n} = \begin{cases} \cos x, & \text{if } n=4k \\ -\sin x, & \text{if } n=4k+1 \\ -\cos x, & \text{if } n=4k+2 \\ \sin x, & \text{if } n=4k+3 \end{cases} \quad k \in \mathbb{N}}$$

Problem 5. From the top of a building $160ft$ high, a ball is thrown upward with an initial velocity of 64 ft/sec.

(a) When does it reach its maximum height?

(b) What is its maximum height?

(c) When does it hit the ground?

(d) With what speed does it hit the ground?

(e) What is its acceleration at $t = 2$?

Since $S_0 = 160$ ft, $V_0 = 64$ ft/sec and $a = -32$ ft/sec^2

$V = -32t + 64$, $\quad S = -16t^2 + 64t + 160$.

(a) It reaches its max height when $V = 0$

$\therefore -32t + 64 = 0$, $\boxed{t = 2 \text{ sec}}$

(b) $s(2) = \boxed{224 \text{ ft}}$

(c) It hits the ground when $s = 0$

$\therefore -16t^2 + 64t + 160 = 0$

$t^2 - 4t - 10 = 0$

$t = 2 + \sqrt{14} \approx \boxed{5.74 \text{ sec}}$

(d) $V(2+\sqrt{14}) = -32\sqrt{14}$ ft/sec

(e) It is always $\boxed{-32 \text{ ft/sec}^2}$

Problem 6. For the implicitly defined curve $\sin\left(\frac{x^2 y\pi}{2}\right) = xy$, find the equation of the perpendicular line to the curve at the point (1, 1).

$$\cos\left(\frac{x^2y\pi}{2}\right)\cdot\left(\frac{2xy\pi}{2} + \frac{x^2\pi}{2}\frac{dy}{dx}\right) = y + x\frac{dy}{dx} \Big|_{x=1,\, y=1}$$

$$\cos\left(\frac{\pi}{2}\right)\cdot\left(\pi + \frac{\pi}{2}\frac{dy}{dx}\right) = 1 + \frac{dy}{dx}$$

$$0 = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = -1$$

Therefore

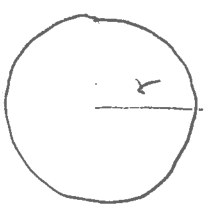
slope of the perpen $= 1$

$$y - 1 = (x - 1)$$

$$\boxed{y = x}$$

Problem 7. Oil spilling from a ruptured tanker spreads in a circle on the surface of the ocean. The area of the spill increases at a rate of $9\pi\ m^2/min$. How fast is the radius of the spill increasing when the radius is $10m$?

(1) Drawing



A = Area of the circle

(2) Given derivative

$$\frac{dA}{dt} = 9\pi\ m^2/min$$

(3) Equation

$$A = \pi r^2$$

(4) Goal : $\frac{dr}{dt}\Big|_{r=10}$

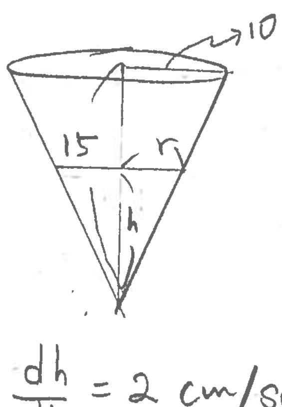
$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}\Big|_{r=10}$$

$$9\pi = 2\pi\, 10\, \frac{dr}{dt}$$

$$\therefore \boxed{\frac{dr}{dt}\Big|_{r=10} = \frac{9}{20}\ m/min}$$

Problem 8. A conical paper cup is 15 cm tall with a radius of 10 cm. The cup is being filled with water so that the water level rises at a rate of $2cm/sec$. At what rate is water being poured into the cup when the water level is $9cm$?

(1)



(2) $\frac{dh}{dt} = 2\ cm/sec$

(3) $V = \frac{1}{3}\pi r^2 h$ ⊛

$$\frac{r}{h} = \frac{10}{15} = \frac{2}{3} \Rightarrow r = \frac{2}{3}h$$

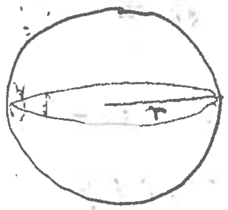
⊛ $V = \frac{4}{27}\pi h^3$

(4) $$\frac{dV}{dt}\Big|_{h=9} = \frac{4}{9}\pi h^2\frac{dh}{dt}\Big|_{h=9}$$

$$= \boxed{72\pi\ cm^3/sec}$$

Problem 9. A spherical balloon is inflated so that its radius(r) increases at a rate of $\frac{2}{r}$ cm/sec. How fast is the volume of the balloon increasing when the radius is 4 cm?

(1)



(2) $\frac{dr}{dt} = \frac{2}{r}$ cm/sec

(3) $V = \frac{4}{3}\pi r^3$

(4) $\left.\frac{dV}{dt}\right|_{r=4} = \left. 4\pi r^2 \frac{dr}{dt} \right|_{r=4}$

$= \left. 4\pi r^2 \cdot \frac{2}{r} \right|_{r=4}$

$= \left. 8\pi r \right|_{r=4}$

$= \boxed{32\pi \text{ cm}^3/\text{sec}}$

Problem 10. Given the following functions, find dy and then compute its value at $x = \frac{\pi}{2}$ for $dx = 0.1$

$dy = f'(x)\,dx$

(1) $y = \frac{1}{x}$

$\boxed{dy = \left(-\frac{1}{x^2}\right)dx}$

$dy = \left(-\frac{1}{\left(\frac{\pi}{2}\right)^2}\right)\cdot 0.1 = -\frac{0.4}{\pi^2} = \boxed{-\frac{2}{5\pi^2}}$

(2) $y = (\sin(2x) + \cos(2x))^3$

$\boxed{dy = 3(\sin(2x) + \cos(2x))^2 (2\cos(2x) - 2\sin(2x))\cdot dx}$

$dy = 3(\cancel{\sin\pi}^{\,0} + \cos\pi)^2(2\cos\pi - 2\cancel{\sin\pi}^{\,0})\cdot(0.1)$

$= 3(-1)^2\cdot(2\cdot(-1))(0.1) = \boxed{-0.6}$

Problem 11. Use differentials to approximate the given numbers

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

(1) $\sqrt{35.9}$

step1) Set a f(x) : $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2\sqrt{x}}$

step2) Set x and x_0 : $x = 35.9$, $x_0 = 36$

step3) $f(35.9) = \sqrt{35.9} \approx f(36) + f'(36)(35.9 - 36)$

$= 6 + \frac{1}{2\sqrt{36}}(-0.1)$

$= 6 - \frac{0.1}{12} = 6 - \frac{1}{120}$

$= \boxed{\frac{719}{120}}$

(2) $\sqrt[3]{27.01}$

1) $f(x) = \sqrt[3]{x}$, $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$

2) $x = 27.01$, $x_0 = 27$

3) $f(27.01) \approx f(27) + f'(27)(27.01 - 27)$

$= 3 + \frac{1}{27}(0.01) = 3 + \frac{1}{2700} = \boxed{\frac{8101}{2700}}$

Problem 12. The diameter of a sphere is measured as 20 ± 0.1 centimeters. Find the absolute error and the relative error in the volume . Also estimate ~~esimated~~ the volume of the sphere.

$V = \frac{4}{3}\pi r^3$, $r = 10$, $dr = \pm 0.05$, $V = \frac{4000\pi}{3}$

(1) $A \cdot E = \Delta V \approx dV = 4\pi r^2 \, dr = 4\pi (10)^2 \cdot (\pm 0.05) = \boxed{\pm 20\pi \text{ cm}^3}$

(2) $R \cdot E = \frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{\pm 20\pi}{\frac{4000\pi}{3}} = \boxed{\pm 0.015 = \pm 1.5\%}$

(3) Estimated volume $= V + \Delta V \approx V + dV = \boxed{\left(\frac{4000\pi}{3} \pm 20\pi\right) \text{cm}^3}$

Problem 13. Find the linear approximation to $f(x) = 2x + \cos(3x)$ at $x_0 = \frac{\pi}{3}$. (Write answer in form $y = f(x_0) + f'(x_0)(x - x_0)$.)

$f'(x) = 2 - 3\sin(3x)$, $f(\frac{\pi}{3}) = \frac{2\pi}{3} + \cos(\pi) = \frac{2\pi}{3} - 1$

$f'(\frac{\pi}{3}) = 2$

$\therefore \boxed{y = \frac{2\pi}{3} - 1 + 2(x - \frac{\pi}{3})}$

Problem 14. Identify the critical points and find the maximum value and minimum value on the given interval.

(1) $f(x) = \sin x$ on $[-\frac{\pi}{4}, \frac{\pi}{6}]$

(1) End pts

$(-\frac{\pi}{4}, \sin(-\frac{\pi}{4})) = (-\frac{\pi}{4}, -\frac{\sqrt{2}}{2})$

$(\frac{\pi}{6}, \sin(\frac{\pi}{6})) = (\frac{\pi}{6}, \frac{1}{2})$

(2) No stationary pt on $[-\frac{\pi}{4}, \frac{\pi}{6}]$

(3) No singular pt

$\therefore$ $\boxed{\text{Max} = \frac{1}{2} \text{ at } x = \frac{\pi}{6}, \quad \text{min} = -\frac{\sqrt{2}}{2} \text{ at } x = \frac{\pi}{4}}$

(2) $f(x) = x^3 - 3x + 1$ on $(-\frac{3}{2}, 3)$

(1) No End pts

but to check $(-\frac{3}{2}, \frac{17}{8})$, $(3, 19)$

(2) stationary pt $(1, -1)$, $(-1, 3)$

$f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$

(3) No singular pt

Thus comparing four pts

$\therefore$ MIN $= -1$ at $x = 1$ but no Max

(3) $f(x) = |x - 1|$ on $[0, 3]$

$$f(x) = \begin{cases} x-1 & \text{on } [0,1) \\ -x+1 & \text{on } [1,3] \end{cases}$$

$$f'(x) = \begin{cases} 1 & \text{on } [0,1) \\ -1 & \text{on } [1,3] \end{cases}$$

(1) End pt

$\boxed{(0, 1) \quad (3, 2)}$

(2) No stationary pt

(3) Singular pt

$\boxed{(1, 0)}$

$\boxed{\text{Max} = 2 \text{ at } x = 3, \quad \text{min} = 0 \text{ at } x = 1}$

Problem 15. Where is $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$ increasing, decreasing, concave up, concave down? Find all mininum and maximum points. Sketch the graph.

$f'(x) = x^2 - 2x - 3 = (x-3)(x+1)$

$f'(x)$: $+$ on the left of -1, $-$ between -1 and 3, $+$ on the right of 3

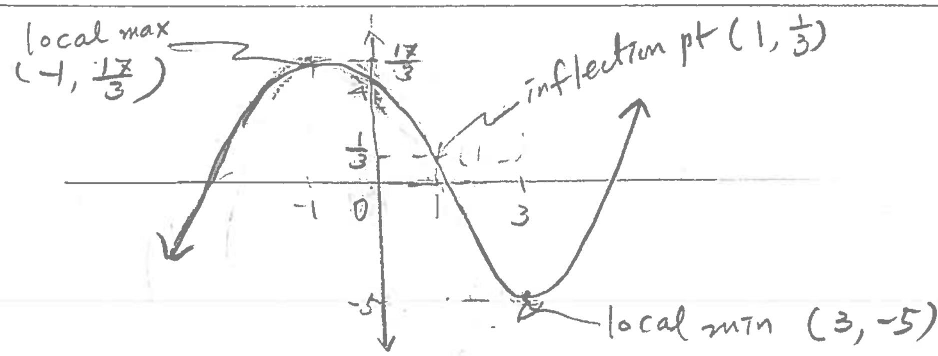
Increasing on $(-\infty, -1) \cup (3, \infty)$
Decreasing on $(-1, 3)$

$f(-1) = \frac{17}{3}$ which is a local max
$f(3) = -5$ which is a local min

$f''(x) = 2x - 2$
$= 2(x-1)$

$f''(x)$: $-$ on the left of 1, $+$ on the right of 1

c. up on $(1, \infty)$
c. Down on $(-\infty, 1)$
$\therefore$ inflection point $(1, f(1)) = (1, \frac{1}{3})$



Problem 16. Find the inflection points of $f(x) = x^{\frac{1}{3}} + 2$.

$f'(x) = \frac{1}{3x^{\frac{2}{3}}}$

$f''(x) = \frac{-2}{9x^{5/3}}$

$f''(x)$: $+$ on the left of 0, $-$ on the right of 0

c. Down on $(0, \infty)$

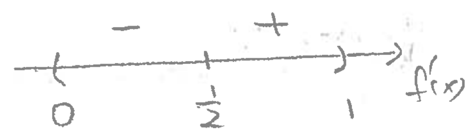
c. up on $(-\infty, 0)$

so inflection pt $(0, f(0)) = (0, 2)$

Problem 17. Find (if any exist) the maximum and minimum values of $f(x) = \frac{1}{x(1-x)}$ on $(0, 1)$.

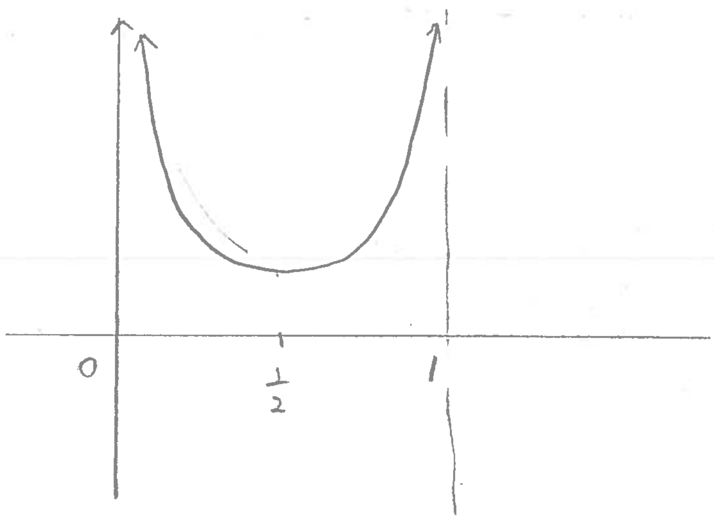
$$f(x) = \frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$$

$$f'(x) = -\frac{1}{x^2} + \frac{1}{(1-x)^2} = \frac{-1+2x}{x^2(1-x)^2}$$



$$f''(x) = \frac{2}{x^3} + \frac{2}{(1-x)^3} > 0 \quad \text{on } (0,1)$$

so $f(x)$ is c. up on (0.1)



(global) minimum value $= f(\frac{1}{2}) = 4$ at $x = \frac{1}{2}$

No Maximum value

Problem 18. Sketch the graphs of the given functions. Include asymptotes, minimum points, maximum points and inflection points.

(1) $f(x) = 2x^3 - 3x^2 - 12x + 3$

$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$

$= 6(x-2)(x+1)$

c.p $(-1, 10)$, $(2, -17)$

$f'(x)$: $+$ on $(-\infty,-1)$, $-$ on $(-1,2)$, $+$ on $(2,\infty)$

Increasing on $(-\infty, -1) \cup (2, \infty)$

Decreasing on $(-1, 2)$

local max $(-1, 10)$

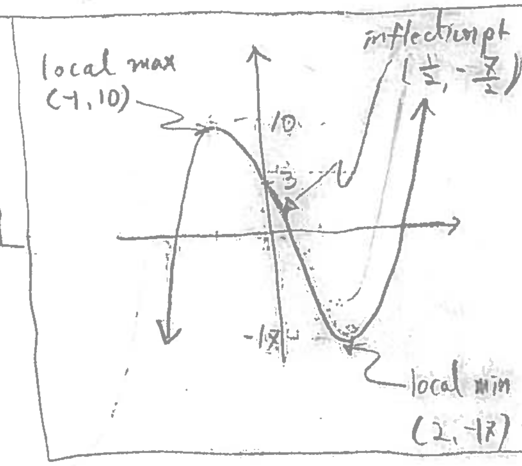
local min $(2, -17)$

$f''(x) = 12x - 6$

$f''(x)$: $-$ before $\frac{1}{2}$, $+$ after $\frac{1}{2}$

C. up on $(-\infty, \frac{1}{2})$

C. Down on $(\frac{1}{2}, \infty)$ so $\boxed{\text{inflection pt } (\frac{1}{2}, -\frac{7}{2})}$



No asymptote

(2) $f(x) = \frac{x^2-2x+4}{x-2}$

$f(x) = \dfrac{x(x-2)+4}{x-2} = x + \dfrac{4}{x-2}$

so $\begin{cases} \text{a vertical asymptote } x = 2 \\ \text{an oblique(=slant) " i.e. } y = x \end{cases}$ $\Rightarrow \lim_{x\to 2^+} f(x) = \infty, \ \lim_{x\to 2^-} f(x) = -\infty$

$f'(x) = \dfrac{x(x-4)}{(x-2)^2}$

- stationary pts $(0, -2)$, $(4, 6)$ (local max, local min)
- singular pts $x = 2$

$f'(x)$: $+$ on $(-\infty,0)$, $-$ on $(0,2)$, $-$ on $(2,4)$, $+$ on $(4,\infty)$

Inc. on $(-\infty, 0) \cup (4, \infty)$

Dec. on $(0, 2) \cup (2, 4)$

$f''(x) = \dfrac{8}{(x-2)^3}$

$f''(x)$: $-$ before 2, $+$ after 2

C. up on $(2, \infty)$

C. Down on $(-\infty, 2)$