Math 1210 Midterm Review

(Sections 3.4, 3.6, 3.8, 3.9, 4.1, 4.2, 4.3, 4.4)

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uID:________________________ Special Number:________________________

Instructions: Please show all of your work. All answers should be completely simplified, unless otherwise stated. Report answers as exact, i.e., no approximations. No calculators or electronics of any kind are allowed.

1. Find the antiderivative of each of the following. Do not forget the constant term.

(a) $f(x) = \frac{2x^7+3}{x^3}$

Solution: $\int f(x)dx = \int(2x^4 + 3x^{-3})dx = \frac{2}{5}x^5 - \frac{3}{2}x^{-2} + C$

(b) $f(x) = \frac{3x^8+x^2+1}{x^2}$

Solution: $\int f(x)dx = \int(3x^6 + 1 + x^{-2})dx = \frac{3}{7}x^7 + x - \frac{1}{x} + C$

(c) $f(x) = (x^2+5)^{10}x$

Solution: Let $u = x^2 + 5$, then
$\int f(x)dx = \int \frac{1}{2}u^{10}du$
$= \frac{u^{11}}{22} + C = \frac{(x^2+5)^{11}}{22} + C$

(d) $f(x) = (x^3+6x)^5(x^2+2)$

Solution: Let $u = x^3 + 6x$,
$\int f(x)dx = \int \frac{1}{3}u^5du$
$= \frac{u^6}{18} + C = \frac{(x^3+6x)^6}{18} + C$

(e) $f(x) = \sin^9 x \cos x$

Solution: Let $u = \sin x$,
$\int f(x)dx = \int u^9du$
$= \frac{u^{10}}{10} + C = \frac{(\sin x)^{10}}{10} + C$

2. **Solve the following differential equations.**

(a) $\frac{dy}{dx} = x^2 + 1$, $y(1) = 1$

> **Solution:** $dy = (x^2 + 1)dx$
> $y = \int (x^2 + 1)dx = \frac{1}{3}x^3 + x + C$
> Since $y(1) = 1$, $1 = \frac{1}{3} + 1 + C$ $C = -\frac{1}{3}$
> The solution is $y = \frac{1}{3}x^3 + x - \frac{1}{3}$

(b) $\frac{dy}{dx} = \sqrt{\frac{x}{y}}, x > 0, y > 0$, $y(1) = 4$

> **Solution:** $\sqrt{y}dy = \sqrt{x}dx$
> $\int \sqrt{y}dy = \int \sqrt{x}dx$
> $\frac{2}{3}y^{3/2} = \frac{2}{3}x^{3/2} + C$
> Since $y(1) = 4$, $\frac{2}{3}4^{3/2} = \frac{2}{3} + C, C = \frac{14}{3}$
> $\frac{2}{3}y^{3/2} = \frac{2}{3}x^{3/2} + \frac{14}{3}$ or $y^{3/2} = x^{3/2} + 7$.
> The solution is explicitly given by $y = (x^{3/2} + 7)^{2/3}$

(c) $\frac{du}{dt} = u^3(t^3 - t)$, $u(1) = 4$

> **Solution:** $u^{-3}du = (t^3 - t)dt$
> $\int u^{-3}du = \int (t^3 - t)dt$
> $-\frac{1}{2}u^{-2} = \frac{1}{4}t^4 - \frac{1}{2}t^2 + C$
> Since $u(1) = 4$, $-\frac{1}{2} = 64 - 8 + C$,$C = -56\frac{1}{2}$
> $u^{-2} = -\frac{1}{2}t^4 + t^2 + 113$
> This is an implicit solution.

(d) $\frac{dz}{dt} = t^2z^2$, $z(1) = \frac{1}{3}$

> **Solution:** $z^{-2}dz = t^2dt$
> $\int z^{-2}dz = \int t^2dt$
> $-\frac{1}{z} = \frac{1}{3}t^3 + C$
> Since $z(1) = \frac{1}{3}$, $-3 = \frac{1}{3} + C$,$C = -\frac{10}{3}$.
> The solution is explictly given by $z = \frac{3}{-t^3+10}$.

(e) $\frac{dy}{dx} = y^2x(x^2 + 2)$, $y(1) = 1$

> **Solution:** $y^{-2}dy = (x^3 + 2x)dx$
> $\int y^{-2}dy = \int (x^3 + 2x)dx$
> $-\frac{1}{y} = \frac{1}{4}x^4 + x^2 + C$
> Since $y(1) = 1$, $-1 = \frac{1}{4} + 1 + C$,$C = -\frac{9}{4}$.
> $-\frac{1}{y} = \frac{1}{4}x^4 + x^2 - \frac{9}{4}$

The solution is explicitly given by $y = -\frac{4}{x^4+4x^2-9}$

3. **An application of the mean value theorem**
   Suppose $f(x)$ is an everywhere differentiable function defined on the whole real line and $|f'(x)| \leq 10$ for all x. Show that $|f(7) - f(5)| \leq 20$.

   Solution: By the mean value theorem, $|f(7) - f(5)| = (7-5)|f'(\theta)|$ for some θ between 5 and 7. Since $|f'(x)| \leq 10$ for all x, $|f(7) - f(5)| = (7-5)|f'(\theta)| \leq (7-5) \times 10 = 20$

4. **An application of the intermediate value theorem**
   Show that $f(x) = 2x^3 - 9x^2 + 1 = 0$ has exactly one solution on each of the intervals $(-1, 0)$,$(0, 1)$, and $(4, 5)$.

   Solution: $f(-1) = -10$,$f(0) = 1$,$f(4) = -15$,$f(5) = 26$.
   So $f(-1)f(0) < 0$, $f(0)f(1) < 0$, $f(4)f(5) < 0$.
   Note that $f(x)$ is continuous everywhere. By the intermediate value theorem, there exists a real root in each of the indicated intervals.
   $f'(x) = 6x^2 - 18x$. It can be checked the derivative does not change sign on each of $(-1, 0)$,$(0, 1)$, and $(4, 5)$. So $f(x) = 0$ has exactly one solution in each of these intervals.

5. **Find the values of the following sums, assuming $\sum_{i=1}^{100} a_i = 30$ and $\sum_{i=1}^{100} b_i = 40$**
   (a) $\sum_{n=1}^{100}(3a_n + 4b_n)$

   Solution: $\sum_{n=1}^{100}(3a_n + 4b_n) = 3\sum_{n=1}^{100} a_n + 4\sum_{n=1}^{100} b_n = 3 \times 30 + 4 \times 40 = 250$

   (b) $\sum_{n=0}^{99}(4a_{n+1} - 3b_{n+1})$

   Solution: $\sum_{n=0}^{99}(4a_{n+1} - 3b_{n+1}) = 4\sum_{n=1}^{100} a_n - 3\sum_{n=1}^{100} b_n = 4 \times 30 - 3 \times 40 = 0$

   (c) $\sum_{n=1}^{100}(n-1)(4n+2)$

   Solution: $\sum_{n=1}^{100}(n-1)(4n+2) = \sum_{n=1}^{100}(4n^2 - 2n - 2) = 4\sum_{n=1}^{100} n^2 - 2\sum_{n=1}^{100} n - 200 = 4 \times \frac{100(101)(201)}{6} + 101(100) - 200 = 1363300$

(d) $\sum_{n=1}^{100}((2n-1)^2+a_n)$

> **Solution:** $\sum_{n=1}^{100}(4n^2-4n+1)+\sum_{n=1}^{100}a_n=4\sum_{n=1}^{100}n^2-4\sum_{n=1}^{100}n+100+30$
> $=4\times\frac{100(101)(201)}{6}-2\times100(101)+130=1353400-20200+130=1333330$

(e) $\sum_{n=1}^{100}(3a_n+n^2-n)$

> **Solution:** $=3\sum_{n=1}^{100}a_n+\sum_{n=1}^{100}n^2-\sum_{n=1}^{100}n=3\times30+100(101)(201)/6+$ $100(101)/2=90+338350-5050=333390$

6. **Definite integrals**
Calculate the following integrals by using (1) the definition and (2) techniques for computing integrals.

(a) $\int_0^2(x^2+1)dx$ by using the definition

> **Solution:** $\int_0^2(x^2+1)dx=\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}[(\frac{2i}{n})^2+1]$
> $=\lim_{n\to\infty}(\frac{2}{n}\sum_{i=1}^{n}4\frac{i^2}{n^2}+n)=\lim_{n\to\infty}[\frac{8}{n^3}\frac{n(n+1)(2n+1)}{6}+2]$
> $=(16/6)+2=(8/3)+2=14/3$

(b) $\int_0^2(x^2+1)dx$ by direct calculation

> **Solution:** $\int_0^2(x^2+1)dx=(\frac{x^3}{3}+x)|_{x=0}^{2}=(16/6)+2=(8/3)+2=14/3$

(c) $\int_{-10}^{10}(x^2+x)dx$ by using the definition

> **Solution:** $\int_{-10}^{10}(x^2+x)dx=\lim_{n\to\infty}\frac{20}{n}[\sum_{i=1}^{n}(-10+\frac{20i}{n})^2+(-10+\frac{20i}{n})]$
> $=\lim_{n\to\infty}\frac{20}{n}[100\sum_{i=1}^{n}(\frac{2i}{n}-1)^2+10\sum_{i=1}^{n}(\frac{2i}{n}-1)]$
> $=\lim_{n\to\infty}\frac{20}{n}[100\sum_{i=1}^{n}(\frac{4i^2}{n^2}-\frac{4i}{n}+1)+20\sum_{i=1}^{n}\frac{i}{n}-10n]$
> $=\lim_{n\to\infty}\frac{20}{n}[\frac{400}{n^2}\sum_{i=1}^{n}i^2-\frac{400}{n}\sum_{i=1}^{n}i+100n+\frac{20}{n}\sum_{i=1}^{n}i-10n]$
> $=\lim_{n\to\infty}\frac{20}{n}[\frac{400}{n^2}\frac{n(n+1)(2n+1)}{6}-\frac{380}{n}\frac{n(n+1)}{2}+90n]$
> $=\frac{20\times400\times2}{6}-\frac{20\times380}{2}+90\times20$
> $=(16000/6)-3800+1800=(8000/3)-2000=2000/3$

(d) $\int_{-10}^{10}(x^2+x)dx$ by direct calculation

> **Solution:** Note that $f(x)=x^2$ is an even function and $g(x)=x$ is an odd function. Furthermore, the domain is symmetric about zero. $\int_{-10}^{10}(x^2+x)dx=$ $2\int_0^{10}x^2dx=\frac{2}{3}x^3|_{x=0}^{10}=2000/3$

7. **Integrals of odd and even functions**
Let f be an odd function and g be an even function, and suppose that $\int_0^1 |f(x)|dx = \int_0^1 g(x)dx = 3$. Use geometric reasoning to calculate each of the following:

(a) $\int_{-1}^1 f(x)dx$

> **Solution:** Since $f(x)$ is an odd function and the domain is symmetric about zero, $\int_{-1}^1 f(x)dx = 0$

(b) $\int_{-1}^1 g(x)dx$

> **Solution:** Since $g(x)$ is an even function and the domain is symmetric about zero, $\int_{-1}^1 g(x)dx = 2\int_0^1 g(x)dx = 6$

(c) $\int_{-1}^1 |f(x)|dx$

> **Solution:** Since $|f(x)|$ is an even function and the domain is symmetric about zero, $\int_{-1}^1 |f(x)|dx = 2\int_0^1 |f(x)|dx = 6$

(d) $\int_{-1}^1 [-g(x)]dx$

> **Solution:** Since $-g(x)$ is an even function and the domain is symmetric about zero, $\int_{-1}^1 [-g(x)]dx = 2\int_0^1 [-g(x)]dx = -2\int_0^1 [g(x)]dx = -6$

(e) $\int_{-1}^1 xg(x)dx$

> **Solution:** Since $xg(x)$ is an odd function and the domain is symmetric about zero, $\int_{-1}^1 xg(x)dx = 0$

(f) $\int_{-1}^1 f^3(x)g(x)dx$

> **Solution:** Since $f^3(x)g(x)$ is an odd function and the domain is symmetric about zero, $\int_{-1}^1 f^3(x)g(x)dx = 0$

8. **Use The First Fundamental Theorem Of Calculus to find the derivatives of the following functions.**

(a) $G(x) = \int_x^1 2tdt$

> **Solution:** $G'(x) = -2x$ by the fisrt fundamental theorem of calculus.

(b) $G(x) = \int_0^x (2t^2 + \sqrt{t})dt$

> **Solution:** $G'(x) = (2x^2 + \sqrt{x})$ by the fisrt fundamental theorem of calculus.

(c) $G(x) = \int_{-x^2}^x \frac{1}{1+t^2}dt$

> **Solution:** $G'(x) = \frac{1}{1+x^2} - \frac{1}{1+(-x^2)^2}(-2x) = \frac{1}{1+x^2} + \frac{2x}{1+x^4}$
> by the first fundamental theorem of calculus and the chain rule.

(d) $G(x) = \int_1^{x^2} x^2tdt$

> **Solution:** Note $G(x) = x^2 \int_1^{x^2} tdt$.
> By the product rule, the first fundamental theorem of calculus, and the chain rule,
> $G'(x) = 2x \int_1^{x^2} tdt + (x^2)(x^2)(2x)$
> $= (2x)\frac{t^2}{2}|_{t=1}^{x^2} + 2x^5 = (2x)(x^4 - 1)/2 + 2x^5 = x^5 - x + 2x^5 = 3x^5 - x$

9. **Newton's method**

Approximate the real root of $f(x) = 4x^3 + x - 5 = 0$ accurate to four decimal places. Choose $x_0 = 2$ as your initial value. Please do use Newton's method to find the real root even if you might or might not see what the real root is. The purpose of this exercise is to give a practice of Newton's method.

Solution: By the Newton's method formula, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{4x_n^3+x_n-5}{12x_n^2+1}$ $= \frac{8x_n^3+5}{12x_n^2+1}$ After a few steps of interation, we will see that x_n stays stable at 1. The following is a Matlab7.0 code to illustrate this.

$y = zeros(10,1);$%Define 10 numbers each of which is zero.

>> $y(1,1) = 2;$%Change the first number to be 2.

>> $fori = 1 : 10$

$y(i+1,1) = (8 * (y(i,1)^3) + 5)/(12 * (y(i,1)^2) + 1);$

end%The iteration process.

>> y% Show the values of the ten numbers after the iteration process.

The computer reports $y =$

2.0000 1.4082 1.1026 1.0087 1.0001 1.0000 1.0000 1.0000 1.0000 1.0000 1.0000

We see that after 5 steps of interation, the approximate solution stays at 1, which is the real solution to the equation.

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10. **Comprehensives**
Show that the rectangle with maximum perimeter that can be inscribed in a circle is a square.

Solution: Without loss of generality, suppose we are working with the unit circle. Assume the dimensions of the rectangle are a and b. Then by the Pythogorean theorem, the diameter of the circle $= \sqrt{a^2 + b^2} = 2$. Our goal is to maximize the perimeter of the rectangle $= 2a + 2b$. By the relation between a and b we have just got, we can write $f(a) = 2a + 2b = 2(a + \sqrt{4 - a^2})$. Set $f'(a) = 2(1 - \frac{a}{\sqrt{4-a^2}}) = 0$ to get $a = \sqrt{2}$. Since we can check $f'(a) > 0$ for $a \in (0, \sqrt{2})$ and $f'(a) < 0$ for $a > \sqrt{2}$, it is justified that $a = \sqrt{2}$ is a global maximum point for $f(a)$. Note that when $a = \sqrt{2}$, we have $a = b = \sqrt{2}$ by the Pythogorean theorem, which implies the rectangle is a square when its perimeter is maximized.

11. **Comprehensives**
Find the equation of the line that is tangent to the ellipse $25x^2 + 16y^2 = 400$ in the first quadrant and forms with the coordinate axes the triangle with smallest possible area.

Solution: First, take the derivative with respect to x at a point on the ellipse that is also in the first quadrant say (x_0, y_0) for both sides of the equation.We get $50x_0 + 32y_0 \frac{dy}{dx}|_{x=x_0} = 0$. The slope of the tangent line at (x_0, y_0) is therefore $\frac{dy}{dx}|_{x=x_0} = -\frac{25x_0}{16y_0}$. The equation of the tangent line is given by $y - y_0 = -\frac{25x_0}{16y_0}(x - x_0)$. Set $y = 0$, then $-y_0 = -\frac{25x_0}{16y_0}(x - x_0)$. We get that the $x-intercept$ is $(\frac{16y_0^2+25x_0^2}{25x_0}, 0) = \frac{400}{25x_0} = 16/x_0$. $(16y_0^2 + 25x_0^2 = 400$ since (x_0, y_0) is on the ellipse.) Similarly, we can get that the $y-intercept$ is $(0, 25/y_0)$. So the area of the indicated triangle $= (16/x_0)(25/y_0)/2 = 200/(x_0y_0)$.To minimize the area of the indicated triangle is equivalent to maximize the function $f(x, y) = xy$ under the restricition $25x^2+16y^2 = 400$. Use a substitution to write $g(x) = f(x, y) = xy = x\sqrt{\frac{400-25x^2}{16}} = x\sqrt{400 - 25x^2}/4 = \frac{5x\sqrt{16-x^2}}{4}$. We can equivalently maximize $h(x) = x\sqrt{16 - x^2}$. Set $h'(x) = \sqrt{16 - x^2} - \frac{x^2}{\sqrt{16-x^2}} = 0$ to get $x = 2\sqrt{2}$.(The equation has a negative root also which is ignored since we only consider points on the first quadrant. We can check $h'(x) > 0$ for $x < 2\sqrt{2}$ and $h'(x) < 0$ for $x > 2\sqrt{2}$. This justifies that $x = 2\sqrt{2}$ is a global maximum point for $h(x)$. The corresponding y-coodinate is $y = \sqrt{\frac{400-25x^2}{16}} = \frac{5\sqrt{2}}{2}$. Plug $(2\sqrt{2}, \frac{5\sqrt{2}}{2})$ for (x_0, y_0) into the tangent line equation $y - y_0 = -\frac{25x_0}{16y_0}(x - x_0)$. We get the tangent line that makes an indicated triangle with the smallest possible area is $y = -\frac{5}{4}x + 5\sqrt{2}$

12. **Comprehensives**
A flower bed will be in the shape of a sector of a circle(a pie-shaped region) of radius r and vertex angle θ. Find r and θ if its area is a constant A and the perimeter is a minimum.

Solution: This question requires us to minimize $f(r, \theta) = (2 + \theta)r$ under the restriction $A = \theta r^2/2$. By replacing θ with $\frac{2A}{r^2}$, we can write $g(r) = f(r, \theta) = (2 + \theta)r = (2 + \frac{2A}{r^2})r = 2r + \frac{2A}{r}$. We can instead minimize $h(r) = r + A/r$. Set $h'(r) = 1 - A/(r^2) = 0$ to get $r = \sqrt{A}$. We can check $h'(r) < 0$ for $r < \sqrt{A}$ and $h'(r) > 0$ for $r > \sqrt{A}$. This justifies $r = \sqrt{A}$ is a global minimum point for $h(r)$. The required radius is thus $r = \sqrt{A}$ and correspondingly, $\theta = 2$.