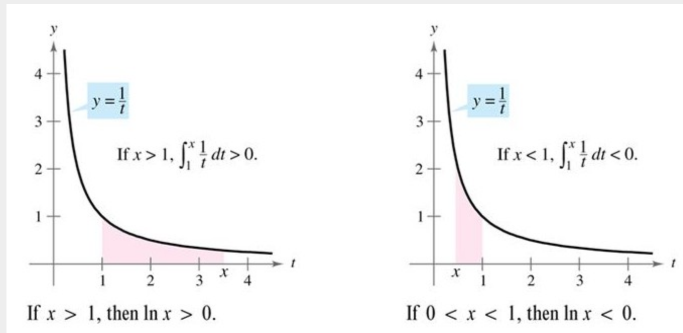
6.1 Natural Logarithm

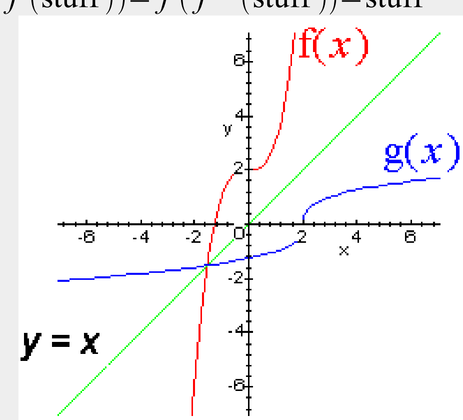
Ex 1: Evaluate the following derivatives or integrals. (a) $D_x\left(\ln\left(3x-\sqrt{x^3+1}\right)\right)$	**Definition:** $\ln x=\int_1^x \frac{1}{t}\,dt$, $x>0$
	
(b) $\int_0^{\frac{\pi}{3}} \tan x\,dx$	$\Rightarrow D_x(\ln x)=\frac{1}{x}$, $x>0$ and $D_x(\ln\lvert x\rvert)=\frac{1}{x}$, $x\neq 0$ $\Rightarrow \int \frac{1}{\text{heart}}\,d(\text{heart})=\ln\lvert\text{heart}\rvert+C$
	$a, b\in\mathbb{R}^+$, $r\in\mathbb{Q}$ (1) $\ln 1=0$ (2) $\ln(ab)=\ln(a)+\ln(b)$ (3) $\ln\left(\frac{a}{b}\right)=\ln(a)-\ln(b)$ (4) $\ln(a^r)=r\ln(a)$

Ex 2: Compute these derivatives. (a) $D_x\left(\dfrac{x^2+4}{9-\ln(3x^2-5)}\right)$	(b) $D_x(\ln(\sec^2 x))$
Ex 3: Compute these integrals. (a) $\int \dfrac{9y}{5y^2-3}\,dy$	(b) $\int \dfrac{x^3+x^2}{x+2}\,dx$

6.1 (continued)

Ex 4: Explain why $\lim_{x \to 0} \ln\left(\frac{\sin x}{x}\right) = 0$.

6.2 Inverse Functions and Their Derivatives

Ex 1: Show that $f^{-1}(x)$ exists for $f(x) = \int_x^1 \sin^4 t \, dt$.	If f is strictly monotonic on its domain, then $f^{-1}(x)$ exists.
	$f^{-1}(f(\text{stuff})) = f(f^{-1}(\text{stuff})) = \text{stuff}$ 
	If f is differentiable and monotonic on interval I and $f'(x) \neq 0$, then $f^{-1}(x)$ is differentiable at $y = f(x)$ and $(f^{-1})'(y) = \frac{1}{f'(x)}$, i.e. $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$.

Ex 2: (a) Prove that $f(x) = -x^9 - 4x^3 + 10$ has an inverse.	(b) Find $(f^{-1})'(15)$ for $f(x) = -x^9 - 4x^3 + 10$.

Ex 3: Do these functions have an inverse, i.e. does $f^{-1}(x)$ exist? Explain your reasoning.

(a) $y = f(x) = x^2 + 10x^3$

(b) $y = f(x) = 4$

(c) $y = f(x) = x^2 + 10$

(d) $y = f(x) = x^3 + 10$

6.3 Natural Exponential Function

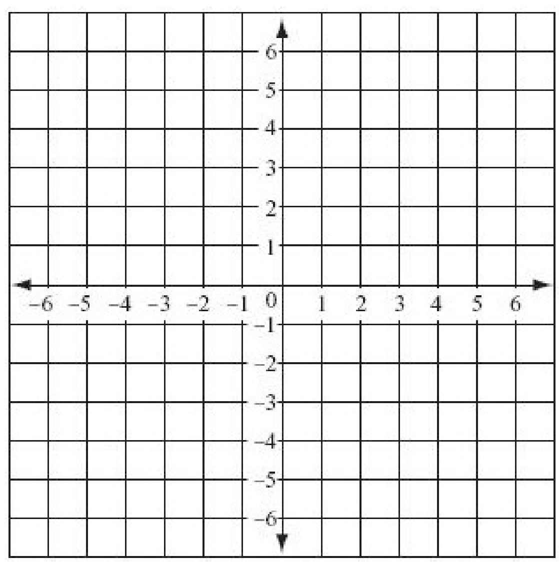
Ex 1: (a) Simplify this expression. $e^{\ln w^3 - 4\ln u}$	$e^{\ln x} = x \ , \ x > 0$ $\ln(e^x) = x \quad \forall x$
(b) Find the derivative $\frac{dy}{dx}$ for $3y = 4e^{xy} - 3\cos x$	$\int e^{\delta} d\delta = e^{\delta} + C$ $D_x(e^x) = e^x$
(c) Evaluate the integral. $\int_0^1 3x e^{6x^2-1} dx$	

6.3 (continued)

Ex 2: Find the derivative. $D_x\left(e^{x^3 \sin x}\right)$

Ex 3: Compute the integral. $\int \frac{e^x}{e^x - 5} dx$

Ex 4: For $f(x) = 2x + e^x$, find the min/max points, inflection points, any asymptotes and sketch the graph.



6.4 General Exponential/Logarithmic Functions

Ex 1: Find the derivatives. (a) $D_x(\sin^3 x + 3^{\sin x})$	$D_x(a^x) = a^x(\ln a)\ ,\ a>0$ $\int a^x\,dx = \frac{a^x}{\ln a} + C\ ,\ a>0\ ,\ a\neq 1$ $D_x(\log_a x) = \frac{1}{x\ln a}$
(b) $D_x\left((\ln(5\mathrm{x}-1))^{3\mathrm{x}^2}\right)$	Remember the equivalence of these two statements: $y=\log_a x \Leftrightarrow a^y = x$ if $a>0\ ,\ a\neq 1$
(c) $D_x\left((x^2-10)^{\tan x}\right)$	Bonus: We now have the power rule for any value of a, i.e. $\int x^a\,dx = \frac{x^{a+1}}{a+1} + C \quad \forall\, a\neq -1$.

6.4 (continued)

Ex 2: Compute these integrals.

(a) $\int 8^{2x-1}\,dx$	(b) $\int (2x-1)^8\,dx$

Bonus Question: Evaluate these integrals, assuming $m \neq 0$, $a > 0$, $a \neq 1$ and $a, m, b \in \mathbb{R}$.

(a) $\int a^{mx+b}\,dx$	(b) $\int (mx+b)^a\,dx$

6.4 (continued)

Ex 3: Compute these integrals.

(a) $\int (9^{2x} - 3^{4x})\,dx$	(b) $\int_1^{25} \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx$

6.5 Exponential Growth & Decay

Ex 1: A population is growing at a rate proportional to its size. After 10 years, its population was 250,000. After 20 years, the population was 800,000. What was the original population?	$\frac{dy}{dt} = ky \Leftarrow y = y_0 e^{kt}$ If $k > 0$, it's exponential growth. If $k < 0$, it's exponential decay.
	Put on your notecard: $\lim_{h \to 0} (1+h)^{\frac{1}{h}} = e$ and equivalently $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

6.5 (continued)

Ex 2: Evaluate these limits.

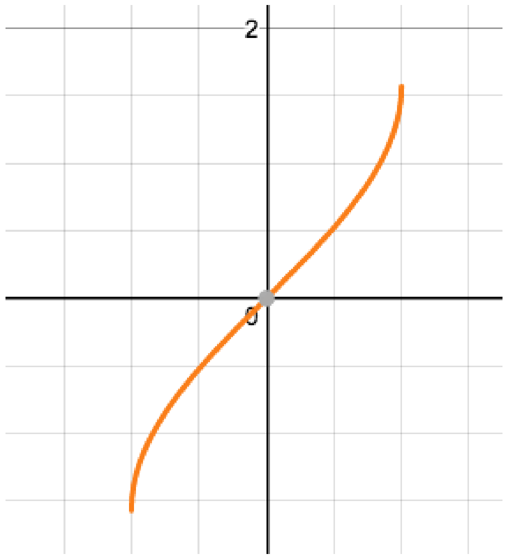
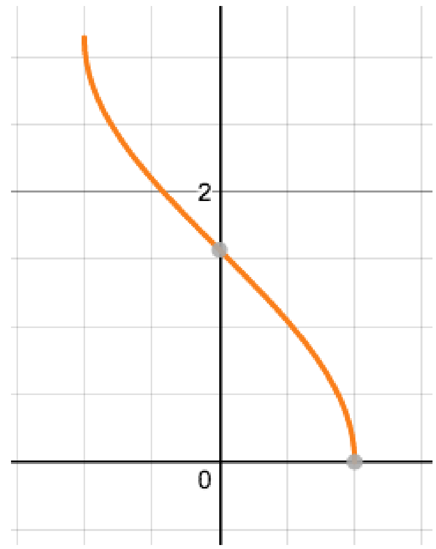
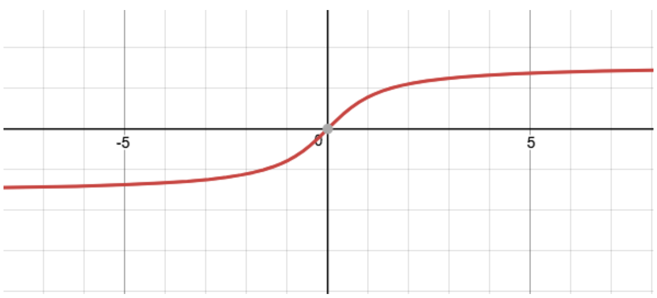
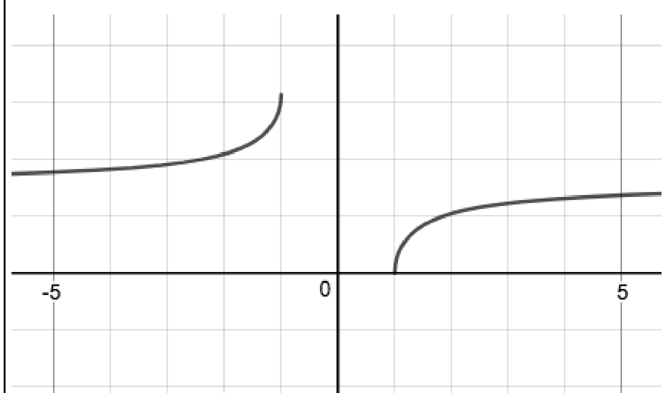
(a) $\lim_{x \to 0} (1+x)^{1,000,000}$	(b) $\lim_{x \to 0} 1^{\frac{1}{x}}$
(c) $\lim_{x \to 0} 1.000001^{\frac{1}{x}}$	(d) $\lim_{x \to 0} 0.99999^{\frac{1}{x}}$
(e) $\lim_{x \to 0^+} (1+\epsilon)^{\frac{1}{x}}$, $\epsilon > 0$	(f) $\lim_{x \to 0^-} (1+\epsilon)^{\frac{1}{x}}$, $\epsilon > 0$
(g) $\lim_{x \to 0} (1+5\text{x})^{\frac{1}{x}}$	(h) $\lim_{x \to 0} (1+mx)^{\frac{p}{x}}$, $m, p \in \mathbb{R}$

6.8 Trigonometric Functions and Their Derivatives

Ex 1: Evaluate the following expressions. (a) $\tan\left(2\tan^{-1}\left(\frac{1}{3}\right)\right)$	A. Trigonometry Review (Inverse functions and their ranges) $x=\sin^{-1}y \Leftrightarrow y=\sin x\ ,\ x\in\left[\frac{-\pi}{2}\ ,\ \frac{\pi}{2}\right]$ $x=\cos^{-1}y \Leftrightarrow y=\cos x\ ,\ x\in[0\ ,\ \pi]$ $x=\tan^{-1}y \Leftrightarrow y=\tan x\ ,\ x\in\left(\frac{-\pi}{2}\ ,\ \frac{\pi}{2}\right)$ $x=sec^{-1}y \Leftrightarrow y=sec\,x\ ,\ x\in[0\ ,\ \pi]\ ,\ x\neq\frac{\pi}{2}$
(b) $\lim\limits_{x\to1}\sin^{-1}x$	B. More Trigonometry Review (Equivalent inverse functions) $\sec^{-1}y=\cos^{-1}\left(\frac{1}{y}\right)$ $\cot^{-1}y=\tan^{-1}\left(\frac{1}{y}\right)$ $\csc^{-1}y=\sin^{-1}\left(\frac{1}{y}\right)$
(c) $\lim\limits_{x\to-\infty}\tan^{-1}x$	C. Equivalent Algebraic and Trigonometric Expressions $\sin(\cos^{-1}(x))=\sqrt{1-x^2}=\cos(\sin^{-1}(x))$ $\sec(\tan^{-1}(x))=\sqrt{1+x^2}$ $\tan(\sec^{-1}(x))=\begin{cases}\sqrt{x^2-1}\ ,\ x\geq1\\-\sqrt{x^2-1}\ ,\ x\leq-1\end{cases}$
E. New: Derivatives of Inverse Trigonometric Functions $D_x(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}\ ,\ x\in(-1,1)$ $D_x(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}\ ,\ x\in(-1,1)$ $D_x(\tan^{-1}x)=\frac{1}{1+x^2}$ $D_x(\sec^{-1}x)=\frac{1}{\lvert x\rvert\sqrt{x^2-1}}\ ,\ \lvert x\rvert>1$	D. Calculus Review (Trigonometric Derivatives) $D_x(\sin x)=\cos x$ $D_x(\cos x)=-\sin x$ $D_x(\tan x)=\sec^2 x$ $D_x(\sec x)=\sec x\tan x$ $D_x(\csc x)=-\csc x\cot x$ $D_x(\cot x)=-\csc^2 x$

6.8 (continued)

Graphs:

$y = \sin^{-1} x$ 	$y = \cos^{-1} x$ 
$y = \tan^{-1} x$ 	$y = \sec^{-1} x$ 

6.8 (continued)

Ex 2: Compute the following derivatives.

(a) $D_x\left(e^x \cos^{-1}\left(x^3\right)\right)$	(b) $D_x\left(\dfrac{x^3}{\sin^{-1}\left(x^2+1\right)}\right)$

Ex 3: Evaluate the following integrals.

(a) $\int \dfrac{x}{\sqrt{12-9x^2}}\,dx$	(b) $\int_{\sqrt{2}}^{2} \dfrac{1}{x\sqrt{x^2-1}}\,dx$

6.8 (continued)

Ex 4: Compute the integral $\int \frac{1}{x^2+8x+18}dx$

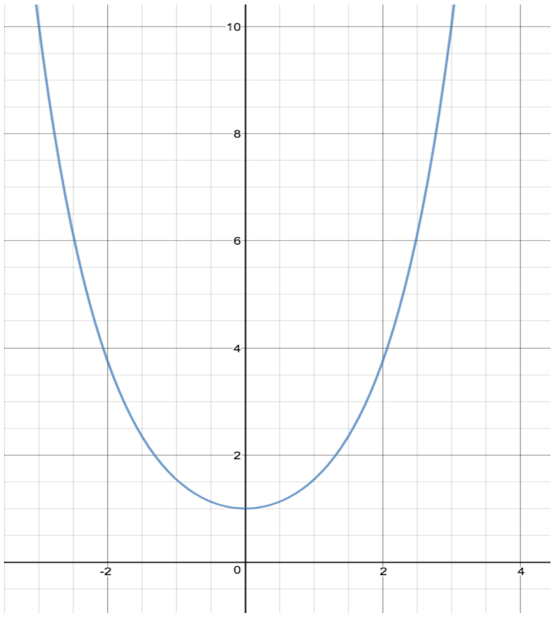
6.9 Hyperbolic Functions and Their Inverses

Ex 1: Find the derivatives. (a) $D_x\left(\ln\left(\operatorname{sech} x+\sinh x\right)\right)$	A. Hyperbolic Identities/Definitions $\cosh^2 x-\sinh^2 x=1$ $\sinh x=\frac{1}{2}\left(e^x-e^{-x}\right)$ $\cosh x=\frac{1}{2}\left(e^x+e^{-x}\right)$
(b) $D_x\left(\frac{\sinh^{-1}\left(x^3\right)}{2^{\tanh x}}\right)$	B. Derivatives of Hyperbolic Functions $D_x(\sinh x)=\cosh x$ $D_x(\cosh x)=\sinh x$ $D_x(\tanh x)=\operatorname{sech}^2 x$ $D_x(\operatorname{sech} x)=-\operatorname{sech} x\tanh x$ $D_x(\coth x)=-\operatorname{csch}^2 x$ $D_x(\operatorname{csch} x)=-\operatorname{csch} x\coth x$
(c) $D_x\left(\cos^{-1}\left(\cosh\left(2\text{x}\right)\right)\right)$	C. Inverse Hyperbolic Function Algebraic Equivalent Statements $\sinh^{-1} x=\ln\left(x+\sqrt{x^2+1}\right)$ $\cosh^{-1} x=\ln\left(x+\sqrt{x^2-1}\right)\ ,\ x\geq 1$ $\tanh^{-1} x=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\ ,\ x\in(-1,1)$ $\operatorname{sech}^{-1} x=\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)\ ,\ x\in(0,1]$
(d) $D_x\left(\tanh^{-1}\left(x^2\right)+\tan^{-1}\left(x^2\right)+\pi^e\right)$	D. Derivatives of Inverse Hyperbolic Functions $D_x\left(\sinh^{-1} x\right)=\frac{1}{\sqrt{x^2+1}}$ $D_x\left(\cosh^{-1} x\right)=\frac{1}{\sqrt{x^2-1}}\ ,\ x>1$ $D_x\left(\tanh^{-1} x\right)=\frac{1}{1-x^2}\ ,\ x\in(-1,1)$ $D_x\left(\operatorname{sech}^{-1} x\right)=\frac{-1}{x\sqrt{1-x^2}}\ ,\ x\in(0,1)$

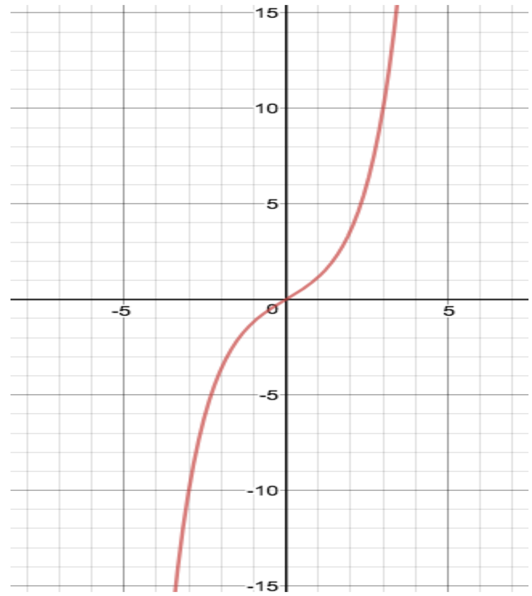
6.9 (continued)

Graphs:

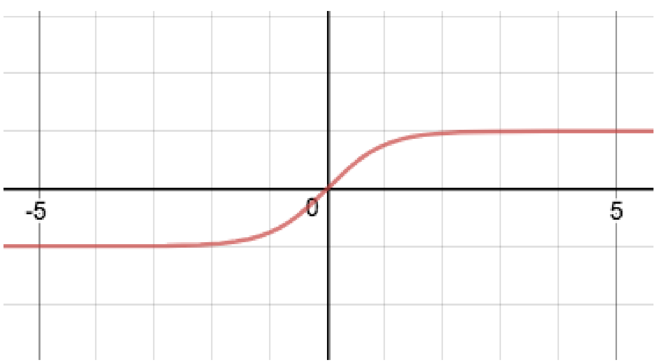
$y = \cosh x$



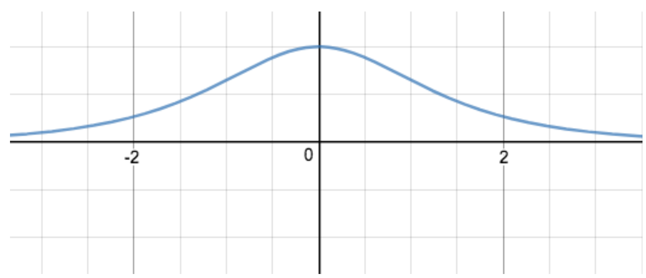
$y = \sinh x$



$y = \tanh x$



$y = \text{sech}\, x$



6.9 (continued)

Ex 2: Compute the integrals.

(a) $\int 2x \operatorname{sech}^2(x^2-3)\,dx$

(b) $\int \frac{e^x}{1-e^{2x}}\,dx$