# 10.4 Plane Curves: Parametric Representation

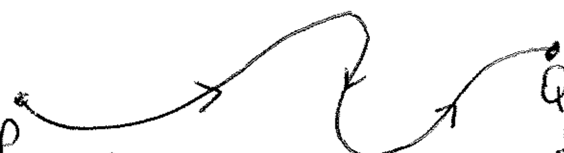
Remember from 5.4 (in Math 1210), a <u>plane curve</u> is a 2d curve given by

$$x = f(t) \quad \& \quad y = g(t) \quad \text{w/} \quad t \in I$$

& $f$ + $g$ are continuous functions on interval $I$ $[a,b]$

$t$ = <u>parameter</u> (we can think of it measuring time)

$t \in [a,b]$

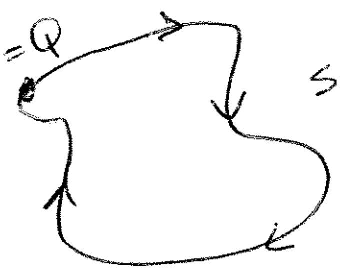


P $(x(a), y(a))$ <u>initial endpt</u>

$Q(x(b), y(b))$ <u>final endpt</u>

arrowheads indicate $t$ going from $a$ to $b$.

simple curve
not closed

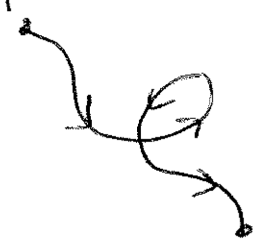


$P=Q$

simple, closed curve

$P=Q$

closed, not simple curve



P

Q

Not simple, not closed curve.

It's harder to recognize curve shape when given parametrically. Sometimes, it's possible to eliminate the parameter.

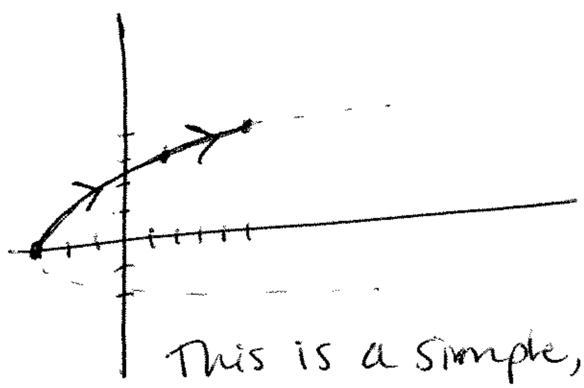
**Ex 1** $x = t - 3$ $y = \sqrt{2t}$ $0 \le t \le 8$

$$\Downarrow$$

$$y^2 = 2t$$

$$\frac{y^2}{2} = t$$

$\Rightarrow x = \frac{y^2}{2} - 3$ which we know is a parabola (facing right)



But, we only want that piece of the sideways parabola when $0 \le t \le 8$

$\Rightarrow -3 \le x \le 5$ + $0 \le y \le 4$.

This is a simple, not closed curve.

**Ex 2** For $x = 3\sqrt{t-3}$ + $y = 2\sqrt{4-t}$ $3 \le t \le 4$, eliminate the parameter $t$, graph the curve + tell if it's simple + closed.

10.4 (continued)

**Ex 3** For $x = \sin\theta$ & $y = 2\cos^2(2\theta)$ $\theta \in \mathbb{R}$, eliminate the parameter $\theta$, graph the curve & tell if it's simple & closed.

10.4 (continued)

**Thm A**

Let f + g be continuously differentiable w/ $f'(t) \neq 0$ on $t \in (\alpha, \beta)$. Then the parametric eqns

$$x = f(t) \quad + \quad y = g(t)$$

define y as a differentiable fn of x + where

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} \quad y' = \frac{dy}{dx}}$$

**Ex 4** Find $\frac{dy}{dx}$ + $\frac{d^2y}{dx^2}$ (w/o eliminating parameter)

(a) $x = \sqrt{3}\,\theta^2$ + $y = -\sqrt{3}\,\theta^3$ $\quad \theta \neq 0$

(b) $x = \frac{2}{1+t^2}$ $\quad y = \frac{2}{t(1+t^2)}$ $\quad t \neq 0$

10.4 (continued)

Note: $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

arc-length formula (sectn 6.4)

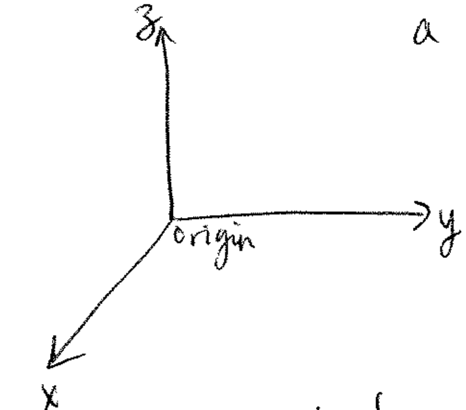
Ex 5

Find length of curve given by

$x = \sin\theta - \theta\cos\theta$ $\quad \theta \in [\pi/4, \pi/2]$

$y = \cos\theta + \theta\sin\theta$

# 11.1 Cartesian Coordinates in 3-Space

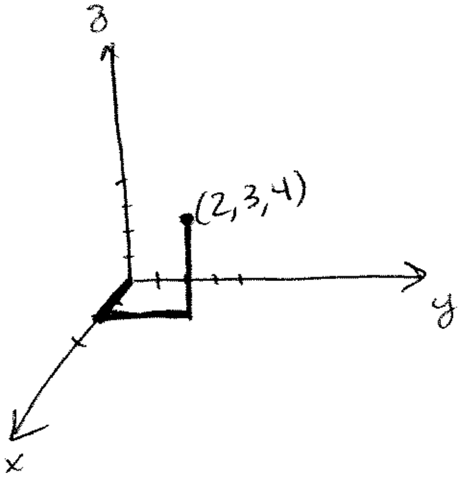



a point in 3-space is given by an ordered triple $(x, y, z)$

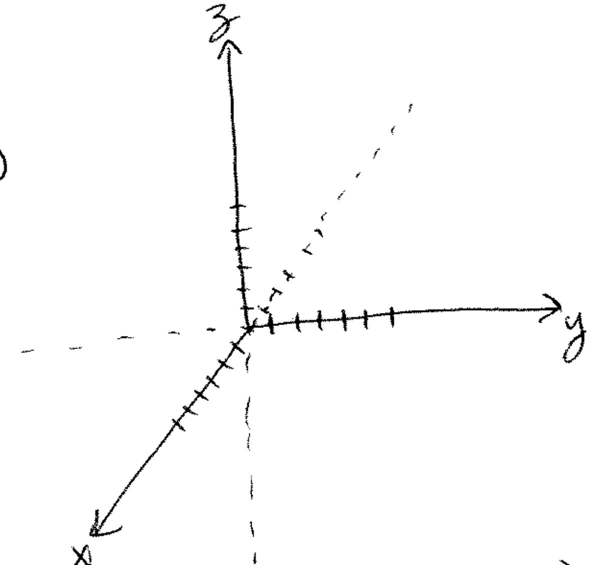
This is a rt-handed system (because if I curl my fingers from the +ve x-directn to the +ve y-directn, then my thumb pts in the +ve z-directn).

x-axis comes out of paper.

☆ A rt-handed system is standard.

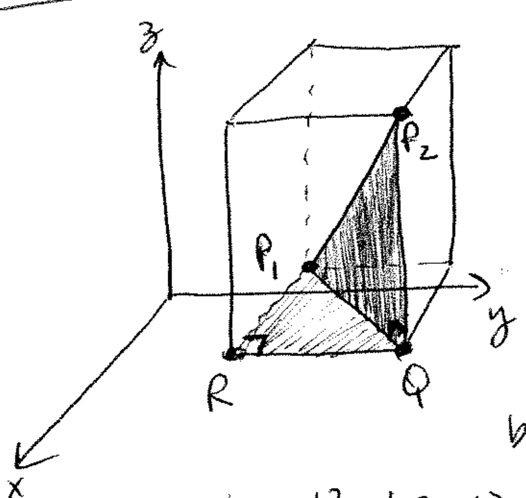



Plot $(2, 3, 4)$




Plot $(-1, 4, -3)$

## Distance Formula




$\Delta P_1P_2Q$ + $\Delta P_1QR$ are both right $\Delta$s.

$\Rightarrow |P_1Q|^2 + |QP_2|^2 = |P_1P_2|^2$

and $|P_1R|^2 + |RQ|^2 = |P_1Q|^2$

by Pythagorean Thm

$\Rightarrow |P_1R|^2 + |RQ|^2 + |QP_2|^2 = |P_1P_2|^2$

11.1 (continued)

If $P_1 = (x_1, y_1, z_1)$ + $P_2 = (x_2, y_2, z_2)$, then
$R = (x_2, y_1, z_1)$ + $Q = (x_2, y_2, z_1)$.

$$\Rightarrow |P_1 R|^2 = \left(\sqrt{(x_2 - x_1)^2}\right)^2 = (x_2 - x_1)^2$$

and likewise $|RQ|^2 = (y_2 - y_1)^2$, $|QP_2|^2 = (z_2 - z_1)^2$

$$\Rightarrow \quad |P_1 P_2|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

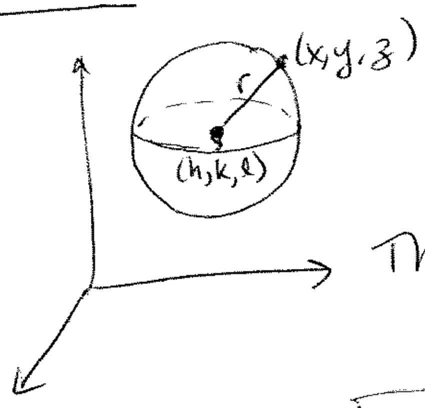
$$\Leftrightarrow \quad \boxed{|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

distance between $P_1$ + $P_2$

Ex 1 Show that $(4,5,3)$, $(1,7,4)$ + $(2,4,6)$ are vertices of an equilateral triangle.

# 11.1 (continued)

## Spheres




For a sphere, all pts $(x,y,z)$ on surface of sphere are a fixed distance, $r$, from its center.

That is,

$$r = \sqrt{(x-h)^2 + (y-k)^2 + (z-l)^2}$$

$$\Rightarrow \boxed{r^2 = (x-h)^2 + (y-k)^2 + (z-l)^2}$$ Sphere Eqn.

If anything fits this formula, then it's either a sphere (if $r>0$), a single pt (if $r=0$) or the empty set (if $r^2<0$).

**Ex 2** Given $x^2+y^2+z^2+2x-6y-10z+34=0$, find the center & radius of this sphere.

## 11.1 (continued)

**Midpt** $M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$ is midpt between $(x_1, y_1, z_1)$ + $(x_2, y_2, z_2)$.

**Linear Eqns** (in 3-space)

a linear eqn of the form $Ax + By + Cz = D$ $\ni A^2 + B^2 + C^2 \neq 0$ graphs into a plane

★ trace ⇒ line of intersectn between plane w/ coordinate planes

**Ex 3** Graph $3x - 4y + 2z = 24$

Get x-intercept:
y-intercept:
z-intercept:

**Ex 4** Graph $3x + 4y = 12$

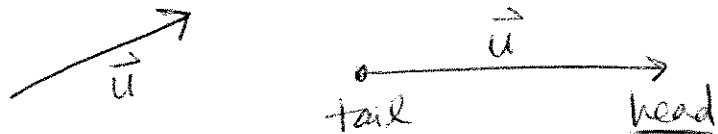
# 11.2 Vectors (Geometric Approach)

<u>scalars</u> $\Rightarrow$ $\mathbb{R}$ numbers

<u>vector</u> $\Rightarrow$ has ① direction + ② magnitude; in book, they show up as boldfaced lowercase letters, like **u**; in writing, its notation is $\vec{u}$.

<u>magnitude</u> $\Rightarrow$ length of vector; denoted by $|\vec{u}|$.

<u>direction</u> $\Rightarrow$ of vector is geometrically indicated by arrowhead

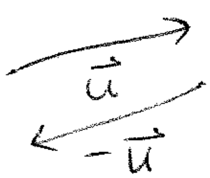



$\vec{u} = \vec{v}$ if magnitude + direction of $\vec{u}$ + $\vec{v}$ are the same.

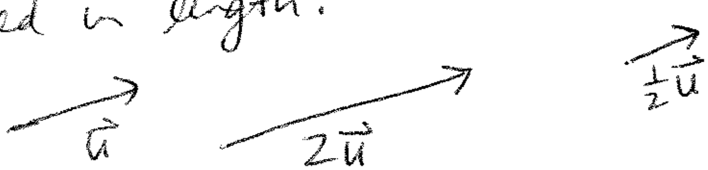
Notice that location doesn't matter for a vector, only magnitude (a.k.a. length) + direction.

<u>zero vector</u> $\Rightarrow$ $\vec{0}$ and $\vec{0} + \vec{u} = \vec{u} + \vec{0} = \vec{u}$
(a zero vector has zero magnitude)

$-\vec{u}$ $\Rightarrow$ means same vector as $\vec{u}$, only pointing in opposite direction

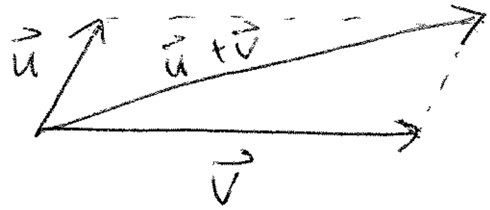



scalar multiple of $\vec{u}$ $\Rightarrow$ $c\vec{u}$, where $c \in \mathbb{R}$, means we have a vector in the direction of $\vec{u}$ but scaled in length.

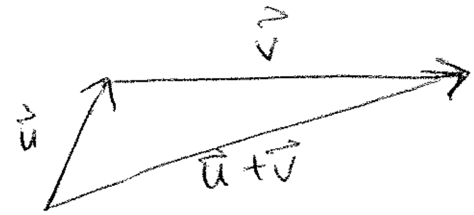


## 11.2 (continued)

Adding vectors $\Rightarrow$ $\vec{u}+\vec{v}$



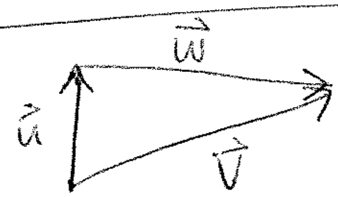

OR




Put $\vec{u}$ + $\vec{v}$ so tails are coincident. Then, they form 2 sides of a parallelogram. And $\vec{u}+\vec{v}$ is diagonal of that parallelogram w/ its tail at same location as tails of $\vec{u}$ + $\vec{v}$.

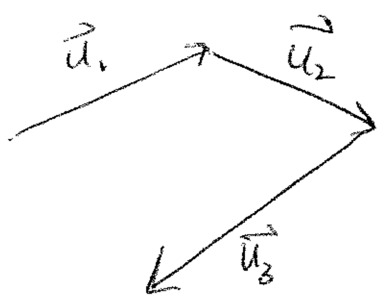
Place $\vec{u}$ + $\vec{v}$ tail-to-head. Then, $\vec{u}+\vec{v}$ initiates at tail of $\vec{u}$ + ends at head of $\vec{v}$.

**Ex 1**

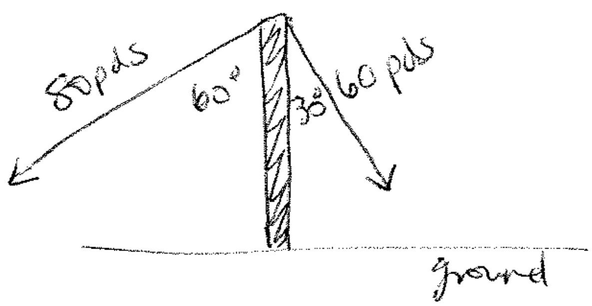



Express $\vec{w}$ in terms of $\vec{u}$ + $\vec{v}$.

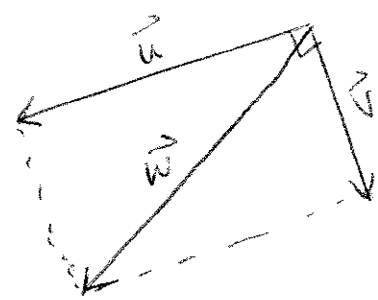
**Ex 2** Draw $\vec{w}$ $\ni$ $\vec{w}=\vec{u}_1+\vec{u}_2+\vec{u}_3$



**Ex 3** (#10) Mark pushes on a post in the direction S 30° E (30° East of South), with a force of 60 pds. Dan pushes on the same post in the direction S 60° W with a force of 80 pds. What are the magnitude + direction of the resulting force?




We have




$|\vec{u}| = 80$

$|\vec{v}| = 60$

resultant force is

$\vec{w} = \vec{u} + \vec{v}$ as drawn

## 11.2 (continued)

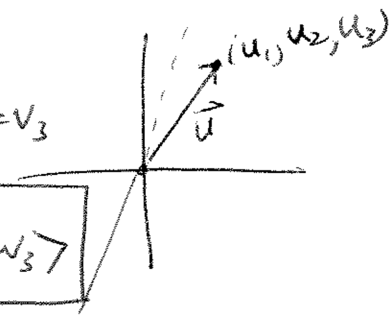
☆ We treat velocities as vectors since they have both magnitude + direction.
$|\vec{v}| =$ "speed"

### Ex 4 (#14)

A ship is sailing due south at 20 mph. A man walks west across the deck at 3 mph. What are the magnitude + direction of his velocity relative to the surface of the water?

# 11.2 Vectors (Algebraic Approach)

If we place our vector $\vec{u}$ on a Cartesian coordinate system with its tail at the origin $(0,0,0)$, then its head will end at some pt $(u_1, u_2, u_3)$. Then, we say $\boxed{\vec{u} = \langle u_1, u_2, u_3 \rangle}$ (Notice the different brackets which distinguish between pts + vectors!)




$u_1, u_2 + u_3$ are called components of $\vec{u}$.

$\vec{u} + \vec{v}$ are equal iff $u_1 = v_1$, $u_2 = v_2$ + $u_3 = v_3$

$$\boxed{\vec{u} + \vec{v} = \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle}$$

$$\boxed{-\vec{u} = \langle -u_1, -u_2, -u_3 \rangle} \qquad \boxed{c\vec{u} = \langle cu_1, cu_2, cu_3 \rangle}$$

$$\boxed{\vec{0} = 0\vec{u} = \langle 0, 0, 0 \rangle}$$

**Thm A**

$\forall$ vectors $\vec{u}, \vec{v} + \vec{w}$ + $a, b \in \mathbb{R}$

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$

$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

$\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$

$\vec{u} + (-\vec{u}) = \vec{0}$

$a(b\vec{u}) = (ab)\vec{u} = \vec{u}(ab)$

$a(\vec{u} + \vec{v}) = a\vec{u} + b\vec{v}$

$(a+b)\vec{u} = a\vec{u} + b\vec{v}$

$1\vec{u} = \vec{u}$

i.e. ① vector addition is commutative + associative ② we still have an additive identity element + additive inverses ③ scalar multiplication is commutative, associative + distributive and ④ there's a scalar multiplicative identity! We get to play a cool new algebra game with vectors now!!

11.2 (Continued)

$$\boxed{|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}}$$ (Pythagorean Thm disguised again)

$|c\vec{u}| = |c|\,|\vec{u}|$ $\quad |c|$ is abs. value of $c$

but $|\vec{u}|$ is magnitude of vector $\vec{u}$

Ex 5 let $\vec{u} = \langle 4, 5, 2 \rangle$. Find $|\vec{u}|$ + $|-3\vec{u}|$.
Also, find a vector $\hat{u}$ (w/ same direction as $\vec{u}$ but with length of 1).

$\hat{u}$ = unit vector of $\vec{u}$

$|\hat{u}| = 1$

$\hat{u} = \dfrac{\vec{u}}{|\vec{u}|}$

Basis vectors $\Rightarrow$ $\hat{i} = \langle 1, 0, 0 \rangle$, $\hat{j} = \langle 0, 1, 0 \rangle$ and $\hat{k} = \langle 0, 0, 1 \rangle$.

All three are unit vectors in $\vec{x}$, $\vec{y}$ + $\vec{z}$ directions, respectively.

$\Rightarrow$ $\vec{u} = \langle u_1, u_2, u_3 \rangle = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ (Just different notation for same vector)

# 11.3 The Dot Product

Dot product $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$
(aka scalar product)

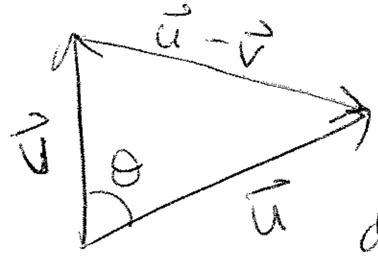
This is a type of multiplication between vectors.

**Thm A** Let $\vec{u}, \vec{v}$ + $\vec{w}$ be vectors, $c \in \mathbb{R}$.

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$
$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$
$$c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$$
$$\vec{0} \cdot \vec{u} = 0$$
$$\vec{u} \cdot \vec{u} = |\vec{u}|^2$$

i.e. dot products are ① commutative + ② distribute thru addition/subtraction, ③ scalar mult. w/ dot products is associative + commutative; ④ the zero property of dot products gives us 0 + ⑤ the dot product of a vector w/ itself is the length squared.

**Thm B**
Geometrically, we can think of dot product as $\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta$ where $\theta$ = smallest nonnegative angle between $\vec{u}$ and $\vec{v}$




**Pf**
We can use this $\Delta$ made by the drawn vectors. Apply Law of Cosines,

$$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}|\,|\vec{v}| \cos\theta. \quad \text{(A)}$$

Also, we know (from dot product properties) that

$$|\vec{u} - \vec{v}|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})$$
$$= \vec{u} \cdot (\vec{u} - \vec{v}) - \vec{v} \cdot (\vec{u} - \vec{v})$$
$$= \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}$$

11.3 (continued)

So $\quad |\vec{u}-\vec{v}|^2 = |\vec{u}|^2 - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + |\vec{v}|^2$

$$= |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u}\cdot\vec{v} \quad (B)$$

$\Rightarrow$ (Equate (A) + (B).)

$$|\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta = |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u}\cdot\vec{v}$$

$$-2|\vec{u}||\vec{v}|\cos\theta = -2\vec{u}\cdot\vec{v}$$

$$|\vec{u}||\vec{v}|\cos\theta = \vec{u}\cdot\vec{v}$$

Thm C

$\vec{u}$ + $\vec{v}$ are $\perp$ iff $\vec{u}\cdot\vec{v} = 0$

$\perp$ vectors are called orthogonal

Ex 1 For what # c are $\langle 2c, -8, 1\rangle$ + $\langle 3, c, c-2\rangle$ $\perp$?

11.3 (cont)

Ex 2 (a) write vector represented by $\vec{AB}$ in the form $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

$A(-2,3,5) \quad B(1,-2,4)$

(b) Find a unit vector $\vec{u}$ in the direction of $\langle -3,5,6 \rangle$ and express it as $\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$.

# 11.3 (cont)

$$|\vec{u}||\hat{i}|\cos\alpha = \vec{u}\cdot\hat{i} \iff |\vec{u}|\cos\alpha = \langle u_1, u_2, u_3\rangle \cdot \langle 1, 0, 0\rangle$$

$$|\vec{u}|\cos\alpha = u_1$$

$$\boxed{\cos\alpha = \frac{u_1}{|\vec{u}|} \quad \text{and} \quad \cos\beta = \frac{u_2}{|\vec{u}|}, \quad \cos\gamma = \frac{u_3}{|\vec{u}|}}$$

where $\alpha$ = angle between vector $\vec{u}$ + x-axis
$\beta$ = angle between vector $\vec{u}$ + y-axis
$\gamma$ = angle between vector $\vec{u}$ + z-axis

Notice $\cos^2\alpha + \cos^2\beta + \cos^2\gamma =$

$$\left(\frac{u_1}{|\vec{u}|}\right)^2 + \left(\frac{u_2}{|\vec{u}|}\right)^2 + \left(\frac{u_3}{|\vec{u}|}\right)^2$$

$$= \frac{u_1^2 + u_2^2 + u_3^2}{|\vec{u}|^2} = \frac{|\vec{u}|^2}{|\vec{u}|^2} = 1$$

i.e. $\boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$

11.3 (cont)

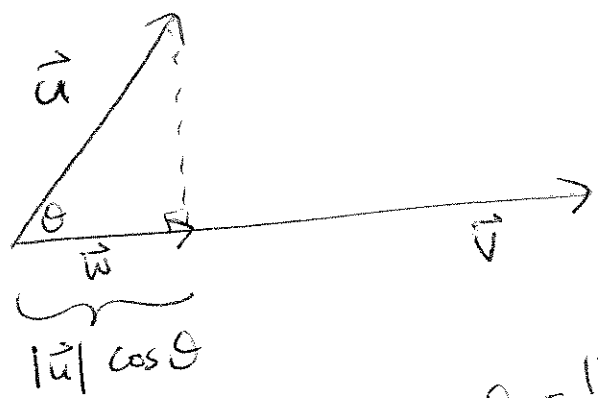
Ex 3 Find the direction cosines for

$\vec{u} = \langle -1, 2, -2 \rangle$.

Ex 4 Find the angle between $-4\hat{i} + 2\hat{j} + 3\hat{k} = \vec{u}$

and $\vec{v} = 2\hat{i} + \hat{j} + 5\hat{k}$.

# 11.3 (continued)




Since we know $\cos\theta = \dfrac{|\vec{w}|}{|\vec{u}|} \Rightarrow |\vec{w}| = |\vec{u}|\cos\theta$ ①

But $\vec{w}$ is in same direction as $\vec{v}$, so

$\vec{w} = c\vec{v}$ must be true for some $c \in \mathbb{R}$.

$\Rightarrow$ $|\vec{w}| = c|\vec{v}|$ combine with ① above

$\Rightarrow$ $|\vec{u}|\cos\theta = c|\vec{v}| \Leftrightarrow c = \dfrac{|\vec{u}|}{|\vec{v}|}\cos\theta$

But $\theta$ is angle between $\vec{u}$ + $\vec{v}$,

$\Rightarrow \vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta \Leftrightarrow \cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$

$\Rightarrow c = \dfrac{|\vec{u}|}{|\vec{v}|}\dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}$

$\Rightarrow \boxed{\vec{w} = c\vec{v} = \left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v}}$ This is the projection of $\vec{u}$ onto $\vec{v}$!

$\boxed{\vec{w} = (\vec{u}\cdot\hat{v})\hat{v}}$

Notation: $\mathrm{pr}_v\vec{u}$ = projection of $\vec{u}$ onto $\vec{v}$

## 11.3 (cont)

**Ex 5** Let $\vec{u} = \langle 1, 6, -2 \rangle$ + $\vec{v} = \langle -3, 2, 5 \rangle$.

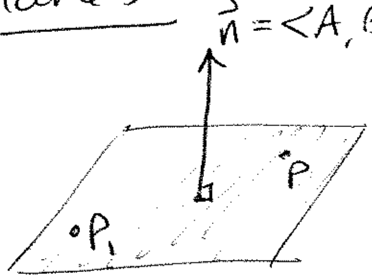
Find the vector projection of $\vec{u}$ onto $\vec{v}$.

**Ex 6** If $\vec{u} = e\hat{i} + \pi\hat{j} + \hat{k}$ + $\vec{v} = \langle 1, 1, 0 \rangle$, express $\vec{u}$ as the sum of a vector $\vec{m} \parallel$ to $\vec{v}$ + $\vec{n} \perp$ to $\vec{v}$.

($\vec{m} = \text{pr}_{\vec{v}}\vec{u} = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\vec{v}$ and $\vec{n} = \vec{u} - \vec{m}$)

# 11.3 (continued)

## Planes




If we have a plane with nonzero normal $\vec{n} = \langle A, B, C \rangle$, then every pt $P(x, y, z)$ will satisfy $\overrightarrow{P_1P} \cdot \vec{n} = 0$ where $P_1(x_1, y_1, z_1)$ is a pt on the plane (and every P is on the plane).

$\Rightarrow \quad \overrightarrow{P_1P} = \langle x - x_1, y - y_1, z - z_1 \rangle$

and $\quad \overrightarrow{P_1P} \cdot \vec{n} = \langle x - x_1, y - y_1, z - z_1 \rangle \cdot \langle A, B, C \rangle$

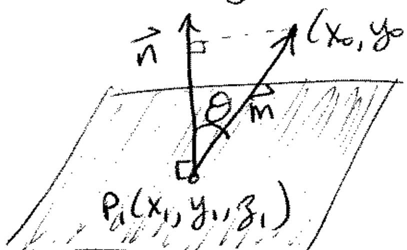
$$= \boxed{\begin{array}{c} A(x - x_1) + B(y - y_1) + C(z - z_1) = 0 \\ \hline \text{standard eqn of a plane} \end{array}}$$

Generally, a plane is given by a linear eqn $\quad Ax + By + Cz = D \quad \ni A^2 + B^2 + C^2 \neq 0$ (i.e. the normal vector can't be the zero vector).

**Ex 4** Find the eqn of the plane through $(1, -3, 4)$ $\perp$ to $\vec{n} = \langle -1, 2, -1 \rangle$.

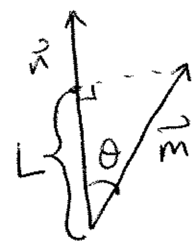
# 11.3 (continued)

Distance from pt $(x_0, y_0, z_0)$ to plane $Ax + By + Cz = D$ $\Rightarrow$ We want the $\perp$ distance from the pt to the plane. So we want to project $\vec{m}$ onto $\vec{n}$ + find that distance.




let $\vec{m}$ = vector from $P_1$ to $(x_0, y_0, z_0)$.

$$\Rightarrow L = \left|\, |\vec{m}| \cos\theta \,\right|$$




$$\Leftrightarrow \quad L = \frac{|\vec{m} \cdot \vec{n}|}{|\vec{n}|} \text{ (A)} \quad \text{and} \quad \vec{n} = \langle A, B, C \rangle$$

$$\vec{m} = \langle x_0 - x_1, y_0 - y_1, z_0 - z_1 \rangle$$

$$L = \frac{|A(x_0 - x_1) + B(y_0 - y_1) + C(z_0 - z_1)|}{\sqrt{A^2 + B^2 + C^2}}$$

$$L = \frac{|(Ax_0 + By_0 + Cz_0) - (Ax_1 + By_1 + Cz_1)|}{\sqrt{A^2 + B^2 + C^2}}$$

But $(x_1, y_1, z_1)$ is on the plane + satisfies

$$Ax_1 + By_1 + Cz_1 = D$$

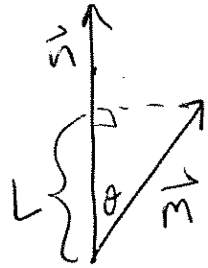
$$\Rightarrow \boxed{L = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}}$$

distance from pt $(x_0, y_0, z_0)$ to plane $Ax + By + Cz = D$

[ $(x_0, y_0, z_0)$ is random pt in space ]

(A) see next "add-on" page of notes for explanation

Math 2210 | Add-on to pg (24) of notes




$$L = \left|\text{proj}_n \vec{m}\right|$$

Using projection (rather than trig)

$$\text{proj}_n \vec{m} = \left(\frac{\vec{m}\cdot\vec{n}}{|\vec{n}|^2}\right)\vec{n}$$

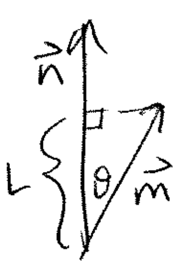
$$\Rightarrow \left|\text{proj}_n \vec{m}\right| = \left|\left(\frac{\vec{m}\cdot\vec{n}}{|\vec{n}|^2}\right)\vec{n}\right|$$

$$= \left(\frac{|\vec{m}\cdot\vec{n}|}{|\vec{n}|^2}\right)|\vec{n}| = \frac{|\vec{m}\cdot\vec{n}|\,|\vec{n}|}{|\vec{n}|^2}$$

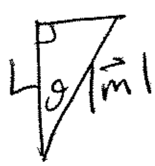
$$\underbrace{\left|\text{proj}_n \vec{m}\right| = \frac{|\vec{m}\cdot\vec{n}|}{|\vec{n}|} = L}$$

---

Using trig (rather than projection)




$\Rightarrow$




$$\cos\theta = \frac{L}{|\vec{m}|}$$

$$\Rightarrow L = |\vec{m}|\cos\theta$$

but $\cos\theta = \dfrac{\vec{m}\cdot\vec{n}}{|\vec{m}|\,|\vec{n}|} \Rightarrow L = |\vec{m}|\left(\dfrac{\vec{m}\cdot\vec{n}}{|\vec{m}|\,|\vec{n}|}\right)$

$$\underbrace{\Leftrightarrow L = \frac{\vec{m}\cdot\vec{n}}{|\vec{n}|}}$$

where we assume we're chosen $\theta$ as the "smaller" angle between $\vec{n}$ + $\vec{m}$ $\Rightarrow \cos\theta$ is +ve

**Ex 5** Find the distance between parallel planes
$-3x+2y+z=9$ & $6x-4y-2z=19$.

**Ex 6** Find the (smaller) angle between two planes
$3x-2y+5z=7$ and $4x-2y-3z=2$.

# 11.4 The Cross Product

$$\boxed{\vec{u} \times \vec{v} = \langle u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1 \rangle \quad \text{where } \vec{u} = \langle u_1, u_2, u_3 \rangle \ \& \ \vec{v} = \langle v_1, v_2, v_3 \rangle}$$

Notice that a dot product yields a scalar, but a cross product yields a vector!

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= (u_2v_3 - v_2u_3)\hat{i} + (u_3v_1 - u_1v_3)\hat{j} + (u_1v_2 - v_1u_2)\hat{k}$$

This format is easier to remember than the above formula. You just need to remember determinants.

Notice $\boxed{\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}}$

**Ex 1** If $\vec{a} = \langle 3, 3, 1 \rangle$ & $\vec{b} = \langle -2, -1, 0 \rangle$ & $\vec{c} = \langle -2, -3, -1 \rangle$, find $\vec{a} \times (\vec{b} \times \vec{c})$

## 11.4 (continued)

**Thm A**
Let $\vec{u}$ + $\vec{v}$ be 3d vectors + $\theta$ is angle between them. Then (1) $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0 = \vec{v} \cdot (\vec{u} \times \vec{v})$, i.e. $\vec{u} \times \vec{v}$ is $\perp$ to both $\vec{u}$ + $\vec{v}$.
(2) $\vec{u}$, $\vec{v}$ + $\vec{u} \times \vec{v}$ form a rt-handed triple
(3) $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$

Pf (1) $\vec{u} \cdot (\vec{u} \times \vec{v}) = \langle u_1, u_2, u_3 \rangle \cdot \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle$

$= u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1)$

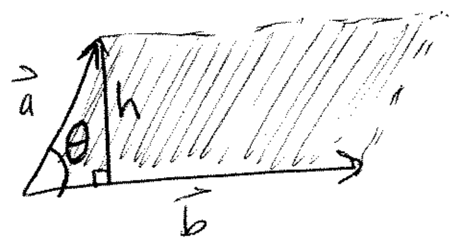
$= u_1u_2v_3 - u_1u_3v_2 + u_2u_3v_1 - u_1u_2v_3 + u_1u_3v_2 - u_2u_3v_1$

$= 0$. //

**Thm B** Two 3d vectors $\vec{u}$ + $\vec{v}$ are $\parallel$ iff $\vec{u} \times \vec{v} = \vec{0}$

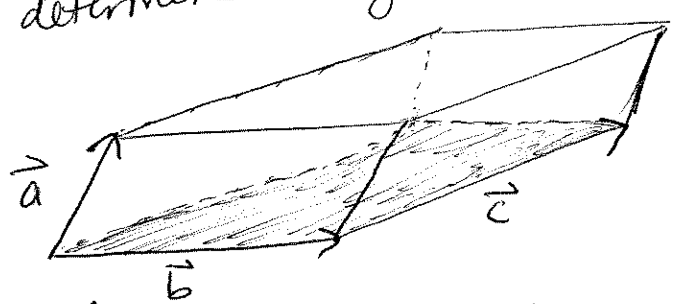
Ex 2 Find the plane through 3 pts $(-1,3,0)$ $(5,1,2)$ + $(4,-3,-1)$.

**Ex 3** Find area of parallelogram w/ $\vec{a}$ + $\vec{b}$ as adjacent sides.



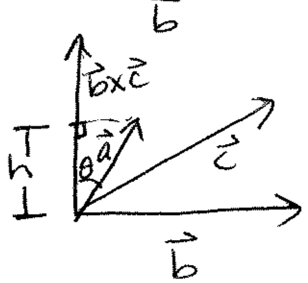

$$\sin\theta = \frac{h}{|\vec{a}|} \Rightarrow h = |\vec{a}|\sin\theta$$

**Ex 4** Find volume of parallelogram prism (box), determined by sides $\vec{a}$, $\vec{b}$ + $\vec{C}$.




Parallelogram base area is $\vec{b} \times \vec{c}$

$V$ = area of base * ht




$$\cos\theta = \frac{h}{|\vec{a}|} \Rightarrow h = |\vec{a}|\cos\theta \quad \text{but } \cos\theta = \frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{|\vec{a}||\vec{b}\times\vec{c}|}$$

**Thm C**

$\vec{u}, \vec{v} + \vec{w}$ are 3d vectors, $k \in \mathbb{R}$:

① $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ (anticommutativity)

② $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$ (left distributivity)

③ $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v})$

④ $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$ + $\vec{u} \times \vec{u} = \vec{0}$

⑤ $(\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w})$

⑥ $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

$\boxed{\hat{\imath} \times \hat{\jmath} = \hat{k} \quad \hat{\jmath} \times \hat{k} = \hat{\imath} \quad \hat{k} \times \hat{\imath} = \hat{\jmath}}$

**Ex 5** Calculate $\vec{u} \times \vec{v}$ if $\vec{u} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}$ + $\vec{v} = -5\hat{\imath} + 4\hat{\jmath}$.

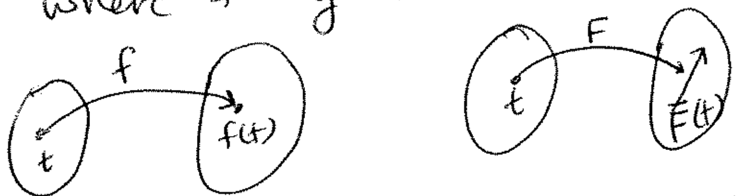
# 11.5 Vector-Valued Functions + Curvilinear Motion

<u>vector-valued function</u> $\Rightarrow$ a function $F$ of $t$ $\ni$ $t \in \mathbb{R}$ +

$F$ associates every input $t$ w/ an output vector $\vec{F}(t)$.

i.e. $\vec{F}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} = \langle f(t), g(t) \rangle$

where $f$ + $g$ are real-valued functions of $t$.




**Defn** $\lim_{t \to c} \vec{F}(t) = \vec{L}$ means that $\forall\, \varepsilon > 0$ $\exists$ a corresponding $\delta > 0$ $\ni$ $|\vec{F}(t) - \vec{L}| < \varepsilon$, provided $0 < |t-c| < \delta$, i.e.

$$0 < |t-c| < \delta \Rightarrow |\vec{F}(t) - \vec{L}| < \varepsilon.$$

**Thm** Let $\vec{F}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath}$. Then $\vec{F}$ has limit at $c$ $\Leftrightarrow$ $f$ + $g$ have limits at $c$.

and $\lim_{t \to c} \vec{F}(t) = \left[\lim_{t \to c} f(t)\right]\hat{\imath} + \left[\lim_{t \to c} g(t)\right]\hat{\jmath}$

<u>Continuity</u> $\Rightarrow$ $\vec{F}(t)$ is cont. if $\lim_{t \to c} \vec{F}(t) = \vec{F}(c)$.

<u>Derivative</u> $\Rightarrow$ $\vec{F}\,'(t) = \lim_{h \to 0} \dfrac{\vec{F}(t+h) - \vec{F}(t)}{h}$

# 11.5 (continued)

$$\Rightarrow \vec{F}'(t) = \lim_{h\to 0} \frac{[f(t+h)\hat{\imath} + g(t+h)\hat{\jmath}] - [f(t)\hat{\imath} + g(t)\hat{\jmath}]}{h}$$

$$= \lim_{h\to 0}\left[\frac{f(t+h)-f(t)}{h}\right]\hat{\imath} + \lim_{h\to 0}\left[\frac{g(t+h)-g(t)}{h}\right]\hat{\jmath}$$

$$\boxed{\vec{F}'(t) = f'(t)\hat{\imath} + g'(t)\hat{\jmath}}$$

## Differentiation Formulas

$\vec{F}$ & $\vec{G}$ are differentiable
$c \in \mathbb{R}$
$h$ is differentiable

1) $D_x[\vec{F}(x) + \vec{G}(x)] = \vec{F}'(x) + \vec{G}'(x)$
2) $D_x[c\vec{F}(x)] = c\vec{F}'(x)$
3) $D_x[h(x)\vec{F}(x)] = h(x)\vec{F}'(x) + h'(x)\vec{F}(x)$
4) $D_x[\vec{F}(x)\cdot\vec{G}(x)] = \vec{F}(x)\cdot\vec{G}'(x) + \vec{G}(x)\cdot\vec{F}'(x)$

"Product" Rules

5) $D_x[\vec{F}(h(x))] = \vec{F}'(h(x))h'(x)$ (chain Rule)

$$\boxed{\int \vec{F}(t)\,dt = \left[\int f(t)\,dt\right]\hat{\imath} + \left[\int g(t)\,dt\right]\hat{\jmath}}$$

**Ex 1** $\displaystyle \lim_{t\to\infty}\left[\frac{t\sin t}{t^2}\hat{\imath} - \frac{7t^3}{t^3-3t}\hat{\jmath}\right]$

11.5 (continued)

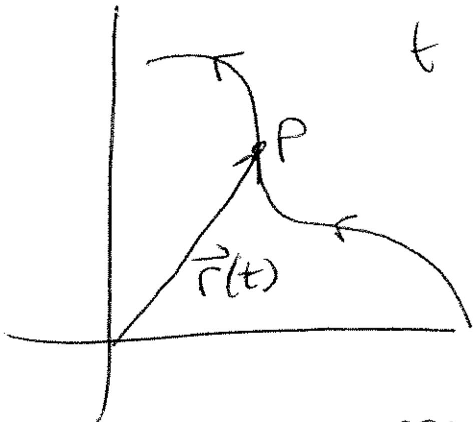
Ex 2 Find $\vec{r}'(x)$ + $\vec{r}''(x)$ for

$$\vec{r}(x) = (e^x + e^{-x^2})\hat{\imath} + 2^x\hat{\jmath}$$

Ex 3 $\vec{f}(y) = \tan^2 y\,\hat{\imath} + \sin^2(\tan^2 y)\,\hat{\jmath}$

Find $\vec{f}'(y)$.

## 11.5 (continued)




$\vec{r}(t)$ is position vector at any time $t$ along a curve given by

$x = x(t)$ & $y = y(t)$

$\Rightarrow \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

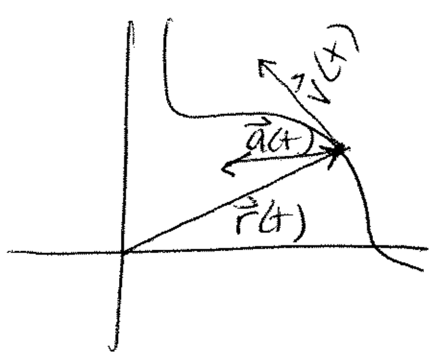
curvilinear motion $\Rightarrow$ the motion associated w/ tracing the path of the moving pt $P$ along the curve.

$\vec{v}(t) = \vec{r}\,'(t) = x'(t)\hat{i} + y'(t)\hat{j}$ (velocity)

(speed = magnitude of velocity vector)

$\vec{a}(t) = \vec{r}\,''(t) = \vec{v}\,'(t) = x''(t)\hat{i} + y''(t)\hat{j}$

(acceleration)




★ velocity vector is along tangent to curve

★ acceleration vector points to concave side of curve

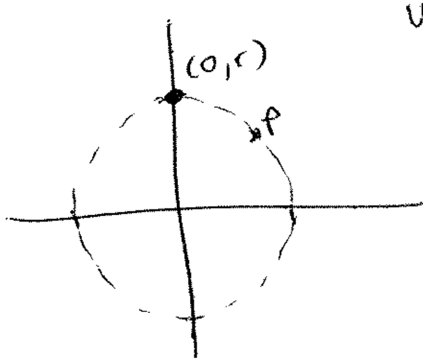
**Ex 4** (a) Given $\vec{r}(t) = 4\sin t\,\hat{i} + 8\cos t\,\hat{j}$, find $\vec{v}(t)$ & $\vec{a}(t)$.

## 11.5 (continued)

(b) (Ex 4 contd) Find speed when $t = \pi/4$.

(c) Sketch a portion of the graph of $\vec{r}(t)$ containing the position $P$ of the particle at $t = \pi/4$. (Draw $\vec{v}$ + $\vec{a}$ at $P$ as well.)

**Ex 5** Suppose that point P moves around a circle w/ center (0,0) & radius r at a constant angular speed of ω radians/sec. If its initial position is (0,r), find its acceleration.




We know

$$\vec{r}(t) = r\cos\omega t\,\hat{\imath} + r\sin\omega t\,\hat{\jmath}$$

or $\vec{r}(t) = r\sin\omega t\,\hat{\imath} + r\cos\omega t\,\hat{\jmath}$

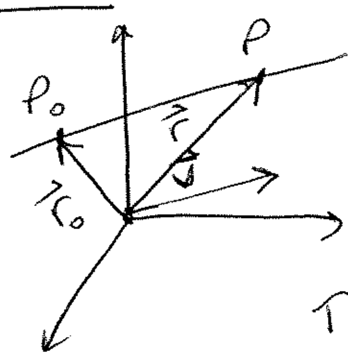
for circular motion.

# 11.6 Lines + Tangent lines in 3-space

A 3d curve can be given parametrically by $x = f(t)$, $y = g(t)$ + $z = h(t)$ $t \in I$, where $f, g$ + $h$ are all continuous on $I$.

We could specify curve as the curve traced out by the position vector
$$\vec{r}(t) = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}.$$

## Lines




Given a pt $P_0$ on a line and a fixed vector $\vec{v} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$ $\ni$ the line is $\parallel$ to $\vec{v}$ ($\overrightarrow{P_0P}$)

Then $\vec{r} = \vec{r}_0 + \vec{v}t$. And if $\vec{r} = \langle x, y, z \rangle$
i.e. $\vec{r} = \vec{r}_0$ + scalar multiple of $\vec{v}$

+ $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ $\quad t \in \mathbb{R}$

$$\Rightarrow \langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \langle a, b, c \rangle t$$

$$\Leftrightarrow \boxed{x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct}$$

$\boxed{\text{Parametric eqns of a line in 3d}}$

$a, b,$ + $c$ are called <u>direction #s</u> for the line (any constant multiple of $a, b + c$ are also direction numbers)

11.6 (continued)

Ex 1 Find parametric eqns of line through $(2,-1,5)$ & $(7,-2,3)$

Symmetric Eqns for a line

$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$ (assume $a \neq 0$, $b \neq 0$ & $c \neq 0$)

$\Rightarrow t = \dfrac{x - x_0}{a} \qquad t = \dfrac{y - y_0}{b} \qquad t = \dfrac{z - z_0}{c}$

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

This is basically the line of intersection between the two planes given by

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} \quad \text{and} \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

11.6 (continued)

Ex 2 Write the symmetric eqns for the line through $(-2, 2, -2)$ + parallel to $\langle 7, -6, 3\rangle$.

Ex 3 Find the symmetric eqns of the line thru $(-5, 7, -2)$ + $\perp$ to both $\langle 2, 1, -3\rangle$ + $\langle 5, 4, -1\rangle$.

Ex 4 Find the symmetric eqns of the line of intersection between the planes

$$x+y-z=2 \quad + \quad 3x-2y+z=3.$$

Tangent line to a Curve

If $\vec{r}=\vec{r}(t)=f(t)\,\hat{\imath}+g(t)\,\hat{\jmath}+h(t)\,\hat{k}$ is position vector along a curve in 3d, then $\vec{r}\,'(t)=\lim\limits_{h\to 0}\dfrac{\vec{r}(t+h)-\vec{r}(t)}{h}$

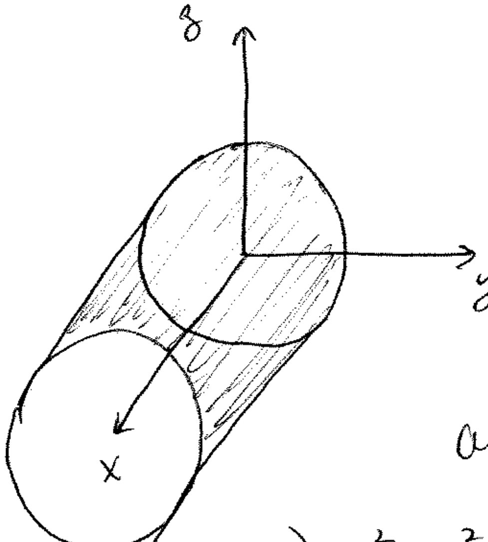
$\Rightarrow \vec{r}\,'(t)=f'(t)\,\hat{\imath}+g'(t)\,\hat{\jmath}+h'(t)\,\hat{k}$ and is in the direction of the tangent line to the 3d curve.

**Ex 5** Find the parametric eqns of tangent line to curve $x=2t^2, y=4t, z=t^3$ at $t=1$.

# 11.8 Surfaces in Three-Space

The graph of a 3-variable eqn is a surface in 3d. One technique for graphing them is to graph cross-sections (intersection of the surface with well-chosen planes) and/or traces (intersectns of the surface with the coordinate planes).

**Ex 1** Sketch a graph of $y^2+z^2=15$.




if $x=0$, $y^2+z^2=15$ is a circle.

In fact, regardless of what $x$ is, $y^2+z^2=15$ will be a circle. So, all cross sections are circles centered about $(x,0,0)$

$\Rightarrow y^2+z^2=15$ graphs into a right circular cylinder.

In computer graphics, it's common to show many cross sections to display the shape of a surface.

11.8 (continued)

a new defn of <u>cylinder</u> ⟹ the set of all pts on lines $\parallel$ to $l$ + that intersect $C$, where $C$ is a plane curve + $l$ is a line intersecting $C$ but not in the plane of $C$.

## Quadric Surfaces

A 3d surface whose eqn is of the second degree. The general eqn is

$$Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Iz+J=0$$

But, w/ rotation + translation, these possibilities can be reduced to 2 distinct types

① $Ax^2+By^2+Cz^2+J=0$

and ② $Ax^2+By^2+Iz=0$

11.8 (continued) Quadric Surfaces

QUADRIC SURFACES

**ELLIPSOID:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

| Plane | Cross Section |
|---|---|
| $xy$-plane | Ellipse |
| $xz$-plane | Ellipse |
| $yz$-plane | Ellipse |
| Parallel to $xy$-plane | Ellipse, point, or empty set |
| Parallel to $xz$-plane | Ellipse, point, or empty set |
| Parallel to $yz$-plane | Ellipse, point, or empty set |

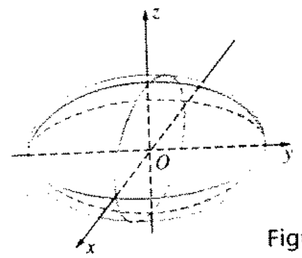


Figure 7

**HYPERBOLOID OF ONE SHEET:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

| Plane | Cross Section |
|---|---|
| $xy$-plane | Ellipse |
| $xz$-plane | Hyperbola |
| $yz$-plane | Hyperbola |
| Parallel to $xy$-plane | Ellipse |
| Parallel to $xz$-plane | Hyperbola |
| Parallel to $yz$-plane | Hyperbola |

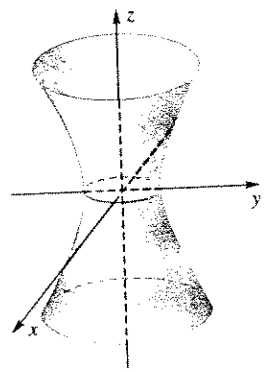


Figure 8

QUADRIC SURFACES (continued)

**HYPERBOLOID OF TWO SHEETS:** $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

| Plane | Cross Section |
|---|---|
| $xy$-plane | Hyperbola |
| $xz$-plane | Hyperbola |
| $yz$-plane | Empty set |
| Parallel to $xy$-plane | Hyperbola |
| Parallel to $xz$-plane | Hyperbola |
| Parallel to $yz$-plane | Ellipse, point, or empty set |

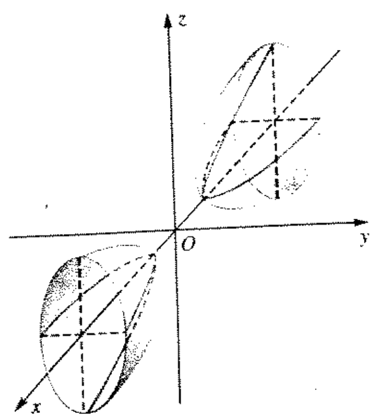


Figure 9

**ELLIPTIC PARABOLOID:** $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

| Plane | Cross Section |
|---|---|
| $xy$-plane | Point |
| $xz$-plane | Parabola |
| $yz$-plane | Parabola |
| Parallel to $xy$-plane | Ellipse, point, or empty set |
| Parallel to $xz$-plane | Parabola |
| Parallel to $yz$-plane | Parabola |

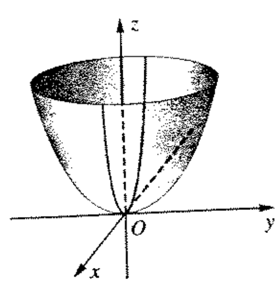


Figure 10

## 11.8 (continued) Quadric Surfaces

**HYPERBOLIC PARABOLOID:** $z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$

| Plane | Cross Section |
|---|---|
| $xy$-plane | Intersecting straight lines |
| $xz$-plane | Parabola |
| $yz$-plane | Parabola |
| Parallel to $xy$-plane | Hyperbola or intersecting straight lines |
| Parallel to $xz$-plane | Parabola |
| Parallel to $yz$-plane | Parabola |

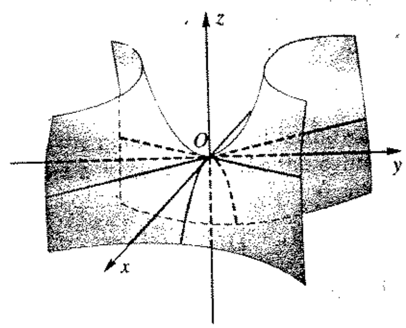



Figure 11

**ELLIPTIC CONE:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

| Plane | Cross Section |
|---|---|
| $xy$-plane | Point |
| $xz$-plane | Intersecting straight lines |
| $yz$-plane | Intersecting straight lines |
| Parallel to $xy$-plane | Ellipse or point |
| Parallel to $xz$-plane | Hyperbola or intersecting straight lines |
| Parallel to $yz$-plane | Hyperbola or intersecting straight lines |

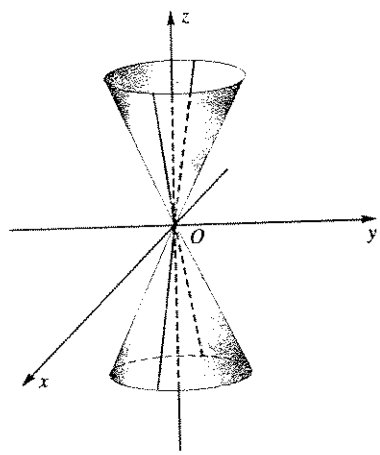



Figure 12

**Ex 2** Name these graphs

(a) $9x^2 + y^2 - 16z^2 = -25$

(b) $9x^2 + y^2 - 16z^2 = 25$

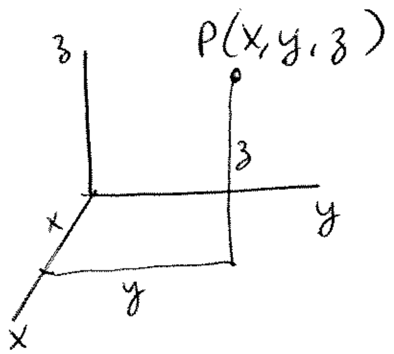
(c) $x^2 + 4y^2 - 100z = 0$

(d) $x^2 - y^2 = 0$

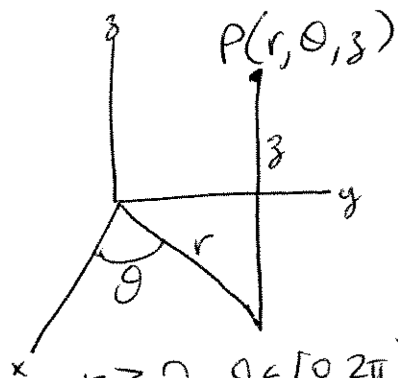
(e) $x^2 - y^2 = 25$

# 11.9 Cylindrical + Spherical Coordinates

**Cartesian**

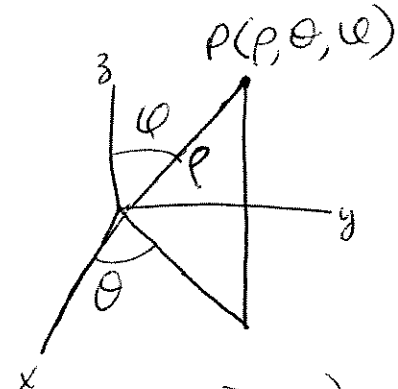



**Cylindrical**




$r \geq 0, \ \theta \in [0, 2\pi)$

**Spherical**




$\rho \geq 0 \quad \theta \in [0, 2\pi)$
$\varphi \in [0, \pi]$

Same pt P can be described 3 different ways!

**Cylindrical Coords**

to Cartesian $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$\left.\begin{array}{l} r = \sqrt{x^2+y^2} \\ \tan\theta = y/x \\ z = z \end{array}\right\}$ to Cylindrical

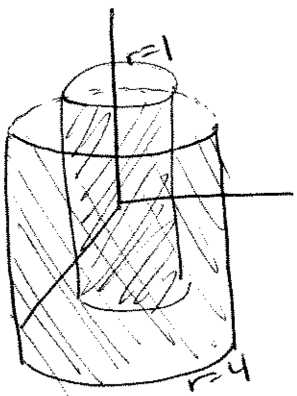
**Spherical Coords**

to Cartesian $\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$

$\left.\begin{array}{l} \rho = \sqrt{x^2+y^2+z^2} \\ \tan\theta = y/x \\ \cos\varphi = \dfrac{z}{\sqrt{x^2+y^2+z^2}} \end{array}\right\}$ to spherical

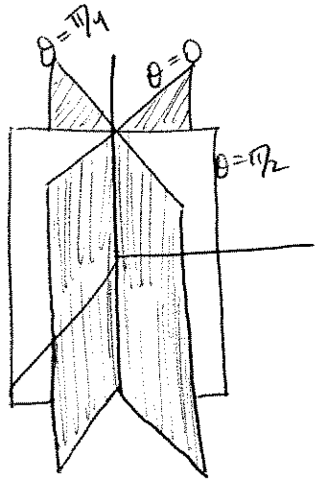
# 11.9 (continued)

## Examples



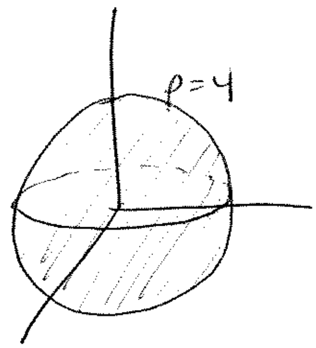

cylinders
(in cylindrical coords)

$r = 1$
and $r = 4$



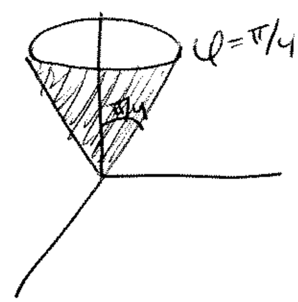

planes
(in cylindrical coords)

$\theta = 0$, $\theta = \pi/4$,
and $\theta = \pi/2$




sphere
(in spherical coords)

$\rho = 4$




rt circular cone
(in spherical coords)

$\varphi = \pi/4$

**Ex 1** change cylindrical coords to Cartesian

(a) $(3, \pi/3, -4)$

change Cartesian to cylindrical
(b) $(2, 2, 3)$

Ex 2

(a) change from spherical to Cartesian

$(8, \pi/4, \pi/6)$

(b) change from Cartesian to spherical

$(2\sqrt{3}, 6, -4)$

**Ex 3** Change from cylindrical to spherical

$(1, \pi/2, 1)$

We can figure out relationship from cylindrical to spherical + vice versa.

spherical to cylindrical $\begin{cases} r = \rho \sin \varphi \\ \theta = \theta \\ z = \rho \cos \varphi \end{cases}$

$\left. \begin{array}{l} \rho = \sqrt{r^2 + z^2} \\ \theta = \theta \\ \cos \varphi = \dfrac{z}{\sqrt{r^2 + z^2}} \end{array} \right\}$ cylindrical to spherical

11.9 (continued)

Ex 4 make required change in given eqn.

(a) $x^2 - y^2 = 25$ to cylindrical coords

(b) $x^2 + y^2 - z^2 = 1$ to spherical coords

(c) $\rho = 2\cos\varphi$ to cylindrical coords

## 11.9 (continued)

(d) $x+y+z=1$ to spherical coords

(e) $r=2\sin\theta$ to Cartesian coords

(f) $\rho\sin\varphi=1$ to Cartesian coords