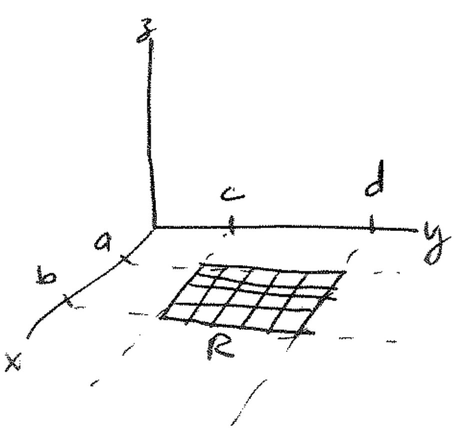
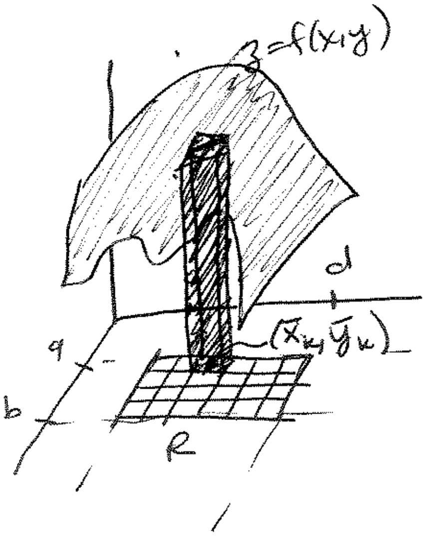
13.1 Double Integrals Over Rectangles

Remember $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx$ is one defn of the definite integral, i.e. the area under the curve $y=f(x)$ from $x=a$ to $x=b$.

Basically, we added up lots of rectangles to get our area. It should be too surprising, then, for $z=f(x,y)$ that finding the volume under the surface requires adding up volumes of rectangular boxes. :)



choose a rectangular region in xy-plane + cut it into small rectangles.



volume of box

$\Rightarrow V = f(\bar{x}_k, \bar{y}_k)\Delta A_k$

where $f(\bar{x}_k, \bar{y}_k) =$ height

$\Delta A_k =$ area of base of box $= \Delta x_k \Delta y_k$

$\Rightarrow$ Volume under $z=f(x,y)$ over R = sum of all the rectangular boxes (like one in figure)

Defn Double Integral

Let $z = f(x,y)$ defined on a closed rectangle R.
If $\lim_{|P| \to 0} \sum_{k=1}^{n} f(\bar{x}_k, \bar{y}_k) \Delta A_k$ exists, then f is integrable over R, and the double integral

$$\iint_R f(x,y)\, dA = \lim_{|P| \to 0} \sum_{k=1}^{n} f(\bar{x}_k, \bar{y}_k) \Delta A_k.$$

Integrability Thm

If f is bounded on the closed rectangle R + if it is continuous there, except for a finite # of smooth curves, then f is integrable on R. If f is continuous on all of R, then f is integrable there.

Properties of the Double Integral

A) It's linear ⇒ ① $\iint_R k f(x,y)\, dA = k \iint_R f(x,y)\, dA$

and ② $\iint_R [f(x,y) + g(x,y)]\, dA = \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$

B) additive on rectangles

$$\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$$

where R_1 + R_2 overlap only on a line segment + comprise all of R.

C) If $f(x,y) \le g(x,y)$, then

$$\iint_R f(x,y)\, dA \le \iint_R g(x,y)\, dA$$

13.1 (continued)

$$\boxed{\iint_R k\,dA = k\iint_R dA = k\,A(R)}$$

Ex 1 For $f(x,y) = \begin{cases} -1 & 1 \le x \le 4,\ 0 \le y < 1 \\ 2 & 1 \le x \le 4,\ 1 \le y \le 2 \end{cases}$.

find $\iint_R f(x,y)\,dA$ where $R = \{(x,y) \mid 1 \le x \le 4,\ 0 \le y \le 2\}$.

13.1 (continued)

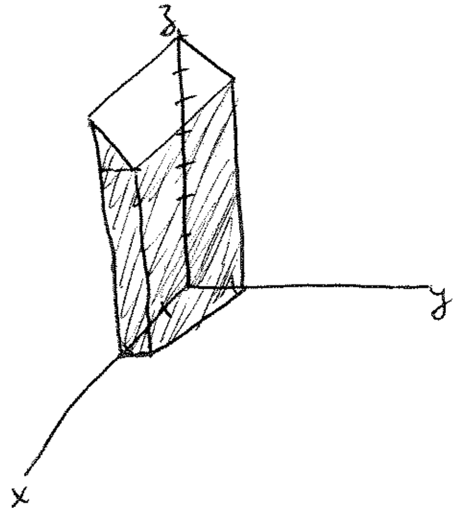
Ex 2 Let $R = \{(x,y) \mid 0 \le x \le 6, 0 \le y \le 4\}$ + $f(x,y) = 10 - y^2$.
Partition R into 6 equal squares by lines $x=2$, $x=4$, + $y=2$. Approximate $\iint_R f(x,y)\,dA$ as $\sum_{k=1}^{6} f(\bar{x}_k, \bar{y}_k)\,\Delta A_k$ where $(\bar{x}_k, \bar{y}_k)$ are centers of squares.

13.1 (continued)

Ex 3 Calculate $\iint_R f(x,y)\,dA$ where $f(x,y) = 7 - y$.

$R = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 1\}$.

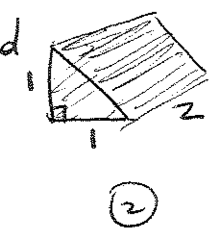
(Hint: Sketch it & see if you recognize it.)



It's a rectangular box w/ some of the top lopped off.

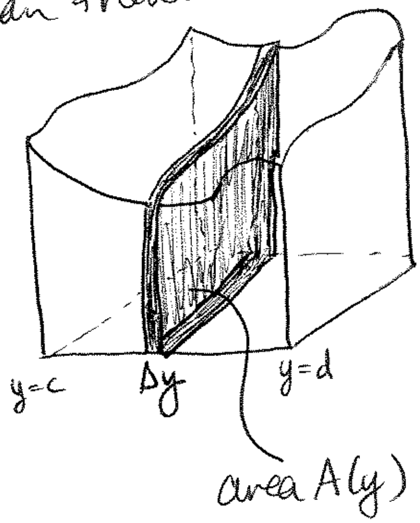
So, we have

and



13.2 Iterated Integrals

We can think about the volume slightly differently.



To find this volume, we can take thin "slab" cross-sections and add them up.
Each slab has volume $A(y)\,\Delta y$

$\Rightarrow V = \int_c^d A(y)\,dy$ but

$$A(y) = \int_a^b f(x,y)\,dx$$

$$\Rightarrow V = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

(If we had taken slabs parallel to yz-plane, then we'd get $V = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$.)

$$\Rightarrow \boxed{\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy}$$

Ex 1 Evaluate $\int_0^4 \left[\int_{-1}^2 (x^2 - 3y)\,dx\right] dy$

13.2 (continued)

Ex 2 $\int_0^1 \int_0^1 \frac{y}{(xy+1)^2}\,dx\,dy$

Ex 3 $\iint_R xy\sqrt{1+x^2}\,dA$ where $R=\{(x,y) \mid 0 \le x \le \sqrt{3},\ 1 \le y \le 2\}$

13.2 (continued)

Ex 4 Find the volume of the solid in the 1^{st} octant enclosed by $z = 4 - x^2$ & $y = 2$

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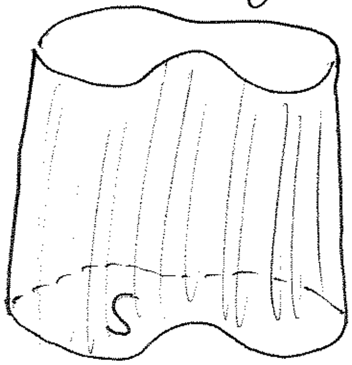
13.3 Double Integrals over Nonrectangular Regions

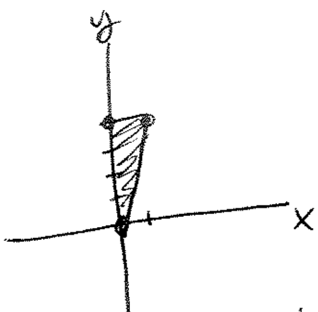
What if the region we're integrating over is not a rectangle, but a simple, closed curve region instead?



$$V = \iint_S f(x,y)\,dA = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy$$

Ultimately, we'll have variables (functions) in our inner integration limits.

Ex 1 Find $\iint_S (x+y)\,dA$ where S is the Δ w/ vertices $(0,0)$, $(0,4)$ & $(1,4)$



$$V = \int_?^? \int_?^? (x+y)\,dy\,dx$$

Then, we need to find limits for y first which will be dependent on x.

The line from $(0,0)$ to $(1,4)$ is $y = 4x \Rightarrow y$ goes from $y=4x$ up to $y=4$. And given that, then x goes from 0 to 1.

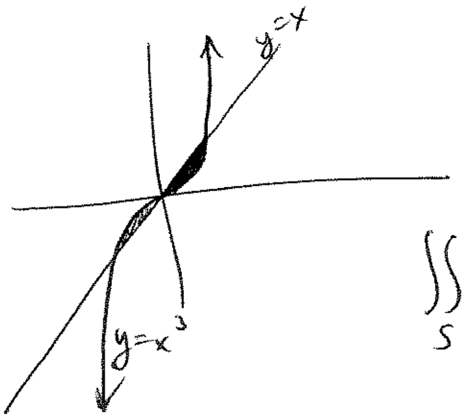
$$\Rightarrow V = \int_0^1 \int_{4x}^4 (x+y)\,dy\,dx = \int_0^1 \left(xy + \tfrac{1}{2}y^2\right)\Big|_{4x}^4 dx$$

$$= \int_0^1 (4x+8) - (4x^2 + 8x^2)\,dx = \int_0^1 -12x^2 + 4x + 8\,dx$$

$$= \left(-4x^3 + 2x^2 + 8x\right)\Big|_0^1 = 6$$

13.3 (continued)

Ex 2 Evaluate $\iint_S x\, dA$ where S is region between $y=x$ & $y=x^3$.



$$x^3 \le y \le x$$

$$0 \le x \le 1$$

$$\iint_S x\, dA =$$

Ex 3 Write these integrals as iterated integrals w/ the order of integration switched.

(a) $\int_0^2 \int_{y^2}^{2y} f(x,y)\, dx\, dy$

(b) $\int_{1/2}^1 \int_{x^3}^{x} f(x,y)\, dy\, dx$

(c) $\int_0^1 \int_{-y}^{y} f(x,y)\, dx\, dy$

13.3 (continued)

Ex 4 Evaluate

(a) $\int_1^5 \int_0^x \frac{3}{x^2+y^2}\, dy\, dx$

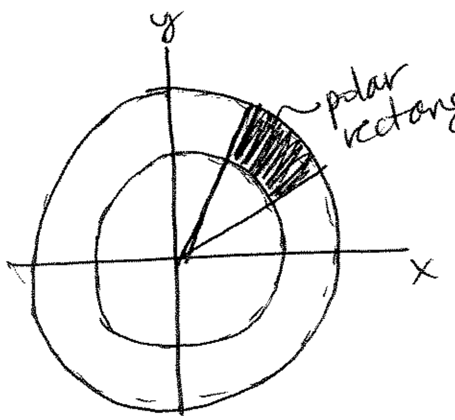
(b) $\int_{\pi/6}^{\pi/2} \int_0^{\sin\theta} 6r\cos\theta\, dr\, d\theta$

13.3 (continued)

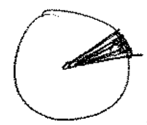
Ex 5 Find the volume of the solid bounded by the parabolic cylinder $x^2 = 4y$ & the planes $z = 0$ & $5y + 9z - 45 = 0$

13.4 Double Integrals In Polar Coordinates

Rather than finding the volume over a rectangle (for Cartesian coords), we'll use a "polar rectangle" for polar coords.



The area of a sector of a circle



is given by $A_{sector} = \pi r^2 \left(\frac{\Delta\theta}{2\pi}\right) = \frac{1}{2}\Delta\theta r^2$

where $\Delta\theta$ is the angle of the pie piece.

$$\Rightarrow A_{\substack{polar\\rect}} = \frac{1}{2}\Delta\theta r_o^2 - \frac{1}{2}\Delta\theta r_i^2$$

where r_o = outer rad.
r_i = inner rad.

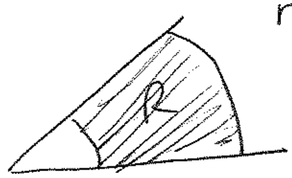
$$= \frac{\Delta\theta}{2}(r_o^2 - r_i^2) = \frac{\Delta\theta}{2}(r_o - r_i)(r_o + r_i)$$

$$= \Delta\theta (r_o - r_i)\left(\frac{r_o + r_i}{2}\right)$$

$$A_{\substack{polar\\rect}} = \Delta\theta \, \Delta r \, \bar{r}$$

where $\bar{r}$ = avg radius $= \frac{r_o + r_i}{2}$

and $\Delta r = r_o - r_i$



$\Rightarrow$ Volume of surface $f(x,y)$ over R is

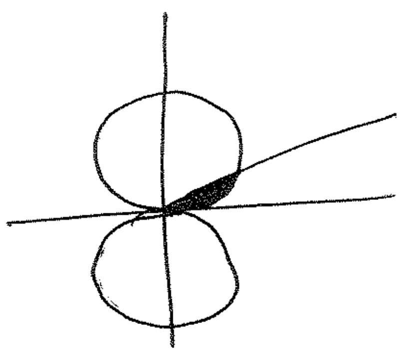
$$V \approx \sum_{k=1}^{n} f(\bar{r}_k, \bar{\theta}_k) \underbrace{\bar{r}_k \Delta r_k \Delta\theta_k}_{\text{Area of each polar rectangle}}$$

13.4 (continued)

$$\Rightarrow \boxed{\iint_R f(x,y)\,dA = \iint_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

Ex 1 Find the area of the given region S by calculating $\iint_S r\,dr\,d\theta$.

(a) S is smaller region bndd by $\theta = \pi/6$ + $r = 4\sin\theta$.



$$\left.\begin{array}{l} 0 \le r \le 4\sin\theta \\ 0 \le \theta \le \pi/6 \end{array}\right\} S$$

$$\Rightarrow \iint_S r\,dr\,d\theta = \int_0^{\pi/6}\int_0^{4\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/6} \frac{1}{2}r^2\Big|_0^{4\sin\theta}\,d\theta$$

$$= \int_0^{\pi/6} 8\sin^2\theta\,d\theta$$

$$= \frac{8}{2}\int_0^{\pi/6} 1-\cos(2\theta)\,d\theta$$

$$= 4\left(\theta - \frac{1}{2}\sin(2\theta)\right)\Big|_0^{\pi/6}$$

$$= 4\left(\pi/6 - \frac{1}{2}\sin(\pi/3)\right) - 0$$

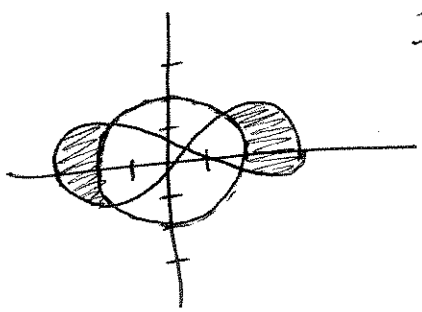
$$= 4\left(\pi/6 - \sqrt{3}/4\right) = \frac{2\pi}{3} - \sqrt{3}$$

13.4 (continued)

(☆ Refer to 10.6 in book if you need to know how to graph some of these polar eqns.)

Ex1 (cont)

(b) S is region outside circle $r=2$ + inside lemniscate $r^2 = 9\cos 2\theta$.



It's symmetric, so we can just double one piece.

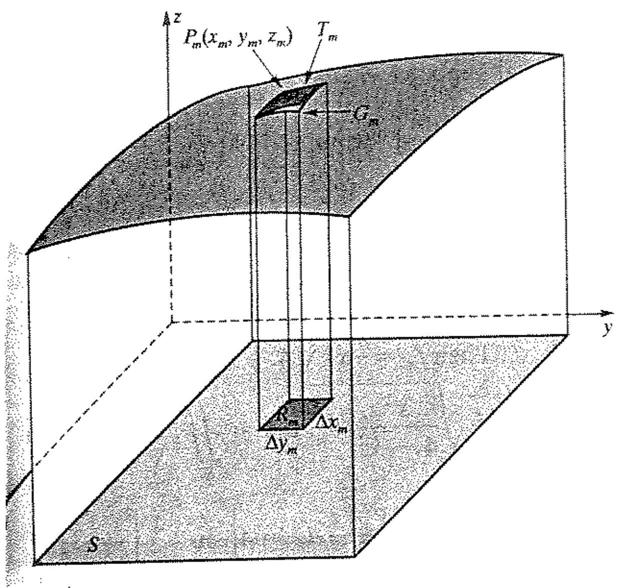
13.4 (continued)

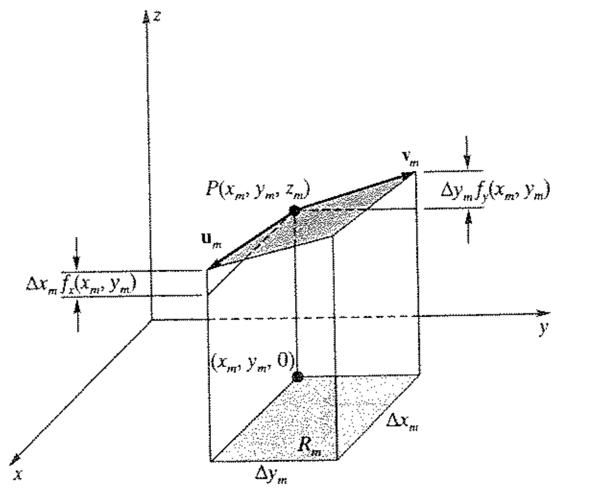
EX 2 Evaluate using polar coords.

(a) $\iint_S y \, dA$ where S is 1st Quadrant polar rectangle inside $x^2+y^2=4$ + outside $\cancel{x}^2+y^2=1$.

(b) $\int_0^1 \int_0^{\sqrt{1-y^2}} \sin(x^2+y^2)\, dx\, dy$

13.6 Surface Area





To find the surface area, we're basically going to add up lots of little parallelograms that are tangent to the surface.

$$\vec{u}_m = \Delta x_m \hat{\imath} + f_x(x_m, y_m)\Delta x_m \hat{k}$$
$$\vec{V}_m = \Delta y_m \hat{\jmath} + f_y(x_m, y_m)\Delta y_m \hat{k}$$

> vectors that make up sides of one tangent parallelogram

We know $A(T_m)$ (the area of the parallelogram) is the magnitude of cross product of its vector sides.

$$\Leftrightarrow A(T_m) = |\vec{u}_m \times \vec{V}_m|$$

and
$$\vec{u}_m \times \vec{V}_m = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \Delta x_m & 0 & f_x \Delta x_m \\ 0 & \Delta y_m & f_y \Delta y_m \end{vmatrix}$$

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$$\vec{u}_m \times \vec{v}_m = \left(-\Delta x_m \Delta y_m f_x(x_m, y_m)\right)\hat{\imath}$$
$$- \left(\Delta x_m \Delta y_m f_y(x_m, y_m)\right)\hat{\jmath} + \Delta x_m \Delta y_m \hat{k}$$
$$= \Delta x_m \Delta y_m \left(-f_x(x_m, y_m)\hat{\imath} - f_y(x_m, y_m)\hat{\jmath} + \hat{k}\right)$$

$$\Rightarrow A(T_m) = |\vec{u}_m \times \vec{v}_m|$$
$$= \underbrace{\Delta x_m \Delta y_m}_{A(R_m)} \sqrt{[f_x(x_m, y_m)]^2 + [f_y(x_m, y_m)]^2 + 1}$$
$$= A(R_m) \sqrt{[f_x(x_m, y_m)]^2 + [f_y(x_m, y_m)]^2 + 1}$$

If we add all those little tangent parallelograms together, we'll have our surface area.

$$\Rightarrow SA = \lim_{|P| \to 0} \sum_{m=1}^{n} A(T_m) = \lim_{|P| \to 0} \sum_{m=1}^{n} \sqrt{[f_x(x_m, y_m)]^2 + [f_y(x_m, y_m)]^2 + 1}\, A(R_m)$$

$$\boxed{SA = \iint_S \sqrt{f_x^2 + f_y^2 + 1}\, dA}$$ surface area

13.6 (continued)

Ex 1 Find the surface area of the plane $3x-2y+6z=12$ that is bndd by $x=0$, $y=0$ + $3x+2y=12$ planes.

13.6 (continued)

Ex 2 Find the surface area for part of the sphere $x^2+y^2+z^2=9$ inside the circular cylinder $x^2+y^2=4$.

13.7 Triple Integrals

$A = \int_a^b f(x)\, dx$ (measures 2d space under curve $f(x)$)

$V = \iint_S f(x,y)\, dA$ (measures 3d space under surface $f(x,y)$)

$\Rightarrow$ We predict that $\iiint_S f(x,y,z)\, dV$ measures 4d space under "hyper surface" $f(x,y,z)$.

Basically, we will extend the pattern we're established for definite integrals. In 4d, we add little boxes of small volume $*$ "height" to the function to get the 4d space under the hyper surface.

$$\boxed{\iiint_S f(x,y,z)\, dV = \int_{a_1}^{a_2} \int_{\varphi_1(x)}^{\varphi_2(x)} \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)\, dz\, dy\, dx}$$

where our integration limits are determined by our 3d region. Notice that innermost integral can depend on 2 variables, middle integral can only depend on 1 variable + last integral can only have constants for its bounds.

13.7 (continued)

Ex 1 Write an iterated integral for $\iiint_S f(x,y,z)\,dV$ where S is region in 1^{st} octant bounded by the surface $z = 9-x^2-y^2$ + the coord planes.

Evaluate $\int_0^{\pi/2}\int_0^{z}\int_0^{y} \sin(x+y+z)\,dx\,dy\,dz$

Ex 2

13.7 (continued)

Ex 3 Find the volume of the solid in the first octant bounded by the elliptic cylinder $y^2+64z^2=4$ & the plane $y=x$.

① Use method from 16.3.

② Use $V=\iiint_S dx\,dy\,dz$

13.7 (continued)

Ex 4 Find the volume of the solid bounded by

$$\underset{\text{(cylinder)}}{y = x^2+2}, \quad \underbrace{y = 4, \; z = 0 \;\&\; 3y - 4z = 0}_{\text{(planes)}}$$

13.8 Triple Integrals (Cylindrical + Spherical Coordinates)

$$\boxed{\iiint\limits_S f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)}\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta}$$

cylindrical coords

Ex 1 Find the volume of the solid bounded above by the sphere $x^2+y^2+z^2=9$, below by the plane $z=0$ + laterally by the cylinder $x^2+y^2=4$. (Use cylindrical coords.)

$$\boxed{\iiint_S f(x,y,z)\,dV = \int_{\varphi_1}^{\varphi_2}\int_{g_1(\varphi)}^{g_2(\varphi)}\int_{\psi_1(\theta,\varphi)}^{\psi_2(\theta,\varphi)} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi = \iiint_S f\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi}$$

Spherical coords

Ex 2 Find the volume of the solid w/in the sphere $x^2+y^2+z^2=16$, outside the cone $z=\sqrt{x^2+y^2}$ and above the xy-plane.

13.8 (continued)

Jacobian

let $x = m(u,v)$ + $y = n(u,v)$ where x, y are old variables + u, v are new variables. I want to change my system from one set of variables to the other.

Define $J(u,v)$ (the Jacobian) as

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

$$\Rightarrow \iint f(x,y)\,dx\,dy = \iint f\left[m(u,v), n(u,v)\right] \left|J(u,v)\right| du\,dv$$

For example, switch from (x,y) to (r,θ).

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta) = r\cos^2\theta + r\sin^2\theta$$

$$= r(\sin^2\theta + \cos^2\theta)$$

$$= r$$

:)

13.8 (continued)

In 3-variables,

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

For example, find $J(\rho, \theta, \varphi)$ where

$x = \rho \sin\varphi \cos\theta \quad y = \rho \sin\varphi \sin\theta \quad z = \rho \cos\varphi$