2.4 Derivatives of Trigonometric fns

Use the defn of the derivative to find $D_x(\sin x)$.

$$D_x(\sin x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \left[\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)\right]$$

$$= \sin x\left[\lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right)\right] + \cos x\left[\lim_{h\to 0}\frac{\sin h}{h}\right]$$

$$= -\sin x\left[\lim_{h\to 0}\frac{1-\cos h}{h}\right] + \cos x(1)$$

$$= -\sin x(0) + \cos x$$

$$= \cos x$$

We can follow the same argument to find $D_x(\cos x)$.

$$\Rightarrow \boxed{D_x(\sin x) = \cos x \quad \text{and} \quad D_x(\cos x) = -\sin x}$$

Also

$$\boxed{\begin{array}{ll} D_x(\tan x) = \sec^2 x & D_x(\cot x) = -\csc^2 x \\ D_x(\sec x) = \sec x \tan x & D_x(\csc x) = -\csc x \cot x \end{array}}$$

2.4 (continued)

Ex 1 Find y' for the following fns.

(a) $y = \sin^2 x = (\sin x)(\sin x)$

(b) $y = \cot x$

(c) $y = \dfrac{x \cos x + \sin x}{x^2+1}$

(d) $y = \sin^2 x + \cos^2 x$

2.4 (continued)

Ex 2 Find the ~~eqn~~ equation of the tangent line to $y = \cot x$ at $x = \pi/4$.

2.5 The Chain Rule

Thm Chain Rule

$$D_x(f(g(x))) = f'(g(x))g'(x)$$

OR $D_x y = (D_u y)(D_x u)$

Basically, we differentiate from the "outside-in".

This is very useful if we need to differentiate something like $f(x) = 3(x^2-2x+1)^{80}$ + you really don't want to multiply it out 80 times.

Ex 1 If $y = (3x^3-4x+5)^{10}$, find y'.

Ex 2 $y = \dfrac{4}{(2x^7+6x^2)^5}$ find y'

2.5 (continued)

Ex 3 Find $f'(x)$.

(a) $f(x) = \sin^3(x^2 - 4x)$

(b) $f(x) = \left(\frac{2x+1}{x-5}\right)^4$

(c) $f(x) = \cos^2(\cos(\cos x))$

(d) $f(x) = \sin^2(4x)\,(2x^5 + 3x^2 - 1)^3$

2.5 (continued)

We can think of the chain rule as

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Ex 4 Find $\frac{dy}{dx}$.

(a) $y = [(2x^2+3)\cos x]^4$

(b) $y = \left(-3x + \frac{5}{x}\right)^{-4}$

(c)

2.6 Higher-Order Derivatives

Derivative	f' notatn	y' notatn	D_x notatn	Leibniz notatn
First	$f'(x)$	y'	$D_x y$	dy/dx
Second	$f''(x)$	y''	$D_x^2 y$	$\frac{d^2y}{dx^2}$
Third	$f'''(x)$	y'''	$D_x^3 y$	$\frac{d^3y}{dx^3}$
Fourth	$f^{(4)}(x)$	$y^{(4)}$	$D_x^4 y$	$\frac{d^4y}{dx^4}$
Fifth	$f^{(5)}(x)$	$y^{(5)}$	$D_x^5 y$	$\frac{d^5y}{dx^5}$
⋮ n^{th}	$f^{(n)}(x)$	$y^{(n)}$	$D_x^n y$	$\frac{d^ny}{dx^n}$

To get the second derivative, you need the first derivative first. It's a recursive process.

Ex 1 Find $f'''(x)$ for $f(x)=(3-5x)^5$

2.6 (continued)

Ex 2 Find $\frac{d^2y}{dx^2}$ for $y = x\sin\left(\frac{\pi}{x}\right)$

Ex 3 What is $D_x^5\left(3x^4 - 2x^3 + x^2 - 7\right)$?

Ex 4 Find a formula for $D_x^n\left(\frac{1}{x}\right)$.

We know $a(t) = v'(t)$, i.e. acceleration = derivative of velocity wrt time.

$\Rightarrow$ $a(t) = v'(t) = s''(t)$

Ex 5 (# 24) An object moves along a horizontal coordinate line according to $s(t) = t^3 - 6t^2$. s = directed distance from origin (in ft.), t = time (in seconds).

(a) what are $v(t)$ + $a(t)$?

(b) when is the object moving to the right?

(c) when is it moving to the left?

2.6 (continued)

(d) When is its acceleration negative?

(e) Draw a schematic diagram that shows the motion of the object.

2.7 Implicit Differentiation

Given $2y^3 - y^2 = x^3 + 5$

This is a function y of x, i.e. $y(x)$, given implicitly (because we cannot solve directly for y in terms of x). So, how do we differentiate it? We start w/ the given eqn + differentiate both sides.

$$\frac{d}{dx}\left(2y^3 - y^2\right) = \frac{d}{dx}\left(x^3+5\right)$$

$$\Leftrightarrow \quad \frac{d}{dx}(2y^3) - \frac{d}{dx}(y^2) = \frac{d}{dx}(x^3) + \frac{d}{dx}(5)$$

$$\Leftrightarrow \quad 6y^2y' - 2yy' = 3x^2$$

$$\Leftrightarrow \quad y'(6y^2 - 2y) = 3x^2$$

$$\Leftrightarrow \quad y' = \frac{3x^2}{6y^2 - 2y}$$

So, we now have the derivative $y' = \frac{dy}{dx}$ of our function. Notice that it is given in terms of x and y.

2.7 (continued)

Let's check to make sure implicit differentiation is reasonable.

If we have $x^2+2x^2y+3xy=0$, we can differentiate it two ways.

① Implicit
$$2x+4xy+2x^2y'+3y+3xy'=0$$
$$2x^2y'+3xy'=-2x-4xy-3y$$
$$y'(2x^2+3x)=-2x-4xy-3y$$
$$y'=\frac{-2x-4xy-3y}{2x^2+3x}$$

② Explicit
$$x^2+2x^2y+3xy=0$$
$$\iff y(2x^2+3x)=-x^2$$
$$\iff y=\frac{-x^2}{2x^2+3x}$$
$$\Rightarrow y'=\frac{(2x^2+3x)(-2x)+x^2(4x+3)}{(2x^2+3x)^2}$$

Are they the same?

2.7 (continued)

Ex 1 Find $\frac{dy}{dx}$ for the following functions.

(a) $x\sqrt{y+1} = xy+1$

(b) $9x^2 + 4y^2 = 36$

Ex2 Find the eqn of the tangent line at the indicated x-value.

(a) $y + \cos(xy^2) + 3x^2 = 4$ at $x=1$

(b) $\sqrt{y} + xy^2 = 5$ at $x=4$

$$\boxed{\begin{array}{l}\underline{\text{Power Rule (revisited)}}\\ \text{Let } r \in \mathbb{Q}. \text{ Then } \forall x > 0\\ \qquad D_x(x^r) = rx^{r-1}.\end{array}}$$

Basically, the power rule now can be used with rational exponents

why? Let $y = x^r = x^{p/q}$ $\quad p, q \in \mathbb{Z},\ q \neq 0$

$$\Rightarrow y^q = x^p$$

$$q y^{q-1} y' = p x^{p-1}$$

$$y' = \frac{p}{q}\left(\frac{x^{p-1}}{y^{q-1}}\right)$$

$$y' = \frac{p}{q}\left(\frac{x^{p-1}}{(x^{p/q})^{q-1}}\right)$$

$$y' = \frac{p}{q}\left(\frac{x^{p-1}}{x^{p-p/q}}\right)$$

$$y' = r\left(x^{p-1-p+p/q}\right)$$

$$y' = r\left(x^{p/q-1}\right)$$

$$y' = r x^{r-1}$$

//

<u>Ex 3</u> Find y' if $y = \sqrt[3]{x} - 2x^{7/2}$

2.8 Related Rates

Remember that derivatives are rates. These problems are basically story problems that show the relevance of using derivatives. :)

<u>Ex 1</u> Assuming that a soap bubble retains its spherical shape as it expands, how fast is its radius increasing when its radius is 3 inches if air is blown into it at a rate of 3 cubic inches per second?

We want to know $\frac{dr}{dt} = ?$ when $r = 3\text{in}$.

And we're given that $\frac{dV}{dt} = 3\,\frac{\text{in}^3}{\text{sec}}$.

$V = \frac{4}{3}\pi r^3$
volume of a sphere

$\Rightarrow$ Since $V = \frac{4}{3}\pi r^3$, then

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

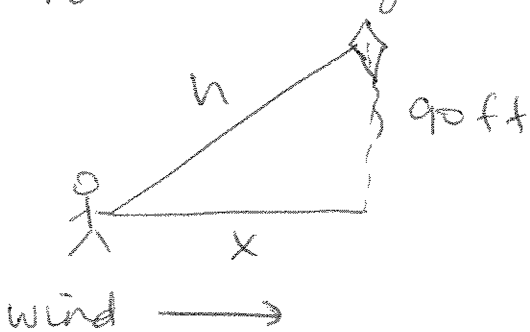
$$\Leftrightarrow \quad \frac{dr}{dt} = \frac{dV}{dt}\left(\frac{1}{4\pi r^2}\right)$$

$\star$ $\frac{dV}{dt}$ + $\frac{dr}{dt}$ are "related rates" here.

$$\Rightarrow \frac{dr}{dt} = \frac{3}{4\pi r^2} \quad \text{+ when } r = 3, \ \frac{dr}{dt} = \frac{3}{4\pi(3^2)} = \frac{1}{12\pi}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{12\pi} \approx 0.0265\,\frac{\text{in}}{\text{sec}}$$

Ex 2 A child is flying a kite. If the kite is 90 ft above the child's hand level & the wind is blowing it on a horizontal course at 5 ft/sec, how fast is the child letting out the cord when 150 ft of cord is out? (Assume that the cord remains straight from hand to kite... not very realistic.)



$\frac{dx}{dt} = 5 \frac{ft}{sec}$

We want to know

$\frac{dh}{dt} = ?$ when $h = 150$ ft

2.8 (continued)

Ex 3 A student is using a straw to drink from a conical paper cup, whose axis is vertical, at a rate of 3 cm^3 per second. If the height of the cup is 10 cm, and the diameter of its opening is 6 cm, how fast is the level of the liquid falling when the depth of the liquid is 5 cm?

2.9 Differentials + Approximations

We've seen the notation $\frac{dy}{dx}$, and we're never separated the symbols. Now, we'll give meaning to dy + dx.

We know $\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = f'(x_0)$ which

gives the derivative (slope) of the functn $f(x)$ at

$x = x_0$.

If Δx is really small, then

$$\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \approx f'(x_0)$$

$$\Longleftrightarrow \quad \underbrace{f(x_0+\Delta x)-f(x_0)}_{\Delta y} \approx \underbrace{f'(x_0)\,\Delta x}_{dy}$$

Δy: actual change in y from x_0 to $x_0+\Delta x$

dy: an approximation of the change in y.

Defn Differentials

Let $y=f(x)$ be a differentiable function of x.

Δx is an arbitrary increment of x,

$dx = \Delta x$ (dx is called differential of x)

Δy is actual change in y as x goes from x to $x+\Delta x$

i.e. $\Delta y = f(x+\Delta x)-f(x)$.

$dy = f'(x)dx$ (dy is called differential of y)

2.9 (continued)

Ex 1 Find dy.

(a) $y = 4x^3 - 2x + 5$

$dy = (12x^2 - 2)\,dx$

(b) $y = 2\sqrt{x^4 + 6x}$

(c) $y = \cos(x^3 - 5x + 11)$

(d) $y = (x^{10} + \sqrt{\sin 2x})^2$

2.9 (continued)

Differentials can be used for approximations!

If $f(x+\Delta x) - f(x) \approx f'(x)\Delta x$, then

$$\boxed{f(x+\Delta x) \approx f(x) + f'(x)\Delta x}$$

Ex 2 Find a good approximation for $\sqrt{9.2}$ (w/o a calculator ☺).

Let $f(x) = \sqrt{x}$. when $x = 9$, $f(x) = 3$. Let $\Delta x = 0.2$.

$\Rightarrow f(x+\Delta x) \approx f(x) + f'(x)\Delta x$ becomes

$f(9.2) \approx f(9) + f'(9)(0.2)$ and we know

$f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow f'(9) = \frac{1}{2}(9^{-1/2}) = \frac{1}{2}\left(\frac{1}{\sqrt{9}}\right) = \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{6}$

$\Rightarrow f(9.2) \approx \sqrt{9} + \frac{1}{6}(0.2) = 3 + \frac{0.1}{3} = 3 + 0.033\overline{3}$

$\Leftrightarrow f(9.2) \approx 3.0\overline{3}$ i.e. $\sqrt{9.2} \approx 3.0\overline{3}$

Ex3 Use differentials to approximate the increase in the surface area of a soap bubble when its radius increases from 4 inches to 4.1 inches.

$A = 4\pi r^2$

2.9 (continued)

Ex 4 The height of a cylinder is measured as 12 cm w/ a possible error of ± 0.1 cm. Evaluate the volume of the cylinder w/ radius 4 cm & give an estimate for the possible error in this value.

$V = \pi r^2 h = \pi (4)^2 h$

$\Rightarrow V = 16\pi h$