3.1 Maxima + Minima

Defn Let S, the domain of f, contain the point c.

Then (i) $f(c)$ is a max. value of f on S if $f(c) \geq f(x)\ \forall x \in S$.

(ii) $f(c)$ is a min. value of f on S if $f(c) \leq f(x)\ \forall x \in S$.

(iii) $f(c)$ is an extreme value of f on S if it is either a maximum or minimum value.

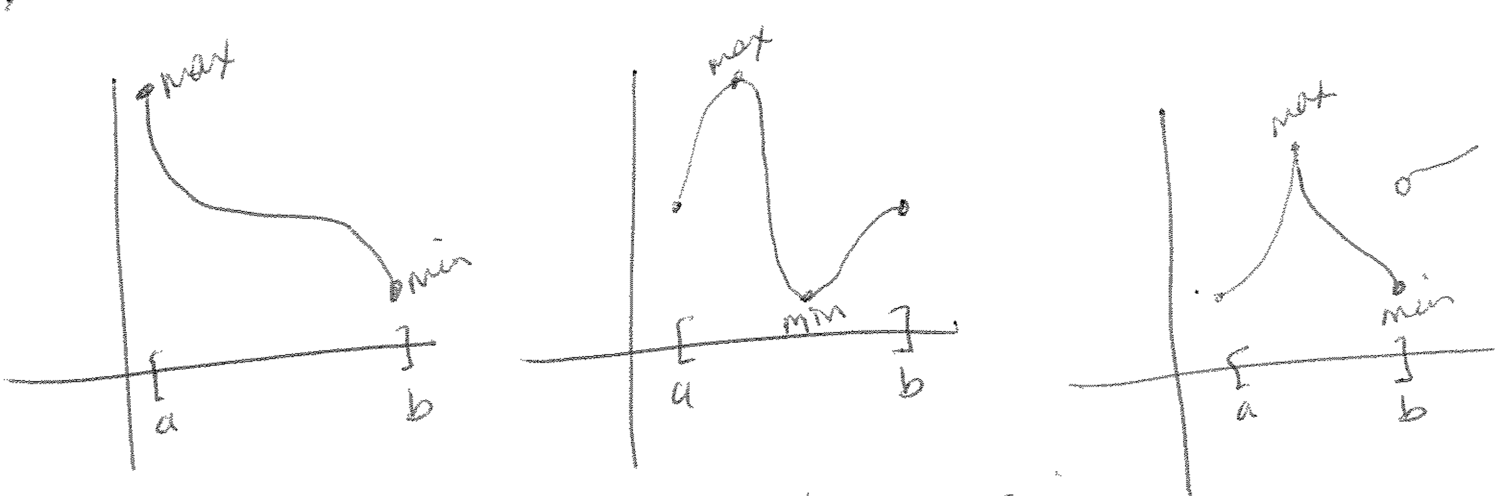
(iv) the function we want to maximize or minimize is called the objective function.

How do we know extreme values exist for a function?

Max-Min Existence Thm

If f is continuous on a closed interval $[a, b]$, then f attains both a max + min value on that interval.

☆ We need ① f continuous + ② a closed interval!



We can have max + min pts occur in one of 3 ways

① endpts of the closed interval

② stationary pts (where $f'(x) = 0$)

③ singular pts (where derivative DNE)

3.1 (continued)

Critical pt Thm

Let f be defined on a closed interval I containing point c. If $f(c)$ is an extreme value, then c is called a critical pt.

c is either ① an endpt of I.
or ② a stationary pt of f, i.e. $f'(c)=0$
or ③ a singular pt of f, i.e. $f'(c)$ DNE.

Ex 1 Find min & max values of

$f(x) = -2x^3 + 3x^2$ on $[-1, 3]$

3.1 (continued)

Ex 2 Find the min & max points for
$f(x) = x^{2/5}$ on $[-1, 32]$.

Ex 3 Show that for a rectangle with perimeter of 30 inches, it has maximum area when it is a square.

3.1 (continued)

Ex 4 Identify critical pts, and specify min and max values.

$f(x) = x - 2\sin x$ on $[-2\pi, 2\pi]$

Ex 5 Sketch the graph of a function that is

1. continuous, but not necessarily differentiable
2. has domain $[0, 6]$
3. reaches a max of 4 (at $x=4$)
4. reaches a min of 2 (at $x=2$)
5. and has no stationary pts.

3.2 Monotonicity + Concavity

Defn Let f be defined on an interval I (open, closed or neither). We say that:

① f is increasing on I if $\forall\ x_1 + x_2$ in I
$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2).$$

② f is decreasing on I if $\forall\ x_1 + x_2$ in I
$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2).$$

③ f is strictly monotonic on I if it is either increasing or decreasing on I.

Monotonicity Thm

Let f be continuous on I + differentiable at every interior pt of I.

① if $f'(x) > 0\ \forall\ x \in I$, then f is increasing on I.

② if $f'(x) < 0\ \forall\ x \in I$, then f is decreasing on I.

Ex 1 For $f(x) = x^3 + 3x^2 - 12$, find where f is increasing + decreasing.

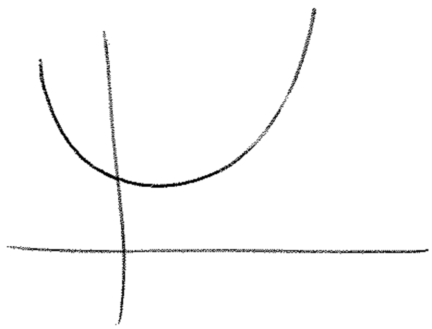
3.2 (continued)

Ex 2 Determine where $f(x) = \frac{x-1}{x^2}$ is increasing + decreasing.

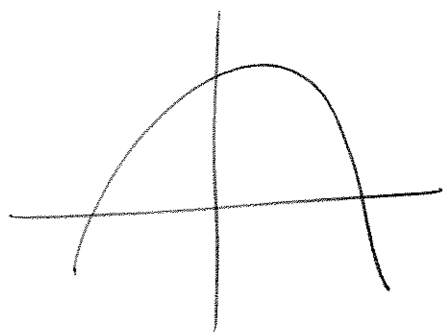
Ex 3 Where is $f(x) = \cos^2 x$ $x \in [0, 2\pi]$ increasing + decreasing?

3.2 (continued)

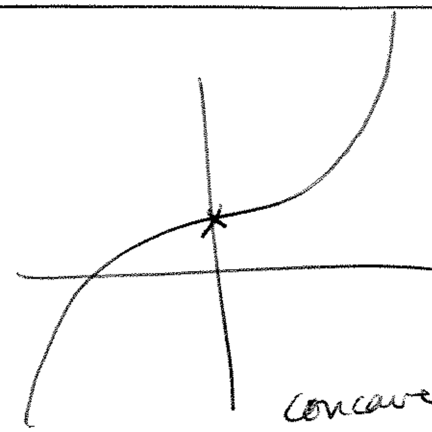
Defn Let f be differentiable on an open interval I.
f is concave up on I if $f'(x)$ is increasing on I
+ f is concave down on I if $f'(x)$ is decreasing on I.



concave up



concave down



concave down

concave up

Concavity Thm

Let f be twice differentiable on an open interval I.

① If $f''(x) > 0$ $\forall x \in I$, f is concave up on I.

② If $f''(x) < 0$ $\forall x \in I$, f is concave down on I.

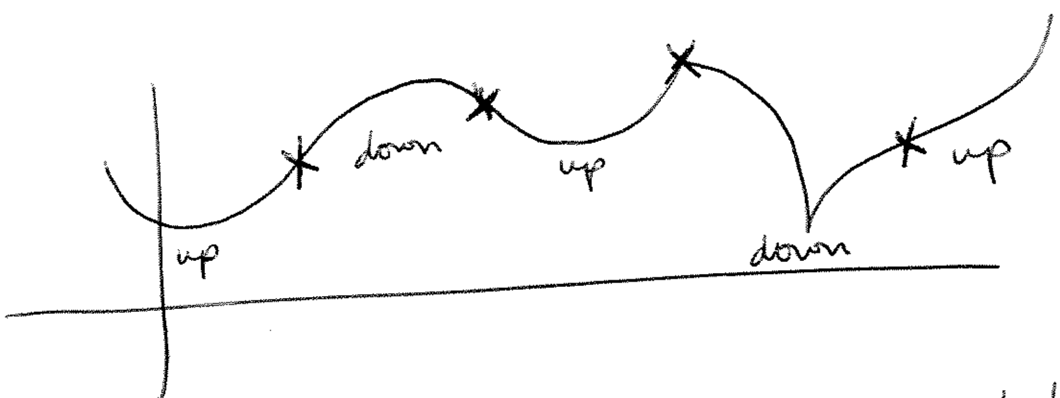
Ex4 Where is $f(x) = 4x^3 - 3x^2 - 6x + 12$ increasing, decreasing, concave up + concave down?

3.2 (Continued)

Ex 5 For $f(x) = 8x^{1/3} + x^{4/3}$, find where it's increasing, decreasing, concave up + concave down. Then, use this info to sketch the graph.

Inflection Point

Let f be continuous at c. We call $(c, f(c))$ an inflection pt of f if f is concave up on one side of c + concave down on the other side of c.



X marks the inflectn pts.

We can find inflection pts by taking the second derivative. The x-values that make $f''(x) = 0$ or $f''(x)$ undefined are the possible x-values for the inflectn pts. You need to check out those possibilities.

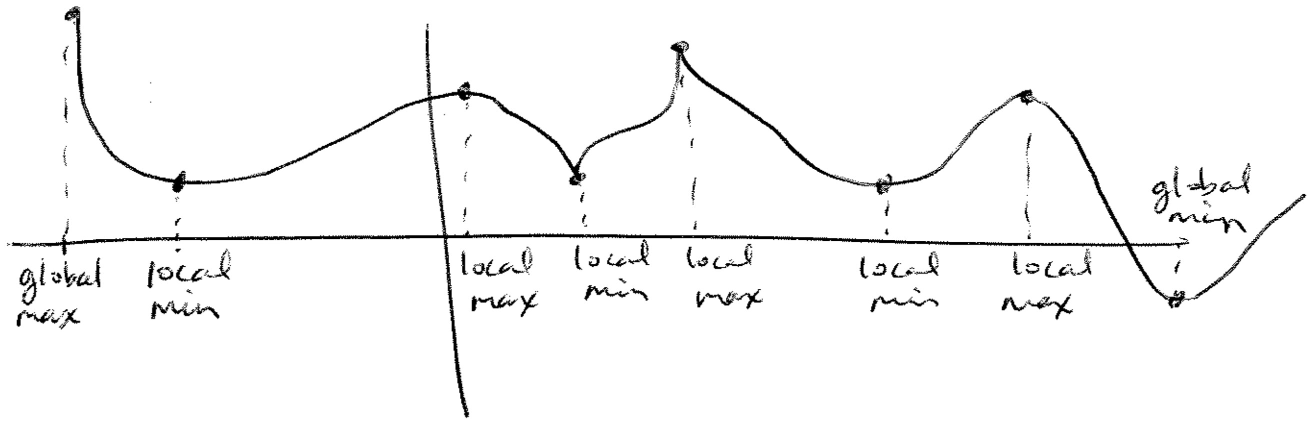
e.g. $f(x) = x^6$

3.2 (continued)

Ex 6 Find all pts of inflection for

$$f(x) = 2x^{1/3} - 1$$

3.3 Local Maxima + Minima (and Extrema on Open Intervals)



Defn Let S = domain of f $\ni$ $c \in S$.

Then ① $f(c)$ is a local max value of f if $\exists$ (a,b) containing c $\ni$ $f(c)$ is max value of f on $(a,b) \cap S$.

② $f(c)$ is a local min value of f if $\exists$ (a,b) containing c $\ni$ $f(c)$ is a min value of f on $(a,b) \cap S$.

③ $f(c)$ is a local extreme value of f if it is either a local min or local max.

How do we find the local extrema?

First Derivative Test

Let f be continuous on an open interval (a,b) that contains a critical pt c.

① If $f'(x) > 0$ $\forall$ $x \in (a,c)$ + $f'(x) < 0$ $\forall$ $x \in (c,b)$, then $f(c)$ is a local max.

② If $f'(x) < 0$ $\forall$ $x \in (a,c)$ + $f'(x) > 0$ $\forall$ $x \in (c,b)$, then $f(c)$ is a local min.

③ If $f'(x)$ has same sign on both sides of c, then it's not a max nor a min.

3.3 (continued)

Ex 1 Find local min + max pts for $f(x) = 2x^2 - 5x + 3$.

Ex 2 Find local min + max pts for $f(x) = \frac{1}{2}x + \sin x$
$0 < x < 2\pi$

3.3 (continued)

Ex 3 Find all extreme values for $f(x) = x^4 + x^2 + 3$

Thm Second Derivative Test

Let f' + f'' exist at every pt in (a,b) containing c, and $f'(c)=0$.

1. If $f''(c) < 0$, $f(c)$ is local max.
2. If $f''(c) > 0$, $f(c)$ is local min.

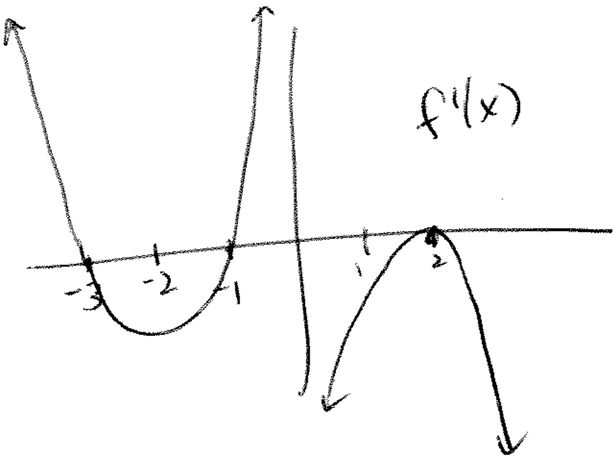
3.3 (continued)

Ex 4 Find all critical pts for $y = (x-2)^5$ + sketch the graph.

Ex 5 Find all min + max x-values for $y = x^2 + \frac{1}{x^2}$

3.3 (continued)

Ex 6 (#29) Let f be continuous & f' has the following graph.



Try to sketch a graph of $f(x)$ & answer these questions.

(a) where is f increasing? decreasing?

(b) where is f concave up? down?

(c) where does f attain a local max? min?

(d) where are the inflectn pts?

3.4 Practical Problems

Ex 1 For what # does the principal square root exceed 8 times the # by the largest amt?

Let x = the #. Then we want to maximize $y = \sqrt{x} - 8x$.

steps

1. Draw a picture or list info given.
2. Write down what needs to be maximized or minimized.
3. If have more than 1 variable, find eqn to eliminate one.
4. Differentiate fn.
5. Set derivative = 0 (or find where it DNE) & solve. (i.e. find critical pts)
6. Check to make sure you found the max or min that you want.

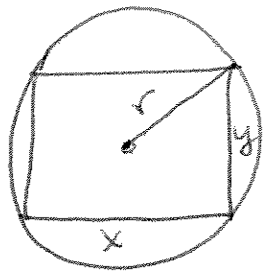
Ex 2 Find 2 #s whose product is −12 + sum of whose squares is a minimum.

let x + y be the #s.

we know $xy = -12$ + we want to minimize

3.4 (continued)

Ex 3 Show that the rectangle w/ max perimeter that can be inscribed in a circle is a square.

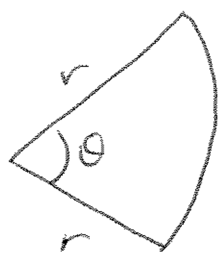


Let r be the radius of the circle + it's fixed.

What do we want to maximize?

What do we know (about relating $x + y$)?

Ex 4 A flower bed will be in the shape of a sector of a circle (a pie-shaped region) of radius r & vertex angle θ. Find r & θ if its area is a constant A & the perimeter is a minimum.



3.4 (continued)

Ex 5 (A true classic!) Find the volume of the largest open box that can be made from a piece of cardboard that is 24" by 9". You'll form the box by cutting out identical squares from the 4 corners + turning up the sides. Also, find the dimensions of the box that yields the max volume.

3.4 (continued)

Ex 6 A farmer has 80 ft of fence. He needs to enclose three identical pens along one side of his barn (the side along the barn needs no fence). What dimensions for the total enclosure make the area of the pens as large as possible?

3.5 Graphing Functions Using Calculus

Ex1 Sketch the graph of $f(x) = x^2(x^2-1)$.

(1) domain:
range:

(2) symmetry:

(3) x-intercepts:

(4) First Derivative info (increasing/decreasing):

(5) Second Derivative info (concavity/inflection pts):

(6) asymptotes:

(7) Sketch graph

3.5 (continued)

Ex 2 Sketch the graph of $f(x) = \frac{4x^4 - 8x^2 - 12}{3}$

3.5 (continued)

Ex 3 Sketch graph of $f(x) = \dfrac{(x-3)^2}{x}$

3.5 (continued)

Ex 4 Sketch the graph of $f(x) = |x|^3$

Since $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \Rightarrow \frac{d|x|}{dx} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

or we can say $\frac{d|x|}{dx} = \frac{x}{|x|}$ which covers it all.

3.6 The Mean Value Thm for Derivatives

Mean Value Thm for Derivatives

f continuous on $[a,b]$ + differentiable on (a,b).

Then $\exists$ at least one $c \in (a,b) \ni$

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\Leftrightarrow \quad f(b)-f(a) = f'(c)(b-a)$$

i.e. if $f(x)$ has nonvertical tangent line everywhere in (a,b), then there's at least one pt where the tangent line is parallel to the secant line connecting the endpts.

Ex 1 Find the # c guaranteed by MVT for

$g(x) = (x+1)^3$ on $[-1,1]$.

Is $g(x)$ cont. on $[-1,1]$?

differentiable on $(-1,1)$?

3.6 (continued)

Ex 2 For $g(x) = \frac{x-4}{x-3}$, decide if we can use the MVT on (a) $[0, 5]$ or (b) $[4, 6]$. If so, use it to find the #c from the MVT. If not, state the reason why?

Ex 3 For $f(x) = \csc x$ on $[-\pi, \pi]$, use MVT to find c.

Thm B

If $f'(x) = g'(x)$ $\forall x \in (a,b)$, then $\exists$ $c \in \mathbb{R}$ $\ni$

$$f(x) = g(x) + c \quad \forall x \in (a,b).$$