3.7 Solving Equations Numerically

(☆ you will definitely need a calculator for this section!)

Sometimes, we need to solve eqns of the form $f(x)=0$, to find the function's roots.

If it's too complicated to solve algebraically, we can use iterative steps numerically to close in on the solutions.

3 methods for solving eqns numerically

1. Bisection method
2. Newton's method
3. Fixed-Point method

1 Bisection method

Algorithm **Bisection Method**

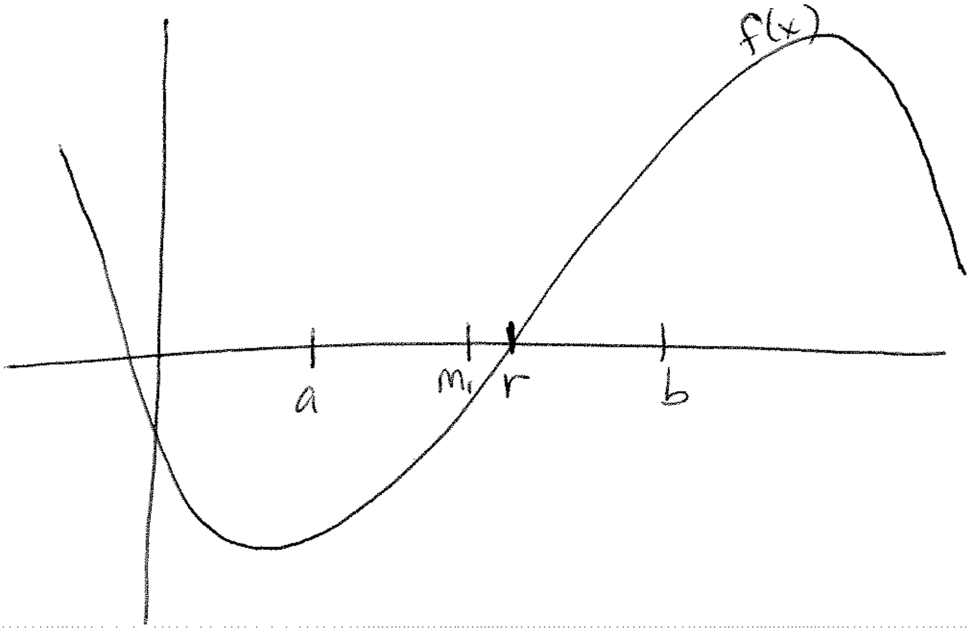
Let $f(x)$ be a continuous function, and let a_1 and b_1 be numbers satisfying $a_1 < b_1$ and $f(a_1)\cdot f(b_1) < 0$. Let E denote the desired bound for the error $|r - m_n|$. Repeat steps 1 to 5 for $n = 1, 2, \dots$ until $h_n < E$:

1. Calculate $m_n = (a_n + b_n)/2$.
2. Calculate $f(m_n)$, and if $f(m_n) = 0$, then STOP.
3. Calculate $h_n = |(b_n - a_n)/2.|$ (for error testing)
4. If $f(a_n)\cdot f(m_n) < 0$, then set $a_{n+1} = a_n$ and $b_{n+1} = m_n$.
5. If $f(a_n)\cdot f(m_n) > 0$, then set $a_{n+1} = m_n$ and $b_{n+1} = b_n$.

⊕
- simple
- reliable

⊖
- can take a lot of time + computations (i.e. it's "computationally expensive")

3.7 (continued)



Basically, we're looking for $r \ni f(r)=0$. We know $f(a)<0$ + $f(b)>0$. (we choose a + b on either side of r.) Find the midpt between a + b (m_1) and find $f(m_1)$. If $f(m_1)=0$, then we're done. If not, then keep going, until you close in on r.

<u>Ex 1</u> Approximate real root of $f(x) = x^4+5x^3+1$ on $[-1,0]$ to two decimal places.

(continued space on next page)

3.7 (continued)

Ex 1 (cont)

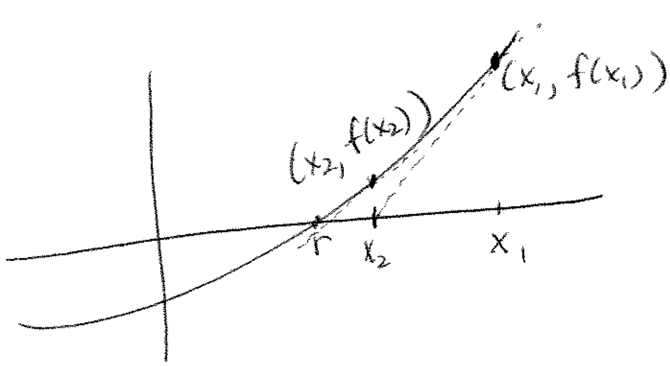
② Newton's method

Algorithm **Newton's Method**

Let $f(x)$ be a differentiable function and let x_1 be an initial approximation to the root r of $f(x) = 0$. Let E denote a bound for the error $|r - x_n|$.

Repeat the following step for $n = 1, 2, \ldots$ until $|x_{n+1} - x_n| < E$:

1. $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



(+)
- finds root more quickly

(−)
doesn't always converge (you have to choose x_1 "close enough" to r)

3.7 (continued)

Basically, we choose x_1 + find tangent line to $f(x)$ at x_1. Find where that tangent line crosses x-axis and use that as x_2. Repeat!

tangent line is $y - f(x_1) = f'(x_1)(x - x_1)$

This crosses x-axis when $y = 0$.

$$\Rightarrow \quad 0 - f(x_1) = f'(x_1)(x - x_1)$$

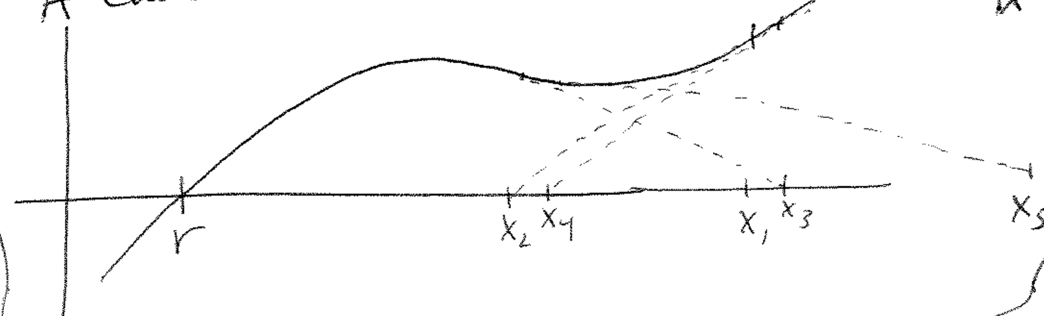
$$\frac{-f(x_1)}{f'(x_1)} = x - x_1$$

$$x_1 - \frac{f(x_1)}{f'(x_1)} = x$$ This is our x_2!

$\Rightarrow$ Newton's method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Note: A case where Newton's method doesn't converge.



* you need to choose a closer start point

3.7 (continued)

Ex 2 Use Newton's method to approximate root of $7x^3+x-5=0$ to 5 decimal places

3.7 (continued)

(3) Fixed-Point Algorithm

Algorithm	**Fixed-Point Algorithm**

Let $g(x)$ be a continuous function, and let x_1 be an initial approximation to the root r of $x = g(x)$. Let E denote a bound for the error $|r - x_n|$.

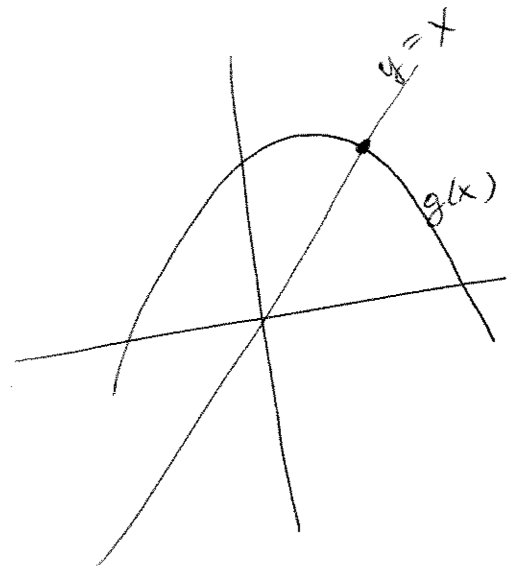
Repeat the following step for $n = 1, 2, \ldots$ until $|x_{n+1} - x_n| < E$:

1. $x_{n+1} = g(x_n)$

(+) simple

(−) Useful only when an eqn can be written in form $x = g(x)$.

Note: may or may not converge, depending on how close first guess is and form of $g(x)$ we choose.



Ex 3 Use the Fixed-Point Algorithm to solve

$x = 2 - \sin x$, given $x_1 = 2$, to 5 decimal places.

3.8 Antiderivatives (Indefinite Integrals)

We already covered the gist of an antiderivative in section 0.4. Here are some reminders of what we did.

Defn

We call F <u>an</u> antiderivative of f on interval I, if $D_x F(x) = f(x)$ on I, i.e. if $F'(x) = f(x)$ $\forall x \in I$.

Power Rule Thm

If $r \in \mathbb{R}$, except $r \neq -1$, then

$$\int x^r \, dx = \frac{x^{r+1}}{r+1} + C$$

Indefinite Integral is a linear Operator

Let f & g have antiderivatives & $k \in \mathbb{R}$. Then

(i) $\int k f(x)\,dx = k \int f(x)\,dx$

and (ii) $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Ex 1 Evaluate the following integrals

(a) $\int (2x^4 + 3x^2 - 7)\,dx$

(b) $\int (u^3 - u^9)\,du$

3.8 (continued)

We know now that the power rule works for all rational exponents (besides -1).

Ex 2 Evaluate.

(a) $\int \left(\frac{1}{y^2} + y^{1/3} \right) dy$

(b) $\int \left(x^{-4} + \sqrt[3]{x^2} - \frac{3}{x^5} \right) dx$

3.8 (continued)

Thm

$$\int \sin x \, dx = -\cos x + C \quad \text{and} \quad \int \cos x \, dx = \sin x + C$$

Ex 3 $\int (t^2 - 2\cos t)\, dt$

Thm Generalized Power Rule

Let g be differentiable + $r \in \mathbb{Q}$, $r \neq -1$. Then

$$\int [g(x)]^r g'(x)\, dx = \frac{[g(x)]^{r+1}}{r+1} + C$$

Ex 4 $\int (4x^3+1)^4 \, 12x^2 \, dx$

3.8 (continued)

Ex 5 $\int (5x^2+1)\sqrt{5x^3+3x-2}\,dx$

Ex 6 $\int \frac{3y}{\sqrt{2y^2+5}}\,dy$

3.9 Intro to Differential Eqns

A differential eqn is an eqn that contains a derivative. We will need to integrate both sides, at some pt, to "undo" the derivative.

<u>Ex 1</u> Find the eqn of the curve that goes through $(2,-4)$ + whose slope at any point on the curve is $3x$.

We know $\frac{dy}{dx} = 3x$. (First Order Separable D.E.)

$$\iff \quad dy = 3x\,dx$$

$$\iff \quad \int dy = \int 3x\,dx$$

$$\iff \quad y = \frac{3}{2}x^2 + C$$

We know it goes thru $(2,-4)$, so

$$-4 = \frac{3}{2}(2^2) + C$$

$$-4 = 6 + C$$

$$C = -10$$

$$\Rightarrow \boxed{y = \frac{3}{2}x^2 - 10}$$

3.9 (continued)

Ex 2 $\qquad \dfrac{dy}{dx} = \sqrt{\dfrac{x}{y}} \qquad y = 4$ when $x = 1$

$$\frac{dy}{dx} = \frac{x^{1/2}}{y^{1/2}}$$

$$y^{1/2}\,dy = x^{1/2}\,dx$$

3.9 (continued)

Ex 3 $\frac{dy}{dx} = -y^2 x (x^2+2)^4$ thru $(0,1)$

3.9 (continued)

Ex 4 The wolf population P in a certain state has been growing at a rate proportional to the cube ~~s~~ root of the population size. The population was estimated at 1000 in 1980 + 1700 in 1990. When will the wolf population reach 4000?