7.1 Basic Integration Rules (Substitution)

u-substitution for Integration

Let g be a differentiable fn + suppose F is an antiderivative of f. Then, if $u = g(x)$,

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du = F(u) + c = F(g(x)) + C$$

☆ see "Standard Integral Forms" (pg 383-384)

Ex 1 $\displaystyle\int \frac{3x}{\sin^2(4x^2)}\,dx$

Ex 2 $\displaystyle\int \frac{5e^{3/x^2}}{x^3}\,dx$

7.1 (cont)

Ex 3 $\displaystyle\int \frac{5}{9+(2x+1)^2}\,dx$

Ex 4 $\displaystyle\int \frac{3x^2-4x+2}{x-2}\,dx$ (Hint: Do long division.)

7.1 (cont)

Ex 5 $\displaystyle\int \frac{2x\,dx}{\sqrt{1-x^4}}$

Ex 6 $\displaystyle\int \frac{\sin(\ln 4x^2)}{x}\,dx$

7.2 Integration by Parts

Look at Product Rule (for Differentiation)

$$D_x[u(x)v(x)] = u'(x)v(x) + v'(x)u(x)$$

$$\Rightarrow \quad u(x)v'(x) = D_x[u(x)v(x)] - v(x)u'(x)$$

$$\Rightarrow \int u(x)v'(x)\,dx = \int \Big(D_x[u(x)v(x)] - v(x)u'(x)\Big)\,dx$$

$$= \int D_x[u(x)v(x)]\,dx - \int v(x)u'(x)\,dx$$

$$= u(x)v(x) - \int v(x)u'(x)\,dx$$

or, written more succinctly,

$$\boxed{\int u\,dv = uv - \int v\,du}$$ Integration by Parts

(The trick here will be choosing u + v properly.)

Ex 1 $\quad \int x\sin(2x)\,dx$

7.2 (cont)

Ex 2 $\int \arctan(5x)\,dx$

Ex 3 $\int \frac{\ln x}{\sqrt{x}}\,dx$

7.2 (cont)

Repeated Integration by Parts

Ex 4 $\int x^3 e^x \, dx$

7.2 (cont)

Ex 5 $\int e^x \cos x \, dx$

Reduction Formula (repeated use of integration by parts)

Ex 6 $\int \cos^n x \, dx$

7.3 Trigonometric Integrals

Combining u-substitution w/ trig identities. :)

3 forms will be addressed

1. $\int \sin^n x\, dx$, $\int \cos^n x\, dx$
2. $\int \sin^m x \cos^n x\, dx$
3. $\int \sin(mx)\cos(nx)\, dx$, $\int \sin(mx)\sin(nx)\, dx$, $\int \cos(mx)\cos(nx)\, dx$

Ex 1 $\int \sin^3 x\, dx$

7.3 (cont)

Ex 2 $\int \cos^4 x \, dx$

type (i)

if n odd,
use
$\sin^2 x + \cos^2 x = 1$

if n even,
use half-angle formula

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^2 x = \frac{1+\cos 2x}{2}$

Ex 3 $\int \cos^5 x \sin^{-4} x \, dx$

7.3 (cont)

Ex 4 $\int \cos^2 x \sin^4 x \, dx$

type 2

If m or n odd + positive, then factor out $\sin x$ or $\cos x$ + use $\sin^2 x + \cos^2 x = 1$.

If m + n even + positive, use half-angle identities.

7.3 (cont)

Ex 5 $\int \sin(4x)\cos(5x)\,dx$

type ③ use product identities:

$\sin(mx)\cos(nx) = \frac{1}{2}\left[\sin((m+n)x) + \sin((m-n)x)\right]$

$\sin(mx)\sin(nx) = -\frac{1}{2}\left[\cos((m+n)x) - \cos((m-n)x)\right]$

$\cos(mx)\cos(nx) = \frac{1}{2}\left[\cos((m+n)x) + \cos((m-n)x)\right]$

Ex 6 $\int_{-4}^{4} \sin\left(\frac{m\pi x}{4}\right)\sin\left(\frac{n\pi x}{4}\right)dx$ (Note: consider both cases ① $m \neq n$ ② $m = n$)

7.4 Rationalizing Substitutions

① Integrands Involving $\sqrt[n]{ax+b}$

(strategy: let $u = \sqrt[n]{ax+b}$)

Ex 1 $\displaystyle\int \frac{x^2+3x}{\sqrt{x+4}}\,dx$

Ex 2 $\displaystyle\int_0^1 \frac{\sqrt{x}}{x+1}\,dx$

7.4 (cont)

② Integrands Involving $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$ $(a \in \mathbb{R})$

Strategy: (a) $\sqrt{a^2-x^2} \rightarrow$ let $x = a\sin\theta$ $\quad \theta \in [-\pi/2, \pi/2]$

(b) $\sqrt{a^2+x^2} \rightarrow$ let $x = a\tan\theta$ $\quad \theta \in (-\pi/2, \pi/2)$

(c) $\sqrt{x^2-a^2} \rightarrow$ let $x = a\sec\theta$ $\quad \theta \in [0,\pi], \theta \neq \pi/2$

Ex 3 $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

Notice

(a) $\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta}$
$= \sqrt{a^2\cos^2\theta}$
$= |a\cos\theta|$
$= |a|\cos\theta$

(b) $\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta}$
$= \sqrt{a^2\sec^2\theta}$
$= |a\sec\theta|$
$= |a|\cos\theta$

(c) $\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2}$
$= \sqrt{a^2\tan^2\theta}$
$= |a\tan\theta|$
$= |a|(\pm\tan\theta)$

7.4 (cont)

Ex 4 $\displaystyle\int_2^3 \frac{dt}{t^2\sqrt{t^2-1}}$

7.4 (cont)

③ Completing the Square (use this strategy when there is a quadratic expression in the radical)

(✩ you may then need to use a trig substitution or some other strategy after that)

Ex 5 $\displaystyle\int \frac{3x}{\sqrt{x^2+4x-5}}\,dx$

7.5 Integration of Rational Fns (using Partial Fraction Decomposition)

rational fn ⇒ quotient of two polynomials

proper rational fn ⇒ rational fn whose numerator has lower degree than denominator

Let's review partial fraction decomposition (pfd)

(★★★ remember to use pfd only on proper rational fns. If you're given an improper rational fn, you will need to do long division first. ☺)

Ex 1 rewrite $\frac{x-7}{x^2-x-12}$ into 2 fractions.

7.5 (cont)

Ex 2 $\displaystyle\int \frac{4x^2-6x+2}{x^2(x-1)(x+3)}\,dx$

7.5 (cont)

Ex 3 $\displaystyle\int \frac{33x^2-7x+70}{(3x-2)(x^2+4)}\,dx$

7.5 (cont)

Ex 4 $\displaystyle\int \frac{\cos x}{\sin^4 x - 16}\,dx$

7.5 (cont)

Ex 5 $$\int \frac{x^6 - 7x^4 + 11x^3 - 13x^2 + x - 6}{x^3 - 2x^2}\,dx$$

7.6 Strategies for Integration

7.1-7.5 ideas:

1. u-substitution
2. integration by parts
3. trig substitutions
4. partial fraction decomposition

5. Integral tables (in back of your book)
6. computer/calculator approximations (especially useful for definite integrals)

Ex 1 $\displaystyle\int_3^4 \frac{1}{t - \sqrt{2t}}\,dt$

7.6 (cont)

Ex 2 $\displaystyle\int \frac{\sqrt{x^2-4x}}{x-2}\,dx$

Ex 3 $\displaystyle\int \frac{\operatorname{sech}(\sqrt{x})}{\sqrt{x}}\,dx$

7.6 (cont)

Ex 4 The density of a rod is given as $\delta(x) = \frac{2}{x^2+1}$. Find $c \ni$ the mass from 0 to c is equal to 1.