7.2 #55) Derive the formula

$$\int x^{\alpha} e^{\beta x}\,dx = \frac{x^{\alpha} e^{\beta x}}{\beta} - \frac{\alpha}{\beta}\int x^{\alpha-1} e^{\beta x}\,dx$$

$$\int x^{\alpha} e^{\beta x}\,dx$$

$u = x^{\alpha}$ $\qquad$ $v = \frac{1}{\beta} e^{\beta x}$

$du = \alpha x^{\alpha-1}\,dx$ $\qquad$ $dv = e^{\beta x}\,dx$

$$= \frac{1}{\beta} x^{\alpha} e^{\beta x} - \frac{\alpha}{\beta}\int x^{\alpha-1} e^{\beta x}\,dx$$

$\boxed{7.5\ \#33}$

$$\int \frac{(\sin^3 t - 8\sin^2 t - 1)\cos t}{(\sin t + 3)(\sin^2 t - 4\sin t + 5)}\,dt$$

Let $u = \sin t$
$du = \cos t\, dt$

$$= \int \frac{u^3 - 8u^2 - 1}{(u+3)(u^2-4u+5)}\,du$$

$$= \int \frac{u^3 - 8u^2 - 1}{u^3 - u^2 - 7u + 15}\,du$$

$(u+3)(u^2-4u+5)$
$= u^3 - u^2 - 7u + 15$

$$\begin{array}{r} 1 \\ u^3 - u^2 - 7u + 15 \,\big)\, u^3 - 8u^2 + 0u - 1 \\ -(u^3 - u^2 - 7u + 15) \\ \hline -7u^2 + 7u - 16 \end{array}$$

$$(\star) \quad = \int 1 + \frac{-7u^2 + 7u - 16}{u^3 - u^2 - 7u + 15}\,du$$

$$\frac{-7u^2 + 7u - 16}{(u+3)(u^2-4u+5)} = \frac{A}{u+3} + \frac{Bu + C}{u^2 - 4u + 5}$$

$$\Rightarrow \quad -7u^2 + 7u - 16 = A(u^2 - 4u + 5) + (Bu + C)(u+3)$$

$u = -3$, $\quad -7(9) - 21 - 16 = A(9 + 12 + 5)$
$-(63 + 21 + 16) = 26A$
$-100 = 26A$
$\boxed{A = \frac{-50}{13}}$

$u = 0$, $\quad -16 = 5\left(\frac{-50}{13}\right) + 3C$

$-16 = \frac{-250}{13} + 3C$

$\frac{-16(13) + 250}{13} = 3C \Rightarrow \frac{42}{13(3)} = C \Rightarrow \boxed{\frac{14}{13} = C}$

7.5 #33 (cont)

$$-7u^2+7u-16 = \frac{-50}{13}(u^2-4u+5) + \left(Bu+\frac{14}{13}\right)(u+3)$$

$u=1$, $\quad -7+7-16 = \frac{-50}{13}(1-4+5) + \left(B+\frac{14}{13}\right)4$

$$-16 = \frac{-100}{13} + 4B + \frac{56}{13}$$

$$-16 = \frac{-44}{13} + 4B$$

$$\frac{-16(13)+44}{13} = 4B \Rightarrow \frac{-164}{13} = 4B \Rightarrow \frac{-164}{13(4)} = B$$

$$\Rightarrow \boxed{\frac{-41}{13} = B}$$

$$(\star) \quad = \int\left[1 + \frac{-50/13}{u+3} + \frac{-41/13\, u + 14/13}{u^2-4u+5}\right]du$$

$$= \int\left[1 - \frac{50}{13}\left(\frac{1}{u+3}\right) + \frac{1}{13}\left(\frac{-41u+14}{u^2-4u+5}\right)\right]du$$

$$(\heartsuit) \quad = u - \frac{50}{13}\ln|u+3| + \frac{1}{13}\int\frac{-41u+14}{u^2-4u+5}\,du$$

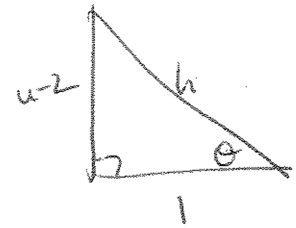
$$\int\frac{-41u+14}{u^2-4u+5}\,du = \int\frac{-41u}{u^2-4u+5}\,du + \int\frac{14}{u^2-4u+5}\,du$$

$$u^2-4u+5 = (u^2-4u+4)+5-4 = (u-2)^2+1$$

$$= -41\int\frac{u}{(u-2)^2+1}\,du + 14\int\frac{1}{(u-2)^2+1}\,du$$

$$= -41\underbrace{\int\frac{u}{(u-2)^2+1}\,du}_{} + 14\arctan(u-2) + C$$

$$\int \frac{u}{(u-2)^2+1}\,du$$



$\tan\theta = \frac{u-2}{1}$

$(u-2)^2+1 = h^2$

$\sqrt{(u-2)^2+1} = h$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{(u-2)^2+1}}$

Let $u-2 = \tan\theta$

$du = \sec^2\theta\,d\theta$

$(u-2)^2+1 = \tan^2\theta + 1 = \sec^2\theta$

$$\Rightarrow \int \frac{(\tan\theta+2)}{\cancel{\sec^2\theta}}(\cancel{\sec^2\theta})\,d\theta = \int(\tan\theta+2)\,d\theta$$

$$= \int \frac{\sin\theta}{\cos\theta}\,d\theta + \int 2\,d\theta$$

$w = \cos\theta$

$-dw = \sin\theta\,d\theta$

$$= -\int \frac{1}{w}\,dw + 2\theta + C = -\ln|\cos\theta| + 2\theta + C$$

$$= -\ln\left|\frac{1}{\sqrt{(u-2)^2+1}}\right| + 2\arctan(u-2)$$

$$= \ln\left(\sqrt{(u-2)^2+1}\right) + 2\arctan(u-2) + C$$

$$(\heartsuit) = u - \frac{50}{13}\ln|u+3| + \frac{1}{13}\Big[-41\Big(\ln\left(\sqrt{(u-2)^2+1}\right) + 2\arctan(u-2)\Big) + 14\arctan(u-2)\Big] + C$$

$$= u - \frac{50}{13}\ln|u+3| - \frac{41}{13}\ln\left(\sqrt{(u-2)^2+1}\right) + \frac{-68}{13}\arctan(u-2) + C$$

$$= \boxed{\sin t - \frac{50}{13}\ln(\sin t + 3) - \frac{41}{13}\ln\left(\sqrt{(\sin t-2)^2+1}\right) \frac{-68}{13}\arctan(\sin t - 2) + C}$$