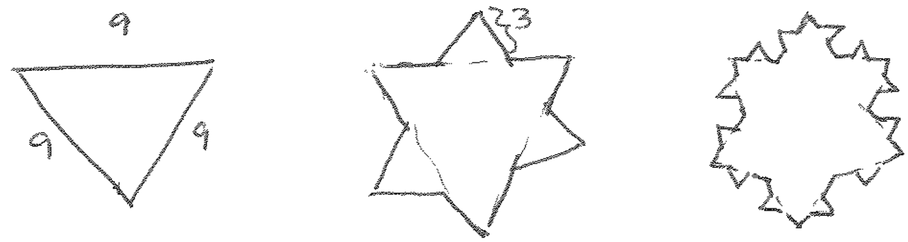
9.2 #33



$n=1$ $\quad$ $n=2$ $\quad$ $n=3$

each little side is length 1

(a) Perimeter

$n=1,\quad P_1 = 3(9) = 3(3^2)$

$n=2,\quad P_2 = 3(9)+3(3) = 12(3) = 3\cdot4(9\div3) = 3\cdot4(3^1)$

$n=3,\quad P_3 = 3(9)+3(3)+12(1) = 48(1) = 3\cdot4\cdot4(3\div3) = 3\cdot4^2(3^0)$

$n=4,\quad P_4 = 3(9)+3(3)+12(1)+48(\tfrac13) = 192(\tfrac13) = 3\cdot4\cdot4\cdot4(1\div3)$
$= 3\cdot4^3(3^{-1})$

$n=5,\quad P_5 = 3\cdot4^4(3^{-2})$

$\vdots$

$n,\quad P_n = 3\cdot4^{n-1}(3^{3-n}) = 4^{n-1}(3^{4-n}) = \dfrac{4^{n-1}}{3^{n-4}} = \left(\dfrac{4^{n-1}}{3^{n-1}}\right)\left(\dfrac{1}{3^{-3}}\right)$

$= 27\left(\frac43\right)^{n-1}$

$\Rightarrow$ Perimeter $P = \lim_{n\to\infty} 27\left(\frac43\right)^{n-1} \to \infty$

(b) Area

$n=1,\quad A_1 = \frac{\sqrt3}{4}(9)^2$

$n=2,\quad A_2 = \frac{\sqrt3}{4}(9^2) + 3\left(\frac{\sqrt3}{4}(3^2)\right)$

$n=3,\quad A_3 = \frac{\sqrt3}{4}(9^2) + 3\left(\frac{\sqrt3}{4}(3^2)\right)$
$+12\left(\frac{\sqrt3}{4}(1^2)\right)$

$n=4,\quad A_4 = \frac{\sqrt3}{4}(9^2) + 3\left(\frac{\sqrt3}{4}(3^2)\right)$
$+12\left(\frac{\sqrt3}{4}(1^2)\right) + 48\left(\frac{\sqrt3}{4}\left(\frac13\right)^2\right)$

$n=5,\quad A_5 = \frac{\sqrt3}{4}(9^2) + 3\left(\frac{\sqrt3}{4}(3^2)\right) + 12\left(\frac{\sqrt3}{4}(1^2)\right) + 48\left(\frac{\sqrt3}{4}\left(\frac13\right)^2\right) + 48(4)\left(\frac{\sqrt3}{4}\left(\frac19\right)^2\right)$

$\vdots$

Area of Equilateral Δ

$A = \frac12 x h$

but by Pythagorean Thm

$(\frac12 x)^2 + h^2 = x^2$

$h^2 = \frac34 x^2$

$h = \frac{\sqrt3}{2}x$

$\Rightarrow A = \frac12 x\left(\frac{\sqrt3}{2}x\right) = \frac{\sqrt3}{4}x^2$

$\boxed{9.2 \# 33 \text{ (cont)}}$

(b) ⋮

$$A_n = \frac{\sqrt{3}}{4}\Big[\,9^2 + \underset{n=1}{3(3^2)} + \underset{n=2}{12(1^2)} + \underset{n=3}{48\left(\tfrac{1}{3}\right)^2} + 192\left(\tfrac{1}{9}\right)^2 + \cdots\Big]$$

$$A_n = \frac{\sqrt{3}}{4}(9^2) + \frac{\sqrt{3}}{4}\Big[3(3^2) + 3(4)(3^0) + 3(4^2)(3^{-2}) + 3(4^3)(3^{-4}) + \cdots\Big]$$

$$= \frac{\sqrt{3}}{4}(9^2) + \frac{\sqrt{3}}{4}\Big[\underset{n=1}{3^3} + \underset{n=2}{4(3^1)} + \underset{n=3}{4^2(3^{-1})} + 4^3(3^{-3}) + 4^4(3^{-5}) + \cdots\Big]$$

$$= \frac{\sqrt{3}}{4}(9^2) + \frac{\sqrt{3}}{4}\sum_{n=1}^{\infty} 4^{n-1}\, 3^{5-2n}$$

$$= \frac{\sqrt{3}}{4}(81) + \frac{\sqrt{3}}{4}\sum_{n=1}^{\infty} \frac{4^{n-1}}{3^{2n-5}}$$

$$= \frac{\sqrt{3}}{4}(81) + \frac{\sqrt{3}}{4}\sum_{n=1}^{\infty} \left(\frac{1}{3^{-3}}\right)\left(\frac{4^{n-1}}{(3^2)^{n-1}}\right)$$

$$= \frac{\sqrt{3}}{4}(81) + \frac{\sqrt{3}}{4}(3^3)\sum_{n=1}^{\infty} \left(\frac{4}{9}\right)^{n-1} \quad \Big\}\text{ geometric series}$$

$$= \frac{81\sqrt{3}}{4} + \frac{27\sqrt{3}}{4}\left(\frac{1}{1-4/9}\right)$$

$$= \frac{81\sqrt{3}}{4} + \frac{27\sqrt{3}}{4}\left(\frac{9}{5}\right)$$

$$= \frac{\sqrt{3}}{4}\left(81 - \frac{243}{5}\right) = \frac{\sqrt{3}}{4}\left(\frac{162}{5}\right) = \boxed{\frac{81\sqrt{3}}{10}}$$

area

9.2 #35

Here's a table to see how far each has travelled.

(counter) n	Achilles	Tortoise	
0	0	100	yds
1	100	10	
2	10	1	
3	1	1/10	
4	1/10	1/100	
$\vdots$	$\vdots$	$\vdots$	
n	10^{-n+3}	10^{-n+2}	

Total distance travelled by Achilles $\Rightarrow$

$$A = \sum_{n=1}^{\infty} 10^{-n+3} = \sum_{n=1}^{\infty} \frac{1000}{10^n}$$

$$= \frac{1000}{10} \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^{n-1}$$

$$= 100 \left(\frac{1}{1-1/10}\right)$$

$$= 100\left(\frac{10}{9}\right) = \frac{1000}{9}$$

Total distance travelled by Tortoise $\Rightarrow$

$$T = 100 + \sum_{n=1}^{\infty} 10^{-n+2}$$

$$= 100 + \sum_{n=1}^{\infty} \frac{100}{10^n}$$

$$= 100 + \frac{100}{10} \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^{n-1}$$

$$= 100 + 10\left(\frac{1}{1-1/10}\right)$$

$$= 100 + 10\left(\frac{10}{9}\right) = \frac{1000}{9}$$

$$= \boxed{111\tfrac{1}{9} \text{ yds}}$$

$\Rightarrow$ when they both travel $111\,1/9$ yds, they meet up exactly