5.6 Moments, Center of Mass

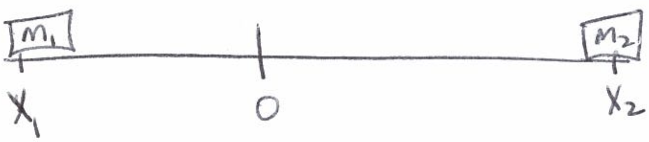
called lever arm

teeter-totter

m_1, m_2 masses (wts)
d_1, d_2 distance from fulcrum

This stays balanced only if $m_1 d_1 = m_2 d_2$.

If we put seesaw on x-axis w/ fulcrum at origin, then to stay balanced we need to satisfy

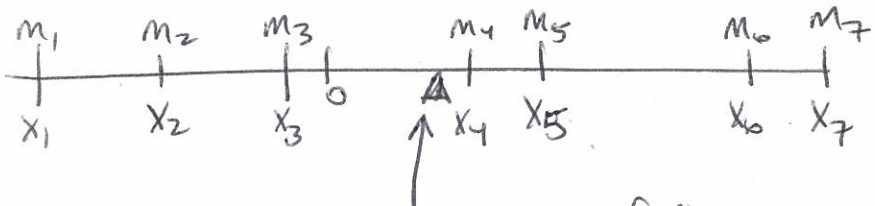


$x_1 m_1 + x_2 m_2 = 0$

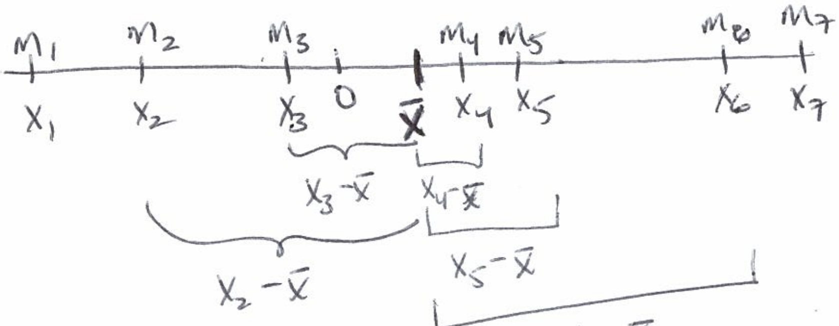
(since $x_1 = -d_1$)

moment of a particle wrt a pt $\Rightarrow$ product of mass m of the particle with its directed distance from a pt. (This measures tendency to produce a rotatn about that pt.)

total moment M for a bunch of masses $= \sum_{i=1}^{n} x_i m_i$



Where does fulcrum need to be placed to balance? Let's call it $\bar{x}$.



5.6 (continued)

Then, for balance at $\bar{x}$, we need

$$(x_1-\bar{x})m_1 + (x_2-\bar{x})m_2 + \dots + (x_n-\bar{x})m_n = 0$$

$$\iff x_1m_1 + x_2m_2 + \dots + x_nm_n = \bar{x}m_1 + \bar{x}m_2 + \dots + \bar{x}m_n$$

$$\iff x_1m_1 + x_2m_2 + \dots + x_nm_n = \bar{x}(m_1+m_2+\dots+m_n)$$

$$\bar{x} = \frac{x_1m_1 + x_2m_2 + \dots + x_nm_n}{m_1+m_2+\dots+m_n} = \boxed{\frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} = \bar{x}}$$

Center of mass

i.e. balance pt.

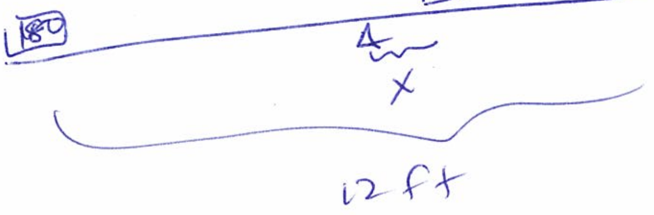
balance pt, a.k.a. center of mass, is just M (total moment w.r.t. origin) divided by m (total mass)

For a continuous mass distribution along a line (like in a wire) $\Rightarrow$

$$\boxed{\bar{x} = \frac{M}{m} = \frac{\int_a^b x\,\delta(x)\,dx}{\int_a^b \delta(x)\,dx}}$$

(where $\delta(x)$ = density function)

Ex 1 John + Mary, weighing 180 lbs + 110 lbs respectively, sit at opposite ends of a 12-ft teeter totter w/ the fulcrum in the middle, where should their 80-lb son sit in order for the board to balance?



$$M = 110(6) + 180(-6) + 80x = 0$$

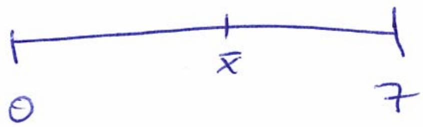
$$x = \frac{21}{4}$$

$$x = 5.25$$

5.6 (continued)

Ex 2 A straight wire 7 units long has density $\delta(x) = 1 + x^3$ at a pt x units from one end. Find the distance from this end to the center of mass.

$$\delta(x) = 1 + x^3 \qquad \bar{x} = \frac{M}{m} = \frac{\int_a^b x\,\delta(x)\,dx}{\int_a^b f(x)\,dx}$$



$$\bar{x} = \frac{\int_0^7 x(1+x^3)\,dx}{\int_0^7 1+x^3\,dx} = \frac{\int_0^7 x+x^4\,dx}{\left(x+\frac{x^4}{4}\right)\Big|_0^7}$$

$$= \frac{\left(\frac{x^2}{2}+\frac{x^5}{5}\right)\Big|_0^7}{\left(7+\frac{7^4}{4}\right)-0} = \frac{\left(\frac{49}{2}+\frac{7^5}{5}\right)-0}{\frac{28+2401}{4}}$$

$$= \frac{\frac{245+33614}{10}}{\frac{2429}{4}} = \frac{33859}{10}\left(\frac{4}{2429}\right)$$

$$= \frac{135436}{24290} \approx 5.576$$

5.6 (continued)

Now, consider a discrete set on 2d masses.

Then, to find the center of mass (i.e. the geometric center) $(\bar{x}, \bar{y})$, we'll have

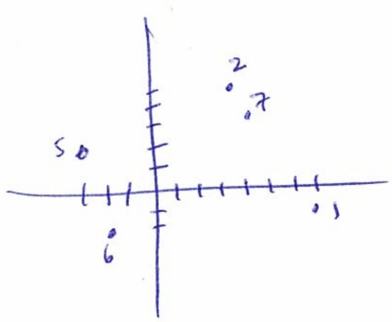
$$\bar{x} = \frac{M_y}{m} \quad \& \quad \bar{y} = \frac{M_x}{m}$$

where $M_y = \sum_{i=1}^{n} x_i m_i \quad M_x = \sum_{i=1}^{n} y_i m_i$

and $m = \sum_{i=1}^{n} m_i$

Ex 3 The masses and coordinates of a system of particles are given by the following: 5, (−3, 2); 6, (−2, −2); 2, (3, 5); 7, (4, 3); 1, (7, −1). Find the moments of this system wrt the coord. axes & find center of mass.

$m = 5+6+2+7+1 = 21$



$M_x = 2(5) + 7(3) + 5(2) + 1(-1) + 6(-2)$

$= 10 + 21 + 10 - 1 - 12$

$= 28$

$M_y = 2(3) + 7(4) + 1(7) + 5(-3) + 6(-2)$

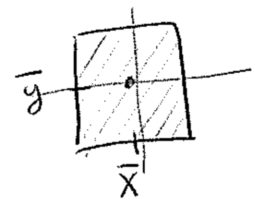
$= 14$

$\bar{y} = \frac{28}{21} = \frac{4}{3} \qquad \bar{x} = \frac{14}{21} = \frac{2}{3}$

$\left(\frac{2}{3}, \frac{4}{3}\right)$

5.6 (continued)

Now, consider a continuous 2d region (we'll call it a lamina) that has constant (homogeneous) density everywhere. Then, to find the center of mass $(\bar{x}, \bar{y})$, we'll have (still)



$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}$$

but $M_y = \delta \int_a^b x[f(x)-g(x)]\,dx$

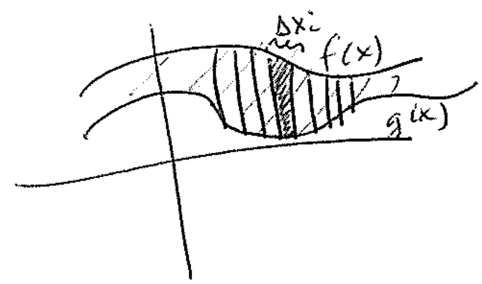
$M_x = \frac{\delta}{2} \int_a^b [f^2(x) - g^2(x)]\,dx$

and $m = \delta \int_a^b [f(x)-g(x)]\,dx$

where δ = density of lamina

because m used to be $m = \sum_{i=1}^{n} m_i$ which becomes

$$m = \sum_{i=1}^{n} \delta \Delta x_i \Delta y_i = \delta \sum_{i=1}^{n} \Delta x_i (f(x_i) - g(x_i))$$



$$\Rightarrow M_y = \sum_{i=1}^{n} x_i m_i = \sum_{i=1}^{n} x_i (f(x_i) - g(x_i))\, \delta \Delta x_i$$
$$= \delta \int_a^b x(f(x) - g(x))\,dx$$

and $\Rightarrow M_x = \sum_{i=1}^{n} y_i m_i = \sum_{i=1}^{n} \underbrace{\left(\frac{f(x_i) + g(x_i)}{2}\right)}_{\text{avg } y\text{-value}} \left(\delta\,(f(x_i) - g(x_i))\,\Delta x_i\right)$

$$= \frac{\delta}{2} \sum_{i=1}^{n} [f^2(x_i) - g^2(x_i)]\,\Delta x_i$$

$$= \frac{\delta}{2} \int_a^b [f^2(x) - g^2(x)]\,dx$$

5.6 (continued)

$$\Rightarrow \bar{x} = \frac{M_y}{m} = \frac{\delta \int_a^b x\,(f(x)-g(x))\,dx}{\delta \int_a^b [f(x)-g(x)]\,dx} = \frac{\int_a^b x\,[f(x)-g(x)]\,dx}{\int_a^b [f(x)-g(x)]\,dx}$$

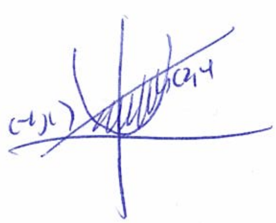
and $$\bar{y} = \frac{M_x}{m} = \frac{1}{2}\left(\frac{\int_a^b [f^2(x)-g^2(x)]\,dx}{\int_a^b [f(x)-g(x)]\,dx}\right)$$

★ Note: It doesn't depend at all on density! Only depends on shape $\Rightarrow$ geometric problem

center of mass = centroid $(\bar{x}, \bar{y})$

Ex 4 Find the centroid of the region bounded by $y = x^2$ and $y = x+2$

$$m = \delta \int_{-1}^{2} x+2-x^2\,dx = \frac{9}{2}\delta$$



$$M_x = \frac{\delta}{2}\int_{-1}^{2} (x+2)^2 - (x^2)^2\,dx = 8.7\delta$$

$$M_y = \delta \int_{-1}^{2} x\,(x+2-x^2)\,dx = \frac{9}{4}\delta$$

$$\Rightarrow \bar{x} = \frac{9/4}{9/2} = \frac{1}{2} \qquad \bar{y} = \frac{87/10}{9/2} = \frac{29}{15}$$

$$\left(\frac{1}{2}, \frac{29}{15}\right)$$