

$$\int \csc x \, dx = \int \csc x \left( \frac{\csc x + \cot x}{\csc x + \cot x} \right) dx$$

$$u = \csc x + \cot x$$

$$du = (-\csc x \cot x - \csc^2 x) dx$$

$$du = -\csc x (\cot x + \csc x) dx$$

$$du = -\csc x (u) dx$$

$$\frac{1}{u} du = \csc x \, dx$$

$$= \int \frac{1}{u} \left( \frac{u}{u} \right) du$$

$$= -\int \frac{1}{u} du = -\ln|u| = -\ln|\csc x + \cot x| + C$$

$$= \ln|(\csc x + \cot x)^{-1}| + C$$

$$= \ln\left| \left( \frac{1 + \cos x}{\sin x} \right)^{-1} \right| + C$$

$$= \ln\left| \frac{\sin x}{1 + \cos x} \right| + C$$

$$= \ln\left| \left( \frac{\sin x}{1 + \cos x} \right) \left( \frac{1 - \cos x}{1 - \cos x} \right) \right| + C$$

$$= \ln\left| \frac{\sin x (1 - \cos x)}{1 - \cos^2 x} \right| + C$$

$$= \ln\left| \frac{\sin x (1 - \cos x)}{\sin^2 x} \right| + C$$

$$= \ln\left| \frac{1 - \cos x}{\sin x} \right| + C = \ln|\csc x - \cot x| + C$$

$$\int \sec x \, dx = \int \sec x \left( \frac{\cos x}{\cos x} \right) dx = \int \frac{\cos x}{\cos^2 x} dx$$

$$= \int \frac{\cos x}{1 - \sin^2 x} dx$$

$$u = \sin x \\ du = \cos x \, dx$$

$$= \int \frac{du}{1 - u^2} = \int \frac{1}{(1-u)(1+u)} du$$

$$\frac{1}{(1-u)(1+u)} = \frac{A}{1-u} + \frac{B}{1+u}$$

$$1 = A(1+u) + B(1-u)$$

$$u=1: \quad 1 = 2A \Rightarrow A = \frac{1}{2}$$

$$u=-1: \quad 1 = -2B \Rightarrow B = -\frac{1}{2}$$

$$\rightarrow = \int \frac{\frac{1}{2}}{1-u} du + \int \frac{-\frac{1}{2}}{1+u} du$$

$$= \frac{1}{2} \frac{\ln|1-u|}{-1} + \frac{-1}{2} \ln|1+u| + C$$

$$= \frac{1}{2} \ln|1-\sin x| - \frac{1}{2} \ln|1+\sin x| + C$$

$$= \frac{1}{2} \ln|(1-\sin x)(1+\sin x)| + C$$

$$= \frac{1}{2} \ln|1 - \sin^2 x| + C$$

$$= \frac{1}{2} \ln|\cos^2 x| + C = \ln|\cos^2 x|^{-\frac{1}{2}} + C$$

$$= \ln|\sec x| + C$$