9.5 Practice (Alternating Series)

Ex 1 Do these series converge absolutely, converge conditionally or diverge?

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^{n-1}}{n!}$

(b) $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$

ART (Absolute Ratio Test)

if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho \begin{cases} \rho<1 & \text{converges absolutely} \\ \rho>1 & \text{doesn't converge absolutely} \\ \rho=1 & \text{inconclusive} \end{cases}$

- Note: If a series does not converge absolutely, then ① if it's all positive terms, it diverges ② if it's alternating, try AST (Alternating Series Test) to see if it conditionally converges

To test for absolute convergence, use all the tests we have so far on $\sum_{n=1}^{\infty} |a_n|$.

AST (Alternating Series Test)

if we have $\sum_{n=1}^{\infty} (-1)^n a_n$ where $a_n > 0$, and if $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges (conditionally at least).

Ex 1 (cont)

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n^{1.01}+5}$

(e) $1 - \frac{5^2}{2!} + \frac{5^4}{4!} - \frac{5^6}{6!} + \cdots$

(f) $\displaystyle\sum_{n=2}^{\infty} \frac{(-5)^n n^3}{8^n}$

EX2 Estimate the error made by using S_{10} as approximation to $\sum_{n=2}^{\infty} (-1)^{n+1}\left(\frac{2n}{n^2-1}\right)$.

$$E_n = |S - S_n| \le a_{n+1}$$

for alternating series

9.6 Practice (Power Series)

Ex 1 Find convergence set.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{(2x-1)^n}{(n+1)!}$

(b) $\displaystyle\sum_{n=1}^{\infty} n x^n$

Power Series (a function of x)

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

converges

① at $x=0$

or ② at $(-R,R)$ (or $[-R,R)$, $(-R,R]$, or $[-R,R]$)

or ③ $\mathbb{R}$

To find convergence set:

(1) use ART & force convergence.

(2) check endpts of convergence interval separately using whatever tests seem appropriate

Note: **Infinite Series of numbers:**
convergence means sum adds to finite #

Power Series
convergence means ∞ polynomial represents some fn of x on the interval of convergence

Find convergence set.

Ex2 (a) $x + 4x^2 + 9x^3 + 16x^4 + \dots$

(b) $(x+3) - 2(x+3)^2 + 3(x+3)^3 - 4(x+3)^4 + \dots$

(c) $\frac{x}{2^2-1} + \frac{x^2}{3^2-1} + \frac{x^3}{4^2-1} + \frac{x^4}{5^2-1} + \dots$

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n (2x-3)^n}{4^n \sqrt{n}}$

9.7 Practice (Operations on Power Series)

Ex 1 Find power series for $f(x)$, and radius of convergence.

(a) $f(x) = \dfrac{2x^3}{1-x^4}$

(b) $f(x) = e^{3x} + 3x^2 - 5$

if $S(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on I, then

$S'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$

and $\int_0^x S(t)\,dt$

$= \sum_{n=0}^{\infty} \dfrac{a_n x^{n+1}}{n+1}$

also converge on interior of I.

① $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \forall\, x \in (-1,1)$

② $\ln(1+x) = \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1} x^n}{n}$

$\forall\, x \in (-1,1)$

③ $\arctan x = \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1} x^{2n-1}}{2n-1}$

$\forall\, x \in [-1,1]$

④ $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} \quad \forall\, x \in \mathbb{R}$

Ex2 Find power series for $f(x)$, and radius of convergence.

(a) $\displaystyle\int_0^x \frac{\arctan t}{t}\,dt = f(x)$

(b) $f(x) = \dfrac{x}{x^2-3x+2}$

(hint: PFD)

Ex3 Find sum of $\displaystyle\sum_{n=1}^{\infty} n x^n$ (ie. find the fn this represents)

9.8 Practice (Taylor & Maclaurin Series)

Ex 1 Find Taylor Series for these fns w/ given center value a, out to x^5 term.

(a) $f(x) = x(\sin(2x) + \sin(3x))$

$a = 0$

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$

$\forall$ x in an interval centered at a, and

$c_n = \dfrac{f^{(n)}(a)}{n!}$

Taylor Series

for Maclaurin series, $a = 0$.

Taylor's Thm

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

on $(a-r, a+r) \iff \lim_{n\to\infty} R_n(x) = 0$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

$c \in (a-x, a+x)$

(5) $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ $\forall x \in \mathbb{R}$

(6) $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ $\forall x \in \mathbb{R}$

(7) $\cosh x = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ $\forall x \in \mathbb{R}$

(8) $\sinh x = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ $\forall x \in \mathbb{R}$

Ex1 (cont)

(b) $f(x) = \sec x$, $a = \pi/4$

(c) $f(x) = e^x$, $a = 2$

(d) $f(x) = -x^3 + 3x^2 - x + 2$

$a = -1$

(e) $f(x) = \begin{cases} 0, & x < 0 \\ x^4, & x \geq 0 \end{cases}$ $a = 0$

explain why we cannot use Maclaurin series to represent this fn. Can we use a different Taylor series?

9.9 Practice (Taylor Approximation to a Fn)

$$f(x) \approx P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

error

$$|R_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \right|$$

$$c \in [a-x, a+x]$$

Ex1 Find a good bound for max value of given expressn.

(a) $\left| \dfrac{c^2 + \sin c}{10 \ln c} \right|$ on $[2,4]$

(b) $|\tan c + \sec c|$ on $[0, \pi/3]$

Ex 2 Find Taylor polynomial of order 3 based at a.

(a) $f(x) = \sqrt{x}$, $a = 2$

(b) $f(x) = \sqrt{1+x}$, $a = 0$

Ex 3 Find formula for $R_6(x)$ & find good bound for $|R_6(\frac{1}{2})|$.

(a) $f(x) = \frac{1}{x^2}$, $a = 1$

(b) $f(x) = \frac{2}{x-3}$, $a = 1$