① $\int \csc x\,dx = \int \csc x \left(\frac{\csc x + \cot x}{\csc x + \cot x}\right) dx$

$= \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\,dx$

$u = \csc x + \cot x$

$du = (-\csc x \cot x - \csc^2 x)\,dx$

$-du = (\csc x \cot x + \csc^2 x)\,dx$

$= -\int \frac{du}{u}$

$= -\ln|u| + C$

$= -\ln|\csc x + \cot x| + C$

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② $\int \sec x\,dx = \int \sec x \left(\frac{\cos x}{\cos x}\right) dx = \int \frac{\cos x}{\cos^2 x}\,dx = \int \frac{\cos x}{1-\sin^2 x}\,dx$

$u = \sin x$

$du = \cos x\,dx$

$= \int \frac{du}{1-u^2} = \int \frac{du}{(1-u)(1+u)}$

$\frac{1}{(1-u)(1+u)} = \frac{A}{1-u} + \frac{B}{1+u}$

$\Rightarrow 1 = A(1+u) + B(1-u)$

if $u=1$: $1 = 2A \Rightarrow A = \frac{1}{2}$

$u=-1$: $1 = 2B \Rightarrow B = \frac{1}{2}$

$\Rightarrow \int \sec x\,dx = \int \frac{\frac{1}{2}\,du}{1-u} + \int \frac{\frac{1}{2}\,du}{1+u}$

$= -\frac{1}{2}\ln|1-u| + \frac{1}{2}\ln|1+u| + C$

$= \frac{1}{2}\ln\left(\left|\frac{1+u}{1-u}\right|\right) + C = \frac{1}{2}\ln\left|\left(\frac{1+\sin x}{1-\sin x}\right)\left(\frac{1+\sin x}{1+\sin x}\right)\right| + C$

$= \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{\cos^2 x}\right| + C = \ln\left|\frac{1+\sin x}{\cos x}\right| + C = \ln|\sec x + \tan x| + C$

$$-\ln|\csc x + \cot x| = -\ln\left|\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right|$$

$$= -\ln\left|\frac{1+\cos x}{\sin x}\right|$$

$$= \ln\left|\frac{\sin x}{1+\cos x}\right|$$

$$= \ln\left|\frac{\sin x}{(1+\cos x)}\left(\frac{1-\cos x}{1-\cos x}\right)\right|$$

$$= \ln\left|\frac{\sin x - \sin x\cos x}{1-\cos^2 x}\right|$$

$$= \ln\left|\frac{\sin x - \sin x\cos x}{\sin^2 x}\right|$$

$$= \ln\left|\frac{1-\cos x}{\sin x}\right|$$

$$= \ln|\csc x - \cot x|$$