Math2210 Final Exam Spring, 2015 Kelly MacArthur

uid number: Key Special number: ________

Instructions:

- Please show all of your work as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown.
- All answers should be completely simplified, unless otherwise stated.
- There are no calculators or any sort of electronics allowed on this exam. Make sure all cell phones are put away and out of sight. If you have a cell phone out at any point, for any reason, you will receive a zero on this exam.
- You will be given an opportunity to ask clarifying questions about the instructions at exactly 8:00 a.m. (for a couple minutes). The questions will be answered for the entire class. After that, no further questions will be allowed, for any reason.
- You must show us your U of U student ID card when finished with the exam.
- The exam key will be posted on Canvas by noon.
- You may ask for scratch paper. You may use NO other scratch paper. Please transfer all finished work onto the proper page in the test for us to grade there. We will not grade the work on the scratch page.
- You are allowed to use one 8.5x11 inch piece of paper with notes for your reference during the exam.

This test totals 210 points, so there are 10 points of extra credit built in to the exam. The score will be computed out of 200 points.

STUDENT—PLEASE DO NOT WRITE BELOW THIS LINE. THIS TABLE IS TO BE USED FOR GRADING.

Problem	Score
1 & 2	
3	
4	
5	
6	
7	
8	
9	
10	

Raw Total:	
Total Percentage:	

Part 1 (Computations)

1. *(15 points)* What type of entity (scalar or vector) does each of these computations return? Assume $\boldsymbol{F}$ is a vector field. *(Note: if the operation is not defined, i.e. it's impossible to do, circle undefined.)*

$\nabla \cdot \boldsymbol{F}$	scalar	vector	undefined	*(circle one)*
$\nabla(\nabla \cdot \boldsymbol{F})$	scalar	vector	undefined	*(circle one)*
$\nabla(\nabla \times \boldsymbol{F})$	*scalar*	*vector*	*undefined*	*(circle one)*
$\nabla \cdot (\nabla \boldsymbol{F})$	*scalar*	*vector*	*undefined*	*(circle one)*
$\nabla \cdot (\nabla \times \boldsymbol{F})$	*scalar*	*vector*	*undefined*	*(circle one)*

2. *(15 points)* Evaluate the integral $\displaystyle\int_0^{\frac{\pi}{16}} \int_{sec(4x)}^{0} \int_{\frac{\pi z}{2}}^{4xz} \cos\left(\frac{y}{z}\right) dy\, dz\, dx$.

$$= \int_0^{\pi/16} \int_{sec(4x)}^{0} z \sin\left(\frac{y}{z}\right)\Big|_{\frac{\pi z}{2}}^{4xz} dz\, dx$$

$$= \int_0^{\pi/16} \int_{Sec(4x)}^{0} z\left[\sin(4x) - \sin\left(\frac{\pi}{2}\right)\right] dz\, dx$$

$$= \int_0^{\pi/16} \int_{sec(4x)}^{0} (z \sin 4x - z)\, dz\, dx$$

$$= \int_0^{\pi/16} (\sin(4x) - 1) \frac{z^2}{2}\Big|_{sec^2(4x)}^{0} dx$$

$$= \int_0^{\pi/16} \left(\frac{1}{2}(\sin(4x) - 1)\right)(-sec^2(4x))\, dx$$

$$= \int_0^{\pi/16} \frac{1}{2}\left(\frac{-\sin(4x)}{\cos^2(4x)} + sec^2(4x)\right) dx = \int_0^{\frac{\pi}{16}} \left(\frac{-1}{2}(\tan(4x) sec(4x)) + \frac{1}{2} sec^2(4x)\right) dx$$

$$= \frac{-1}{2}\left(\frac{sec(4x)}{4}\right) + \frac{1}{2}\frac{\tan(4x)}{4}\Big|_0^{\pi/16} = \frac{1}{8}\left\{\left(-sec\left(\frac{\pi}{4}\right) - -sec(0)\right) + \tan\frac{\pi}{4} - \tan 0\right\}$$

Answer: $\boxed{\frac{1}{4} - \frac{1}{8}\sqrt{2}}$

$$= \frac{1}{8}\left(-\sqrt{2} + 1 + 1 - 0\right) = \boxed{\frac{1}{8}(2 - \sqrt{2})} \text{ or}$$

3. Given $\boldsymbol{F}(x,y,z)=xyz\,\boldsymbol{i}-\ln(yx)\,\boldsymbol{j}+y\sin(xz)\,\boldsymbol{k}$, calculate the following.

(a) *(10 points)* $\nabla(\nabla\cdot\boldsymbol{F})$

$$\nabla\cdot\vec{F}=\frac{\partial}{\partial x}(xyz)+\frac{\partial}{\partial y}(-\ln(xy))+\frac{\partial}{\partial z}(y\sin(xz))$$

$$=yz+\frac{-x}{xy}+yx\cos(xz)=yz-\frac{1}{y}+xy\cos(xz)$$

$$\nabla(\nabla\cdot\vec{F})=\langle y\cos(xz)+xyz(-\sin(xz)),\ z+\frac{1}{y^2}+x\cos(xz),\ y-x^2y\sin(xz)\rangle$$

$$\nabla(\nabla\cdot\boldsymbol{F})=\boxed{\langle y\cos(xz)-xyz\sin(xz),\ z+\frac{1}{y^2}+x^2\cos(xz),\ y-x^2y\sin(xz)\rangle}$$

(b) *(10 points)* $\nabla\cdot(\nabla\times\boldsymbol{F})$

$$\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ xyz & -\ln(xy) & y\sin(xz)\end{vmatrix}=\hat{i}(\sin(xz)-0)-\hat{j}(yz\cos(xz)-xy)+\hat{k}\left(\frac{-1}{x}-xz\right)$$

$$\nabla\cdot\langle\sin(xz),\ xy-yz\cos(xz),\ \frac{-1}{x}-xz\rangle$$

$$=\cancel{z\cos(xz)}+\cancel{x}-\cancel{z\cos(xz)}-\cancel{x}=0$$

$\nabla\cdot(\nabla\times\boldsymbol{F})$ = 0

4. Let $a=\langle -1,5,0\rangle$ and $b=\langle 2,-3,1\rangle$. Find each of the following.

(a) *(10 points)* $4a-3b$

$\langle -4, 20, 0\rangle + \langle -6, 9, -3\rangle$

$2a-3b$ = $\langle -10, 29, -3\rangle$ or $-10\hat{i}+29\hat{j}-3\hat{k}$

(b) *(10 points)* $a\cdot b$

$\vec{a}\cdot\vec{b} = -1(2)+5(-3)+0(1) = -2-15 = -17$

$a\cdot b$ = -17

(c) *(10 points)* projection of a onto b

$pr_{\vec{b}}\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\vec{b} = \dfrac{-17}{14}\langle 2,-3,1\rangle$

$\|\vec{b}\| = \sqrt{4+9+1}$

$= \sqrt{14}$

projection of a onto b = $\left\langle \dfrac{-17}{7}, \dfrac{51}{14}, \dfrac{-17}{14}\right\rangle$

(d) *(10 points)* the parametric equations of the line going through the point (6,7,8) in the direction of the vector that's orthogonal to both a and b

$x = 6 + 5t$

$y = 7 + t$

$z = 8 + -7t$

$\vec{n} = \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 5 & 0 \\ 2 & -3 & 1 \end{vmatrix}$

$= \hat{i}(5-0) - \hat{j}(-1-0) + \hat{k}(3-10)$

$= \langle 5, 1, -7\rangle$

Line: $\begin{cases} x = 6+5t \\ y = 7+t \\ z = 8-7t \end{cases}$

Part 2 (Applications)

5. For this implicitly defined 3-d surface $f(x,y,z)=e^{x}\cos(\pi y)+\sqrt{yz}=0$, answer the following questions.

(a) *(10 points)* Find the gradient.

$$\nabla f = \left\langle e^{x}\cos(\pi y), -\pi e^{x}\sin(\pi y) + \frac{z}{2\sqrt{yz}}, \frac{y}{2\sqrt{yz}} \right\rangle$$

Answer: $\left\langle e^{x}\cos(\pi y), \pi e^{x}\sin(\pi y) + \frac{z}{2\sqrt{yz}}, \frac{y}{2\sqrt{yz}} \right\rangle$

(b) *(10 points)* Describe geometrically what the gradient tells us.

a normal vector to the surface at any given point

(c) *(10 points)* Find the equation of the tangent plane to the surface at the input point (0, 1).

$(0,1,1)$

$e^{0}\cos(\pi)+\sqrt{z}=0$

$-1+\sqrt{z}=0$

$\sqrt{z}=1 \Rightarrow z=1$

$$\nabla f(0,1,1) = \left\langle 1\cdot(-1), -\pi(1)(0) + \frac{1}{2\sqrt{1}}, \frac{1}{2\sqrt{1}} \right\rangle$$

$$= \left\langle -1, \frac{1}{2}, \frac{1}{2} \right\rangle = \frac{1}{2}\langle -2,1,1\rangle$$

$\vec{n} = \langle -2,1,1\rangle$ pt $(0,1,1)$

$-2x+y+z=D$

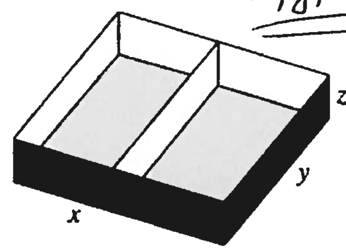
$0+1+1=D$

$D=2$

Tangent plane: $-2x+y+z=2$

6. *(15 points)* We want to make a rectangular open box with one partition in the middle, as illustrated in the picture, from 162 square inches of cardboard. Find the dimensions that would maximize the volume. *(Assume the cardboard is so thin as to have no width to it, for measuring purposes.)*

$\underline{x, y, z > 0}$



$\underbrace{V = xyz}_{\text{fn to maximize}}$

$SA = 162$

$\underbrace{162 = 2xz + 3yz + xy}_{\text{constraint}}$

$\nabla V = \lambda \nabla S$

$\langle yz, xz, xy \rangle = \lambda \langle 2z+y,\ 3z+x,\ 2x+3y \rangle$

④ $162 = 2xz + 3yz + xy$

① $yz = \lambda(2z+y)$

$y(z-\lambda) = 2\lambda z$

$y = \dfrac{2\lambda z}{z-\lambda}$

② $xz = \lambda(3z+x)$

$x(z-\lambda) = 3\lambda z$

$x = \dfrac{3\lambda z}{z-\lambda}$

③ $xy = \lambda(2x+3y)$

assume $z \neq \lambda$

③ $\left(\dfrac{3\lambda z}{z-\lambda}\right)\left(\dfrac{2\lambda z}{z-\lambda}\right) = 2\lambda\left(\dfrac{3\lambda z}{z-\lambda}\right) + 3\lambda\left(\dfrac{2\lambda z}{z-\lambda}\right)$

$6\lambda^2 z^2 = (6\lambda^2 z + 6\lambda^2 z)(z-\lambda)$

$6\lambda^2 z^2 = 12\lambda^2 z^2 - 12\lambda^3 z$

$-6\lambda^2 z^2 = -12\lambda^3 z$

$12\lambda^3 z - 6\lambda^2 z^2 = 0$

$6\lambda^2 z(2\lambda - z) = 0$

Ⓐ $\lambda = 0$

$\Rightarrow x = y = 0$

not interesting $V = 0$

Ⓑ $z = 0$

not interesting

$V = 0$

Ⓒ $z = 2\lambda$

$z = 2\lambda$

$\Rightarrow x = \dfrac{3\lambda(2\lambda)}{2\lambda - \lambda} = 6\lambda$

$y = \dfrac{2\lambda(2\lambda)}{2\lambda-\lambda} = 4\lambda$

④ $162 = 2(12\lambda^2) + 3(8\lambda^2) + 24\lambda^2$

$162 = 72\lambda^2$

$\lambda^2 = \dfrac{162}{72} = \dfrac{18}{8} = \dfrac{9}{4}$

$\lambda = \pm \dfrac{3}{2}$

(if $\lambda < 0$, then $x, y, z < 0$ $\Rightarrow$ N.S.)

$\lambda = 3/2$

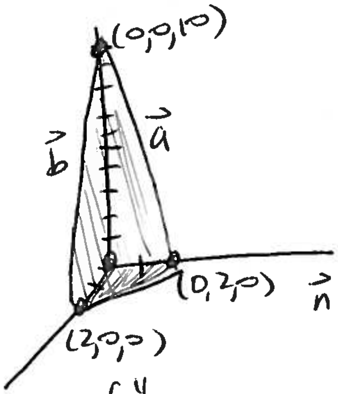
$\Rightarrow z = 3,\ y = 6,\ x = 9$

dimensions of box: ____________________

(remember the inches units)

$x = 9$ in, $y = 6$ in, $z = 3$ in

7. *(15 points)* Compute the volume V of the tetrahedron (triangular pyramid) with vertices (0,0,0), (2,0,0), (0,2,0) and (0,0,10).



we need "roof" plane:

$\vec{a} = \langle 0-0, 2-0, 0-10 \rangle = \langle 0, 2, -10 \rangle$

$\vec{b} = \langle 2-0, 0-0, 0-10 \rangle = \langle 2, 0, -10 \rangle$

$$\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 2 & -10 \\ 2 & 0 & -10 \end{vmatrix} = \hat{\imath}(-20) - \hat{\jmath}(0+20) + \hat{k}(0-4)$$

$= \langle -20, -20, -4 \rangle$

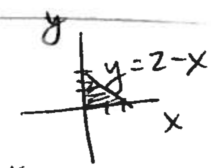
$= -4 \langle 5, 5, 1 \rangle$

"roof" plane: $5x + 5y + z = D$

thru pt $(0,0,10) \Rightarrow D = 10$

$\Rightarrow$ plane: $5x + 5y + z = 10$

$\Leftrightarrow z = 10 - 5x - 5y$



$$V = \int_0^2 \int_0^{2-x} \int_0^{10-5x-5y} dz\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} (10 - 5x - 5y)\, dy\, dx = \int_0^2 \left[(10-5x)y - \frac{5}{2}y^2 \right]_0^{2-x} dx$$

$$= \int_0^2 (10-5x)(2-x) - \frac{5}{2}(2-x)^2\, dx$$

$$= \int_0^2 \left(20 - 20x + 5x^2 - \frac{5}{2}(4 - 4x + x^2) \right) dx$$

$$= \int_0^2 \left(20 - 20x + 5x^2 - 10 + 10x - \frac{5}{2}x^2 \right) dx$$

$$= \int_0^2 \left(10 - 10x + \frac{5}{2}x^2 \right) dx = \left(10x - 5x^2 + \frac{5}{6}x^3 \right) \Big|_0^2$$

$$= \cancel{10(2)} - \cancel{5(4)} + \frac{5}{6}(8) - 0 = \frac{5(4)}{3} = \frac{20}{3}$$

Answer: $\frac{20}{3}$

8. Let the vector field $\boldsymbol{F}$ be defined by

$$\boldsymbol{F}(x,y,z)=(1+2yz^2)\,\boldsymbol{j}+(1+2zy^2)\,\boldsymbol{k}\ .$$

notice: $\hat{\imath}$ component is zero

(a) (*10 points*) Show that $\boldsymbol{F}$ is conservative.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 1+2yz^2 & 1+2zy^2 \end{vmatrix} = \hat{\imath}(4zy-4yz) - \hat{\jmath}(0-0) + \hat{k}(0-0) = \langle 0,0,0\rangle$$

$\Leftrightarrow \vec{F}$ is conservative

(b) (*15 points*) Compute f such that $\boldsymbol{F}=\nabla f$.

we know $f_x=0 \Rightarrow f(x,y,z)=A(y,z)$

$A_y(y,z)=1+2yz^2 \Rightarrow A(y,z)=\int(1+2yz^2)\,dy$

$=(y+y^2z^2)+B(z)$

$\Rightarrow f=y+y^2z^2+B(z)$

$f_z = 2y^2\cancel{z}+B'(z) = 1+2\cancel{z}y^2$

$B'(z)=1$

$\Rightarrow B(z)=\int 1\,dz = z+k$

$\Rightarrow f(x,y,z)=y+y^2z^2+z+k$

$f =$ $y+y^2z^2+z+k$

(c) (*5 points*) Then, compute $\int_C \vec{F}\cdot\vec{dr}$ given C is any path from (0,1,2) to (3,2,5).

$\int_C \vec{F}\cdot d\vec{r} = f(3,2,5)-f(0,1,2) = 2+2^2(5^2)+5-(1+1^2\cdot 2^2+2)$

$=2+100+5-7=100$

$\int_C \vec{F}\cdot\vec{dr} =$ 100

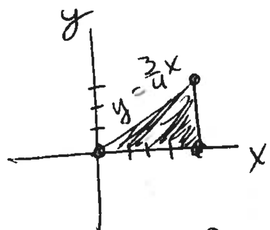
9. *(15 points)* Use Green's Theorem to evaluate $\oint_C \overbrace{(xy-1)}^{M}\,dx+\overbrace{(y^3+\cos y)}^{N}\,dy$ where C is the boundary of the triangle with vertices (0, 0), (4, 0) and (4, 3) oriented counter-clockwise.

$$\oint_C (xy-1)\,dx+(y^3+\cos y)\,dy = \iint_R (N_x - M_y)\,dA$$

(interpreted as circulation along boundary)

$N_x = 0, \quad M_y = x$

$$= \iint_R (-x)\,dA$$



$0 \le y \le \frac{3}{4}x$

$0 \le x \le 4$

$$= \int_0^4 \int_0^{\frac{3}{4}x} (-x)\,dy\,dx$$

$$= \int_0^4 (-x)\left(y \Big|_0^{\frac{3}{4}x}\right)dx$$

$$= \int_0^4 -\frac{3}{4}x^2\,dx$$

$$= \frac{-1}{4}x^3\Big|_0^4 = \frac{-1}{4}(4^3 - 0) = -16$$

OR

$$\oint_C \underbrace{(xy-1)}_{-N}\,dx + \underbrace{(y^3+\cos y)}_{M}\,dy = \iint_R (M_x + N_y)\,dA$$

$M_x = 0, \quad N = 1 - xy$

$N_y = -x$

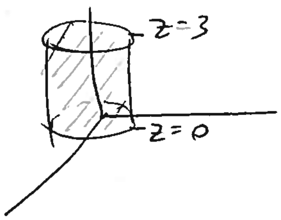
$$= \iint_R (-x)\,dA$$

(& now we're in same place as above)

(interpreted as flux across curve)

Answer: -16

10. *(15 points)* Let S be the solid cylinder bounded by $x^2+y^2=4$, $z=0$ and $z=3$, and let $\hat{\boldsymbol{n}}$ be the outer unit normal vector to the boundary ∂S . If $\boldsymbol{F}(x,y,z)=(x^3+\tan(yz))\boldsymbol{i}+(y^3-e^{xz})\boldsymbol{j}+(3z+x^3)\boldsymbol{k}$, find the flux of $\boldsymbol{F}$ across the surface ∂S .



use Gauss' Divergence Thm to find flux
(especially since $\hat{n}$ would be different for the ① top, ② lateral side, & ③ bottom of cylinder)

$$\iint_{\partial S} \vec{F}\cdot\hat{n}\, dS = \iiint_S \operatorname{div}\vec{F}\, dV$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x^3+\tan(yz)) + \frac{\partial}{\partial y}(y^3-e^{xz}) + \frac{\partial}{\partial z}(3z+x^3)$$

$$= 3x^2+3y^2+3$$

use cylindrical coords

$0 \le z \le 3$
$0 \le r \le 2$
$0 \le \theta \le 2\pi$

$$\Rightarrow \iiint_S \operatorname{div}\vec{F}\, dV = \iiint_S (3x^2+3y^2+3)\, dV$$

$$= \int_0^{2\pi}\int_0^2\int_0^3 (3r^2+3)\, r\, dz\, dr\, d\theta$$

$$= 2\pi(3)(3)\int_0^2 (r^3+r)\, dr$$

$$= 18\pi\left(\frac{r^4}{4}+\frac{r^2}{2}\right)\Big|_0^2 = 18\pi(4+2-0) = 18(6)\pi$$

flux: 108π