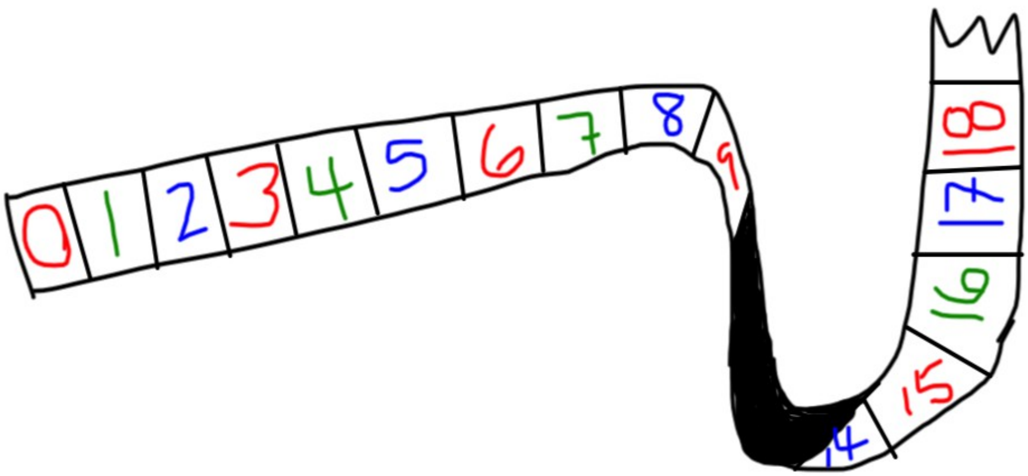
Math6100
Day 1 Notes
1.1 & 1.2, Sequences and Series

Sequences
What is a sequence?

Sequence Notation:

Ex 1: Here is a strip made with a repeating pattern of numbers, in different colors.



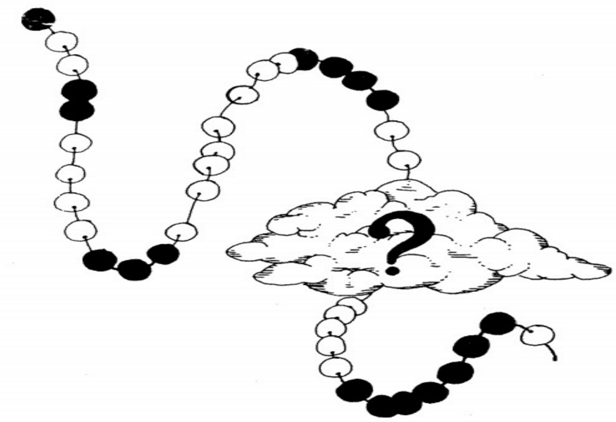
How can we tell the color of the numbers:

(a) 31? (b) 253,679?

(c) any natural number, n?

Ex 2:

How many beads are hidden under the cloud?



Ex 3: The integers greater than 1 are arranged as shown.

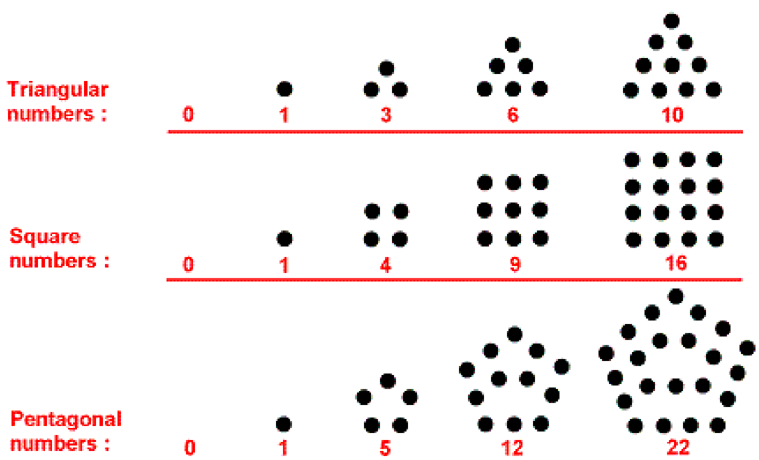
	2	3	4	5
9	8	7	6	
	10	11	12	13
17	16	15	14	

…

(a) In which column will 100 fall?
(b) In which column will 1000 fall?
(c) How about 1999?
(d) How about 99, 997?

Ex 4: It is Thanksgiving and the king decides to let out a few prisoners. He sends out his top man and tells him to unlock every cell beginning with the first cell. Deciding this is too much, he immediately sends out his number two man and says, "lock every second cell, beginning with cell #2." He thinks on it again and sends his number three man and says "Change the position of the lock on every third cell. If it is locked, unlock it and if it is unlocked, lock it." He sends his number four man and tells him to change the position of the lock on every fourth cell...then the fifth, sixth and so on. He continues in this indecisive manner all night long. If his men act on these instructions in the order he gave them, who will eventually get out of jail?

Ex 5: Here are the first four triangular, square and pentagonal numbers. Find the next three. Can you find a formula for the nth triangular, square and pentagonal numbers?



What is the difference between a recursive/implicit and explicit/direct/iterative formula for a sequence?

And, how do we get from a recursive to iterative formula (if that's possible)?

Definitions:

- arithmetic sequence

- geometric sequence

- convergent sequence

- divergent sequence

Weierstrass's Theorem: A nondecreasing sequence is convergent iff it's bounded. Similarly, a nonincreasing sequence is convergent iff it's bounded.

*Note: The limit operator has "super powers" IF the limit exists and is finite. What do I mean by that?

Ex 6: Are these sequences convergent or divergent?

(a) $x_n = \frac{1}{n^{1/2}}$

(b) $y_n = \frac{3n}{n+100}$

(c) $d_n = 1 + (-1)^n$

(d) $b_n = \frac{1+(-1)^n}{n}$

Ex 7: Find the limits, if they exist.

(a) $\lim_{n\to\infty} \frac{(-5)^{n-2}}{3^n}$

(b) $\lim_{n\to\infty} \frac{(-5)^{n-2}}{6^n}$

(c) $\lim_{n\to\infty} (5-2\text{n})$

(d) $\lim_{n\to\infty} \frac{\sin(n)}{3^n}$

Sum of an arithmetic sequence:

Given an arithmetic sequence $a_n = a + (n-1)d$ where $a = a_1$ and d is the common difference, determine, and prove, the formula for the sum of the first p terms.

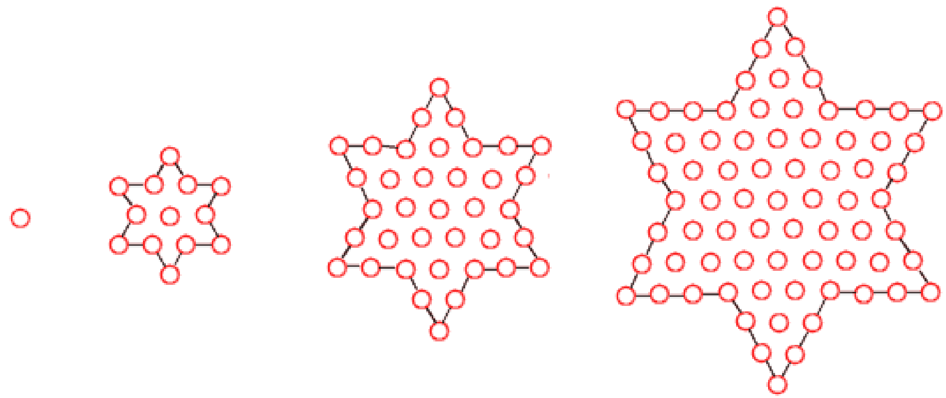
Sum of a geometric sequence:

Given a geometric sequence $a_n = a(r^{n-1})$ where $a = a_1$ and r is the common ratio, determine, and prove, the formula for the sum of the first p terms.

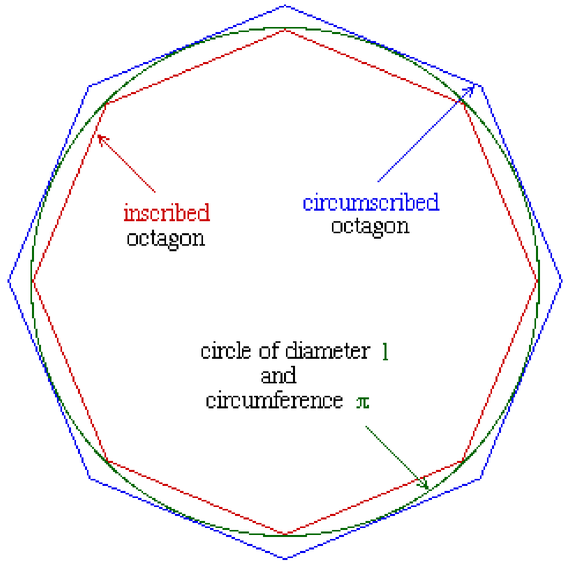
Ex 8: (a) You attend a party where there is a total of 25 people, including you. If each person shakes every other person's hand only once, how many handshakes will there be in total?

(b) Assume that an hour after the party starts, three extra people arrive. They each shake the hand of every other person there. How many handshakes do those three extra people engage in?

Ex 9: For these star numbers, find the iterative formula to indicate the sequence they represent.



Ex 10: Archimedes computation of π .



What is the radius of the circle?

Let P_n be the perimeter of the regular n-gon circumscribed around the circle. Let p_n be the perimeter of the regular n-gon inscribed in the circle.

Use trigonometry to find the explicit/iterative/direct formulas for p_n and P_n . Then explore what happens for the limit of the sequences $\{p_n\}$ and $\{P_n\}$.

1.2 Series

What is a series? How is it related to a sequence?

Ex 1: Three friends decide to divide a rectangular cake up equally, using the following strategy. They cut the cake into four equal pieces and each person gets one of those pieces. They cut the remaining piece into four equal pieces and give one of those smaller pieces to each person. With the leftover piece, they cut that into four equal pieces, giving each of those still smaller pieces to one person. They continue in this fashion forever. How much cake does each person get, in the end? How can you relate this to a series?

Ex 2: What is $0.\overline{9}$? Prove it in some algebraic way.

Ex 3: Two students are arguing over their logic, when they are given a series and they come up with two different answers. Who is right and why?

Student 1: $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots. = 0$

` because $(1 - 1) + (1 - 1) + (1 - 1) + \ldots = 0 + 0 + 0 + 0 + \ldots = 0$

Student 2: $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots. = 1$

because $1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \ldots = 1 + 0 + 0 + 0 + 0 + \ldots = 1$

Who's right and why?

<u>Infinite Geometric Series</u>

Remember that we found the sum of the first p terms of a geometric series. What happens as p goes to infinity?

Ex 4: Revisit $0.\overline{9}$. Prove your answer using infinite series.

Ex 5: What is the fraction form of these numbers. For each problem, give answer using

(1) algebra technique AND

(2) infinite series.

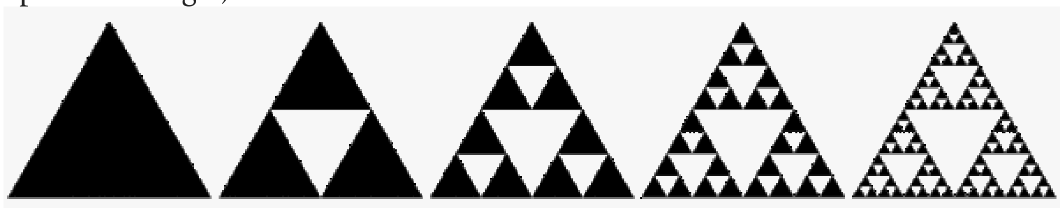
(a) $0.1\overline{5}$

(b) $0.11\overline{5}$

(c) $0.111\overline{5}$

(d) $0.111...1\overline{5}$ (where 1 is repeated n times)

Ex 6: (Sierpinski Triangle)



n = 1 n = 2 n = 3 n = 4 n = 5

(a) Make a table to count the number of solid black triangles in each figure above. (Note: These black triangles are getting smaller and smaller.)
(b) In your table, add a column to keep track of the area that's shaded (black) in each figure. Assume the original triangle has area 1.
(c) If the pattern is continued, how many solid black (very small) triangles will there be at the nth figure? What will the shaded area be?
(d) If you're feeling daring, add another column to your table and figure out the total perimeter for all the black triangles in each figure. Assume the first triangle has sides lengths of 1 (so it's perimeter is 3 for n=1 case).
(e) Do you notice something interesting between the area and perimeter?