Math6100
Day 2 Notes
2.1 & 2.2, Functions and Limits of Functions

2.1 Functions

Functions: $f:\mathbb{R}\to\mathbb{R}$

Vocabulary

- domain

- range

- codomain

Ex 1: Identify/categorize these function types and also state the domain.

(a) $f(x)=\dfrac{x^2-1}{(x-1)(x+3)x^2}$

(b) $f(x)=\dfrac{2x}{\sqrt{4-x}}$

(c) $f(x)=\sqrt[4]{2x+1}$

(d) $f(x)=3x^3-9x+1$

Composite Functions: $(f \circ g)(x) = f(g(x))$

Ex 2: State the domain for both $(f \circ g)(x)$ and $(g \circ f)(x)$

$f(x) = \dfrac{x}{x^2 - 1}$ and $g(x) = \sqrt{x}$

Ex 3: For $f : \mathbb{R} \to \mathbb{R}$, if $f(x) = x^3 - 1$, then find $f(x+2)$. Why do students have such troubles with this?

Even/Odd Functions:

Even function: $f(x)=f(-x)$

Odd function: $-f(x)=f(-x)$

Ex 4: Can the point (-1, 2) be in the domain of an even/odd function? Why or why not?

Ex 5: (a) Is the sum of two odd functions even, odd, neither, or it depends on some other criteria? Prove it. What about the sum of two even functions?

(b) What about the difference, quotient and product of two even/odd functions?

Ex 6: Prove that if f(x) is both even and odd, it must be identically zero.

Ex 7: If f(x) and g(x) are both even, then what is $(f \circ g)(x)$ (i.e. is it even, odd, neither or it depends on some other criteria)? What if they're both odd functions?

<u>Inverse Functions</u>:
Can we “undo” a function? What does this mean anyway?

Ex 8: If $\text{domain}(f)=A$ and $B\neq\text{Image}(f)$, can we undo f? (In other words, does $f^{-1}(x)$ exist? (If $B=\text{Image}(f)$, then f is onto.)

Ex 9: $f^{-1}(x)$ exists if $f(x)$ is 1-1. What does this mean?

Ex 10: What is the relationship between the domain/range of $f(x)$ and $f^{-1}(x)$?

Ex 11: Does $f^{-1}(x)$ exist for these functions? If so, find it.

(a) $f(x)=x^2-x\ ,\ f:\mathbb{R}\to[-\frac{1}{4},\infty)$

(b) $f(x)=x^2-x\ ,\ f:(-\infty,\frac{1}{2}]\to[-\frac{1}{4},\infty)$

(c) $f(x)=x^2-x\ ,\ f:[\frac{1}{2},\infty)\to[-\frac{1}{4},\infty)$

2.2 Limits of Functions

Note: Remember that limits can commute with every mathematical operator as long as the limit exists.

Definition: One-sided limits

$f(x)$ is real-valued function whose domain is subset of the real numbers. If $f(x)$ can be made arbitrarily close to a real number, L, by taking x < a (x > a) sufficiently close to a, then we say L is the limit of $f(x)$ as x approaches a from the left (right), i.e.

$$\lim_{x\to a^-} f(x)=L \qquad \left(\lim_{x\to a^+} f(x)=L\right).$$

Definition: Limit

We say the limit of $f(x)$ as x approaches a exists , i.e. $\lim_{x\to a} f(x)=L$ iff

$$\lim_{x\to a^-} f(x)=L=\lim_{x\to a^+} f(x).$$

Ex 1: For $f(x)=\dfrac{x^2(x-1)(x+3)}{x^2(x-1)(x+1)}$

(a) What is the domain of f?

(b) Is $f(x)=\dfrac{x(x+3)}{(x+1)}$ the simplified form of f?

(c) Find $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 1^-} f(x)$, and $\lim_{x\to 1^+} f(x)$.

(d) Categorize all discontinuities (as a hole, jump or VA).

Ex 2: For $f(x)=\dfrac{(x+2)^2(x-3)}{(x-4)(x+2)}$

(a) What is the domain of f?

(b) Is $f(x)=\dfrac{(x+2)(x-3)}{(x-4)}$ the simplified form of f?

(c) Find $\lim\limits_{x\to 4^-} f(x)$, $\lim\limits_{x\to 4^+} f(x)$, $\lim\limits_{x\to -2^-} f(x)$, and $\lim\limits_{x\to -2^+} f(x)$.

(d) Categorize all discontinuities (as a hole, jump or VA).

Ex 3: True or false?

$$\lim_{x\to 3}\frac{x-3}{x^2-9}=\lim_{x\to 3}\frac{1}{x+3}=\frac{1}{6}$$ A student argues that we can't do this because we divided out zero over zero which is illegal, so the limit DNE.

Ex 4: For the graph of the function, given in this image, explore some interesting limits.