Math6100
Day 3 Notes
2.3, 3.1 & 3.2, Continuity, Average & Instantaneous Rate of Change

2.3 Continuity

$f(x)$ is continuous at $x=c$ iff $\lim_{x\to c} f(x)=f(c)$.

Ex 1: Are these functions continuous or not? If there are discontinuities, categorize them.

(a) $f(x)=\begin{cases} 2x+3 & \text{if } x<0 \\ x^2+3 & \text{if } 0\le x<2 \\ 4 & \text{if } x\ge 2 \end{cases}$

(b) $f(x)=\sqrt{x}$

(c) $f(x)=|x+5|-1$

(d) $f(x)=\dfrac{(x-1)^2(x+2)}{(x-1)^2}$

(e) $f(x)=\dfrac{(x-1)^2(x+2)}{(x-1)^3}$

3.1 & 3.2 Average and Instantaneous Rates of Change

What is the difference between average rate of change and instantaneous rate of change?

Definition: Average Rate of Change
The average rate of change of $f(x)$ with respect to x over the interval [a,b] is

$$\frac{f(b)-f(a)}{b-a} \quad .$$

Ex 1: (from page 116)
Diana walks from home to school in 25 minutes. The distance d in meters she has traveled after a t-minute walk is given by d(t) = 60t.
(a) Compute Diana's average walking rate over the following intervals of time:
[1,4] , [6,7] , [0,25] . What do you observe? Interpret your results.

(b) Make a conjecture about how the average rates of change of d computed over two arbitrary intervals compare. Prove your conjecture.

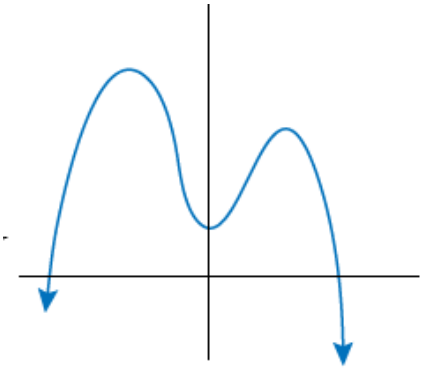
Ex 2: (from #3 on page 116)
(a) Let m and c be arbitrary real numbers and consider the linear function $g(x) = mx + c$. Compute the average rate of change of g over the interval [a,b]. What do you observe?

(b) Consider a function g for which the average rate of change over any interval [a,b] is constant and equal to a real number m. Prove that g is linear.

(c) Can you use parts (a) and (b), above, to argue that:

A function is linear iff the function's average rate of change over any interval is constant.

How can we determine instantaneous rate of change? Use this graph to argue your idea geometrically.



Definition: Instantaneous Rate of Change (Derivative)
The instantaneous rate of change of $f(x)$ with respect to x at some x-value is

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

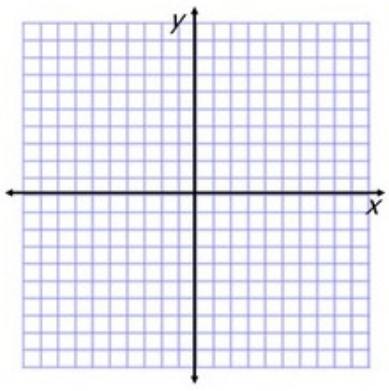
(or it can also be defined as $f'(x)=\lim_{c\to x}\frac{f(c)-f(x)}{c-x}$).

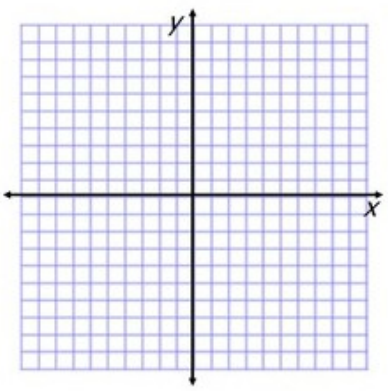
Vocabulary/Definitions:

- f(x) is increasing on (a,b)

- f(x) is decreasing on (a,b)

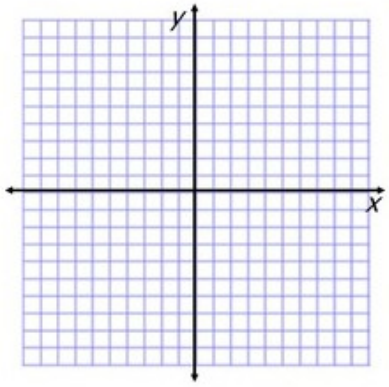
Draw some examples:

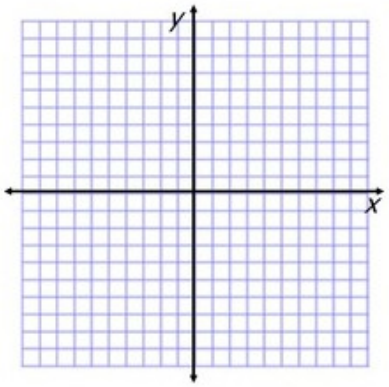




Ex 3: On two separate coordinate axes, sketch these function curves.

$f(x)=x^2$ and $f'(x)=2x$





(a) Where is each function increasing/decreasing? And, what is the sign of f'(x) in each of those intervals?

(b) x_0 = ? such that $f'(x_0)=0$. What does this mean about the graph of $f(x)=x^2$?

(c) What is the geometric correlation of increasing/decreasing sections of $f(x)=x^2$ with the graph of $f'(x)=2x$?

Ex 4: Explain why this statement is true.

For a differentiable function f(x), the tangent line to the graph of y = f(x) at the point (c, f(c)) is horizontal iff f '(c) = 0.

Then tell what part(s) we can remove from that statement that would make it false.

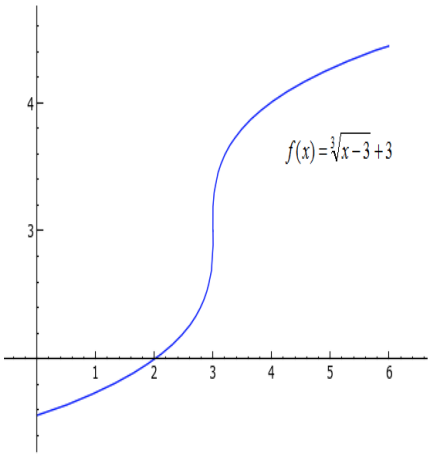
Ex 5: For each of these graphs, answer these questions.

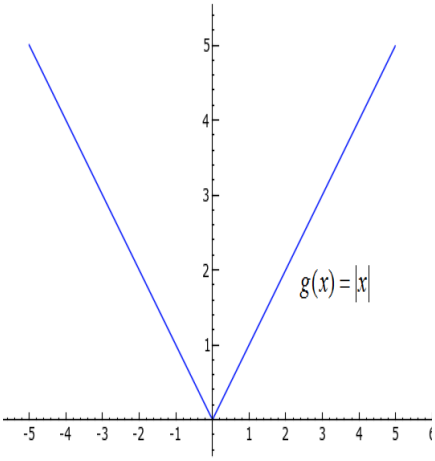
(a) Where is the derivative positive?

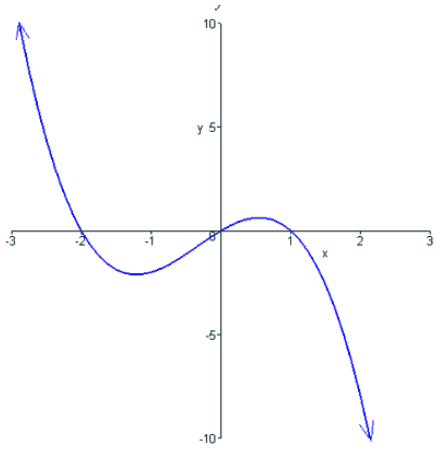
(b) Where is the derivative negative?

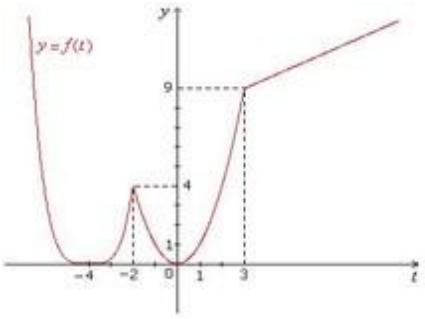
(c) Where is the derivative zero?

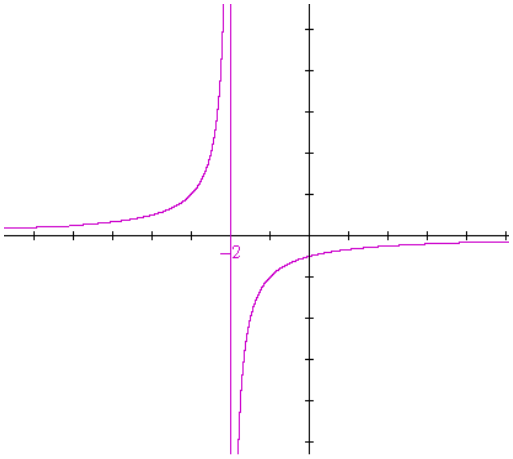
(d) Are there any points where the derivative is undefined? And, if so, what type of "problem" is it?






10
y 5
-3
-2
-1
0
1
2
3
x
-5
-10


y
y = f(t)
9
4
1
−4
−2
0
1
3
t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True or False? Why?

Differentiability ==> Continuity

Continuity ==> Differentiability

Continuity <==> Differentiability

Ex 6: Use the definition of the derivative to find the derivative for these functions.

(a) $f(x)=|x|$

(b) $f(x)=2x^2-1$

(c) $f(x)=\sqrt{x+5}$

(d) $f(x)=\sin x$