

Math6100
Day 4 Notes

3.3, 3.4, 3.5, Motion & Derivatives and Derivative “Shortcuts”
(Power, Product, Quotient, Chain Rules)

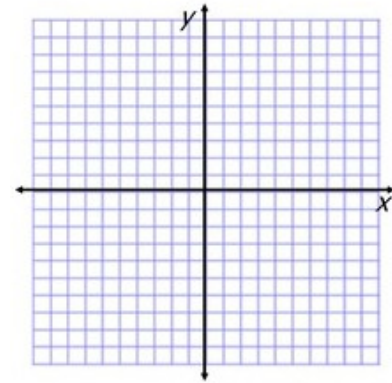
3.3 Motion & Derivatives

(From Classroom Discussion on pages 132-133)

Ex 1: Adrian received a model rocket for his birthday. The launcher is powered by an air pump, and the height attained by the rocket depends on the number of times Adrian pumps air. Adrian knows that five pumps will project the rocket upward according to the formula

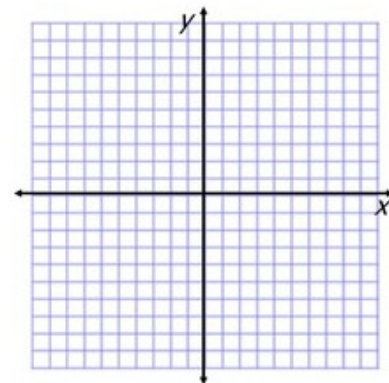
$h(t) = -16t^2 + 128t$, where $h(t)$ is the rocket's height, measured in feet from the ground, t second after take-off.

(a) Sketch the graph of $h(t) = y$ for $t \geq 0$. Is this the trajectory of the rocket? Explain.

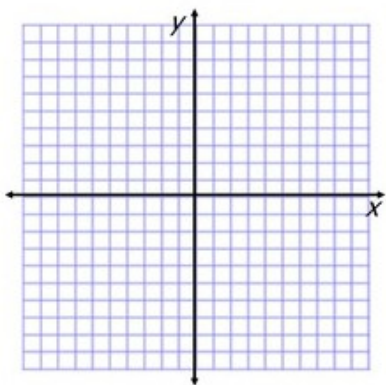


(b) Compute the velocity of the rocket, $v(t)$.

(c) Sketch the graph of $v(t)$ and explain why velocity can sometimes be negative.



(d) Sketch the graph of the function, $s(t) = |v(t)|$. What does this function represent physically?



(e) After how many seconds will the rocket start falling?

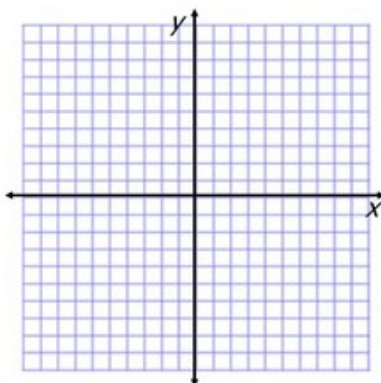
(f) What is the maximum height?

(g) After how many seconds will the rocket hit the ground?

(h) What is the rocket's acceleration?

Ex 2: A rock is thrown up from a cliff. Measured from the cliff's base, the height (in feet) of the rock after t seconds is given by the expression $h(t) = -16t^2 + 64t + 80$.

(a) Sketch the graph of $y = h(t)$.



(b) Compute the rock's velocity $v(t)$.

(c) What does the 64 represent in the velocity function?

(d) What is the rock's velocity when it hits the ground?

(e) What does the 80 represent in the height function?

(f) During which intervals of time is the speed decreasing/increasing?

(g) Determine the acceleration of the rock.

3.4 & 3.5 (Derivative Shortcuts)

1. Prove that the derivative is a linear operator (using the definition of the derivative.)

2. Derive the power rule, using the definition of the derivative, for n-values that are whole numbers.

n	$f(x) = x^n$	$f'(x) = D_x(x^n)$

3. (Prove the Product Rule)

(a) Check if this is true or false. $D_x(x^2 \cdot x^3) = D_x(x^2) \cdot D_x(x^3)$

(b) Prove this identity:

$$\frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \frac{f(x+h) - f(x)}{h}g(x+h) + f(x)\frac{g(x+h) - g(x)}{h}$$

(c) Then prove the product rule. $D_x(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

4. Use the definition of the derivative to find $f'(x)$ for $f(x) = \frac{1}{x}$, $\forall x \neq 0$.

5. What is the derivative of $g(x) = \frac{1}{f(x)}$, $\forall x \in \text{domain}(f)$ s.t. $f(x) \neq 0$?

Do this two ways:

(a) Prove and use the identity $\frac{g(x+h) - g(x)}{h} = \frac{-1}{f(x+h)f(x)} \cdot \frac{f(x+h) - f(x)}{h}$.

(b) Use product rule on $g(x) \cdot f(x) = 1$.

6. What does this mean about $D_x(x^n)$, $\forall n \in \mathbb{Z}$?

7. (Prove the Quotient Rule)

Use the above results to find $D_x\left(\frac{f(x)}{g(x)}\right)$ (i.e. use this hint $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$).

8. Find $D_x(\sqrt{f(x)})$, assuming $f(x)$ is differentiable and is always nonnegative, by proving and using the identity

$$\frac{g(x+h) - g(x)}{h} = \frac{f(x+h) - f(x)}{h} \cdot \frac{1}{\sqrt{f(x+h)} + \sqrt{f(x)}} \quad \text{where } g(x) = \sqrt{f(x)}$$

9. (Prove the Chain Rule)

Given the composite function $(f \circ g)(x)$. Let $x \in \text{domain}(f \circ g) \cap \text{domain}(f' \circ g)$.

Prove and use the identity

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} \text{ to find } D_x((f \circ g)(x)).$$

10. Use the chain rule to derive the power rule for rational exponents.

Let $f(x) = x^{a/b}$, $a, b \in \mathbb{Z}$, $b \neq 0$.

$\Rightarrow (f(x))^b = x^a$ (Now we can differentiate both sides of the equation.)

Ex 1: Use the derivative rules we've proved to compute $f'(x)$ for each of these functions.

(a) $f(x) = \sqrt[3]{x^2 + 20x - \frac{1}{x^2} + \pi^3}$

(b) $f(x) = \frac{x^{3/2} + e^{10} - 2x}{x^7 - 9}$

(c) $f(x) = \left(x^2 - \frac{1}{x}\right)^{20} (\pi x + 10)$

$$(d) \quad f(x) = \left(\frac{2-7x}{x^3-x} \right)^5$$

$$(e) \quad f(x) = \frac{8}{(x^3+5)^6}$$

$$(f) \quad f(x) = \sin(\cos(\sqrt{1+(4-x^2)^3}) + \pi)$$