Math6100
Day 5 Notes
4.1 & 4.2, Optimization & Curve Sketching

4.1 Optimization

Ex 1: (Classroom Discussion from pages 154-156)
In the 19th century, many adventurers traveled to North America to search for gold. A man named Dan Jackson owned some land where gold had been found. Instead of digging for the gold himself, he rented plots of land to the adventurers. The "rent" was to give Dan 50% of any gold found on the plot of land.

Dan gave each adventurer four stakes and a rope that was exactly 100 meters long. Each adventurer had to use the stakes and rope to mark off a rectangle with north-south and east-west sides.
1. Did everyone get the same area to dig for gold? Explain.

2. (a) On the graph paper provided, draw at least five different rectangles each with a perimeter of 100 meters. Draw your rectangles to scale, using one grid unit to represent five meters.
    (b) Cut out your rectangles and tape them on the graph (that's on the back side of this page) so the lower-left corners lie on the origin. For each rectangle, only the upper-right vertex will not touch an axis.

3. All the upper-right vertices seem to lie on a line. What is the equation of that line?

4. Calculate the area of each rectangle you cut out and organize your information in the table.
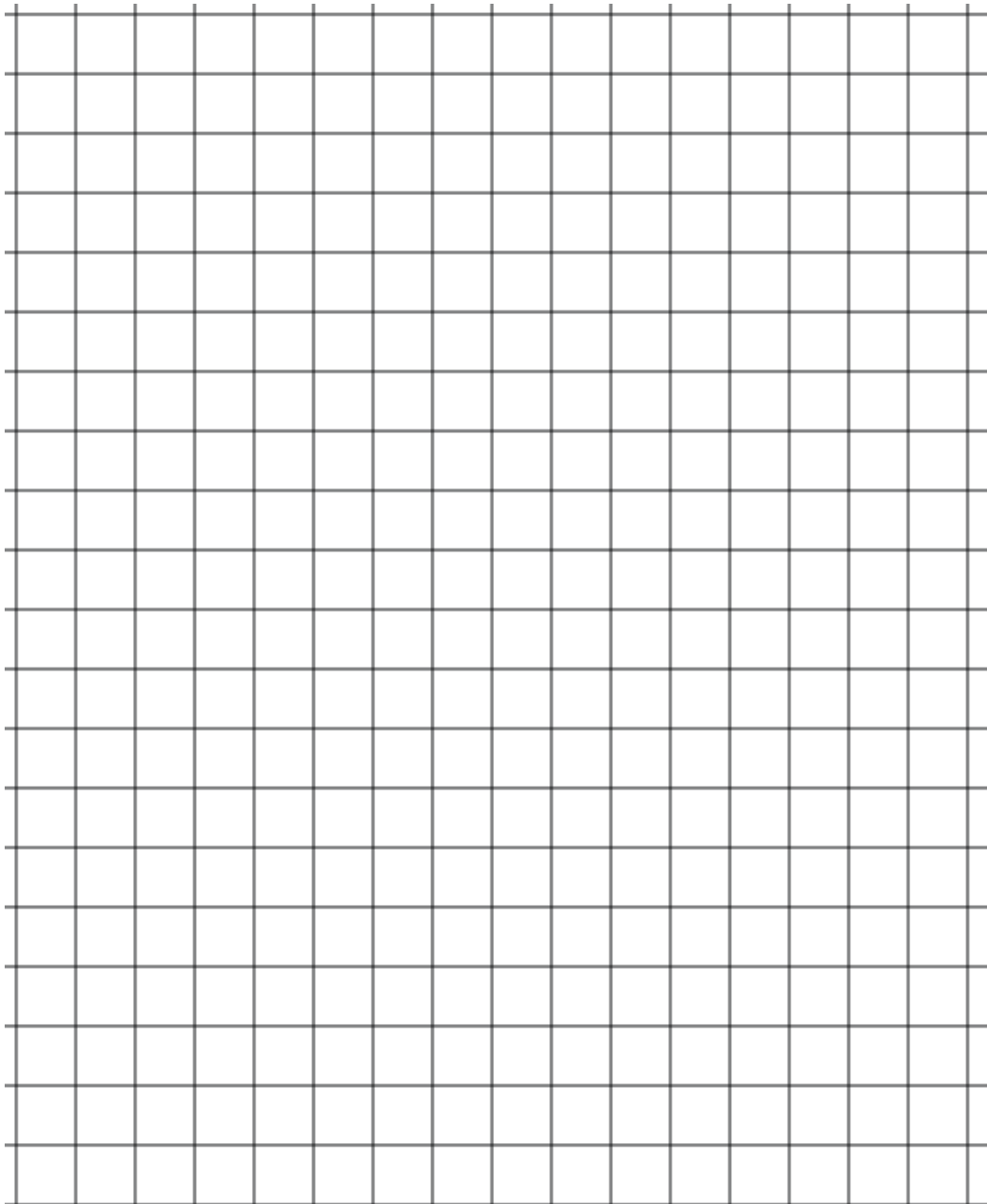
n	area
1	
2	
3	
4	
5	
6	

5. One of the diggers discovered that one kind of rectangle always had the greatest area. He decided to sell the secret to other diggers. What's the secret?
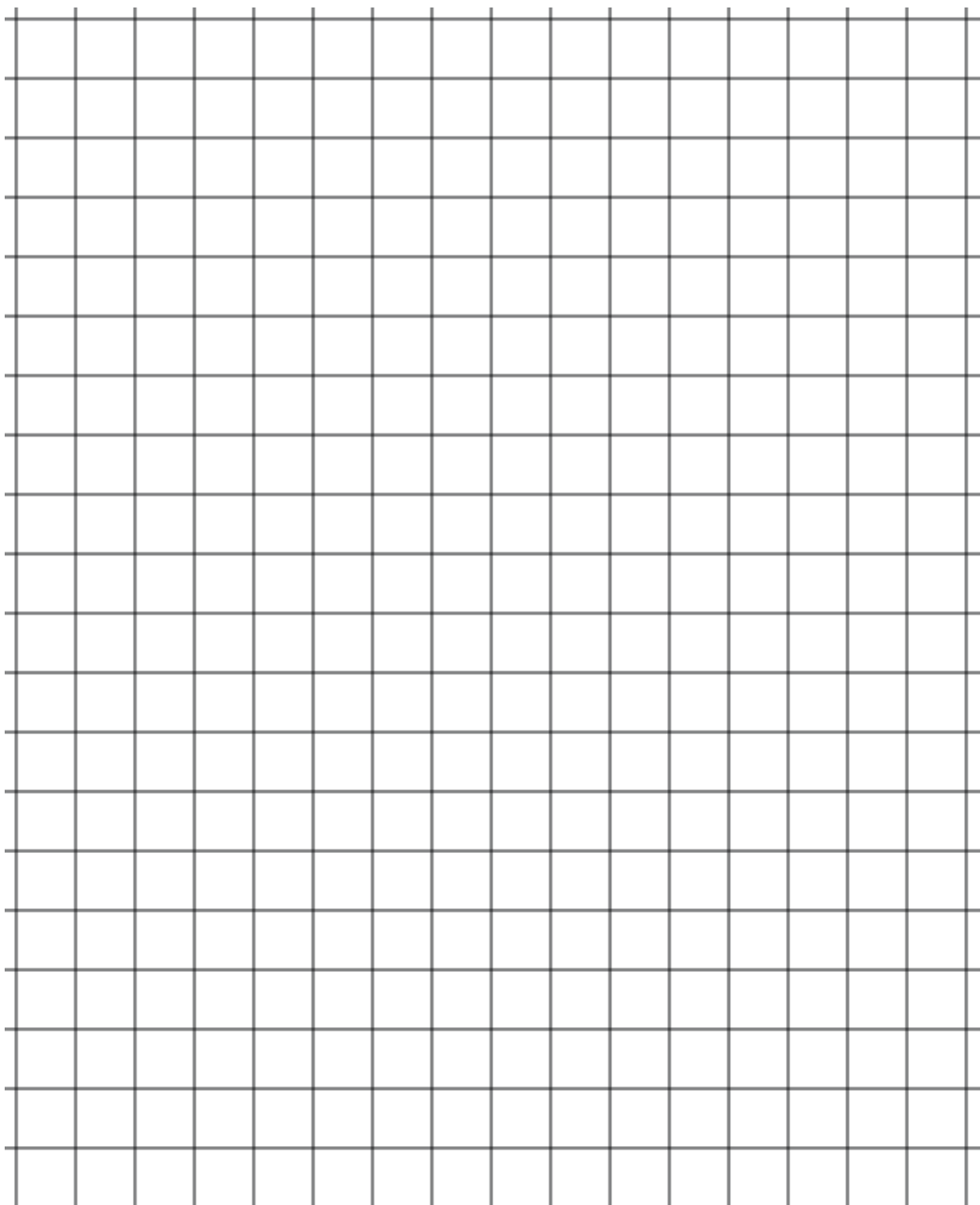
1

Use this for 2(b).

Use this for cutting.

3
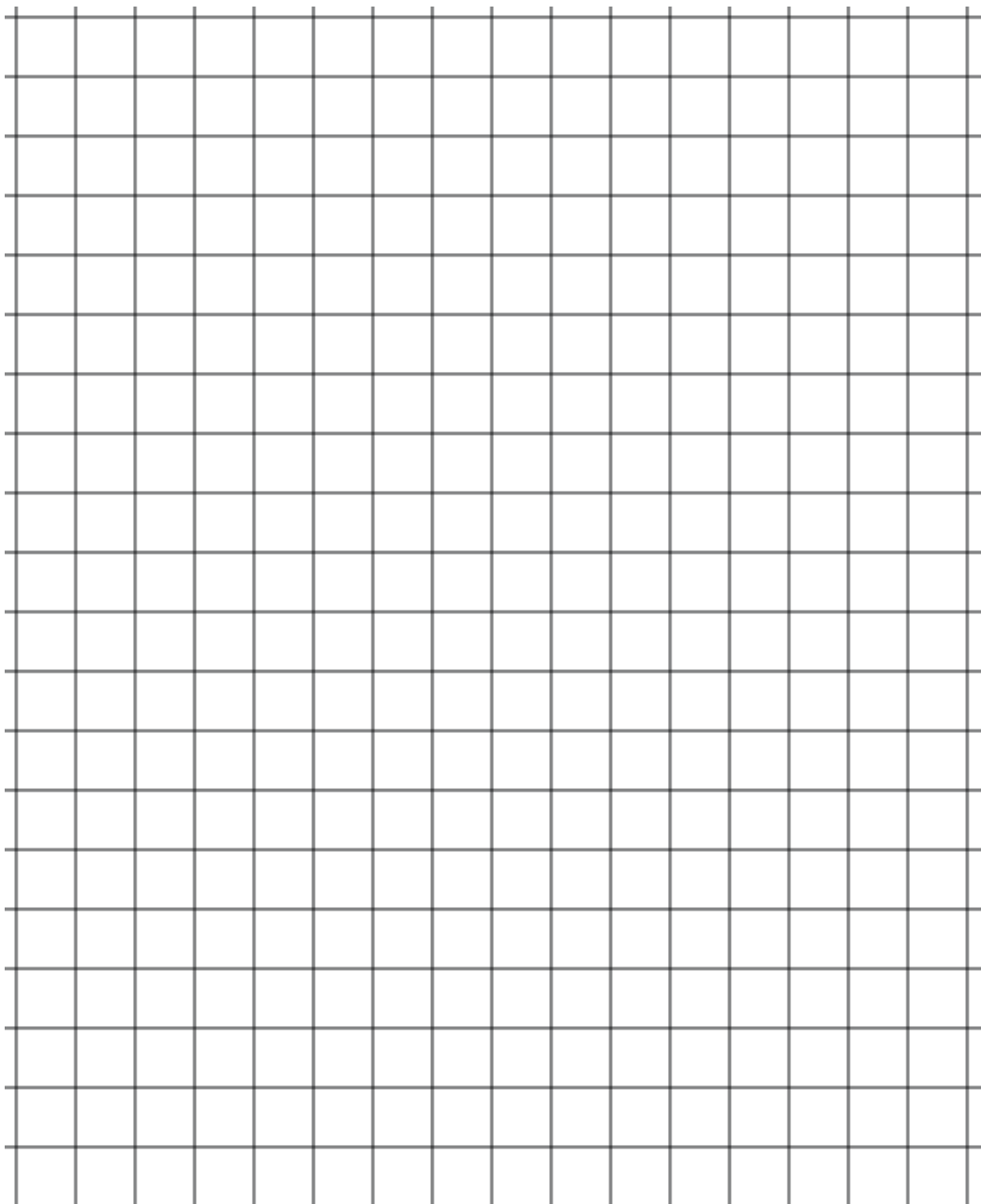
Use this for cutting.

Questions:
1. Does the reasoning you used in finding the "secret" constitute a proof of your result? If not, how can you show that no other rectangle with a perimeter of 100 meters will have an area larger than the rectangle you discovered?

Let's do the proof in three different ways (one geometric, one algebraic, one using calculus).
(a) (geometric argument) Use this figure.
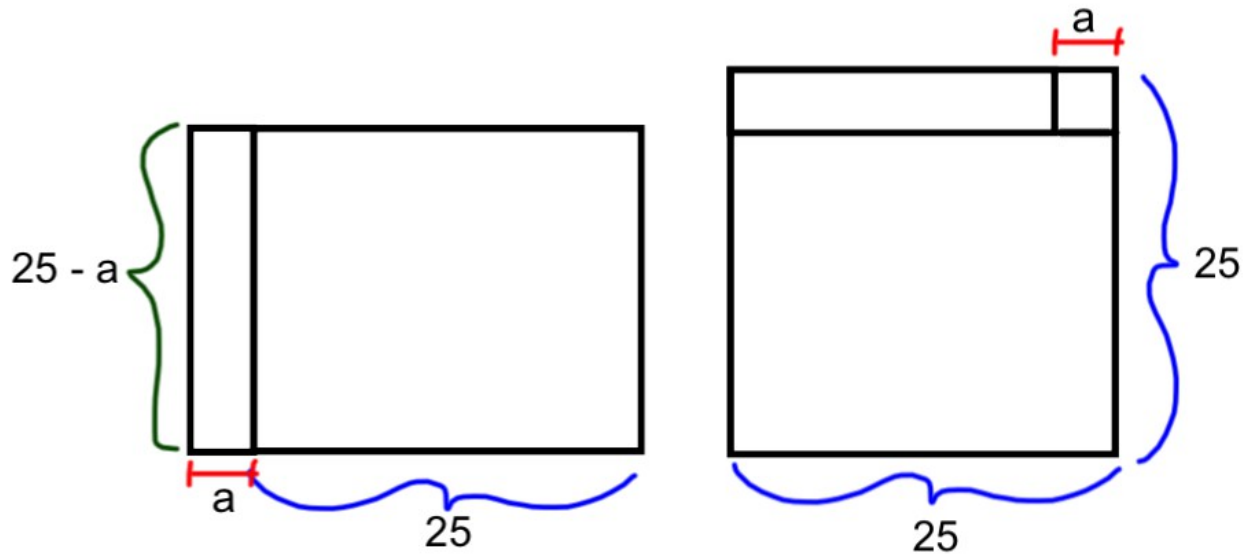
(b) (algebraic argument) Prove and use the fact that $(25-a)(25+a) \leq 625$.

(c) Finally, use calculus to prove that we have maximum area.

Ex 2: Find the critical points for $f(x)=2x^2-3x+1$ on the interval [-1,1]. We can, in fact, find the absolute/global max and min points. Why are we guaranteed that we'll find a min and a max point here?

Ex 3: Find the absolute min/max points for $f(x)=x^{2/5}+2$ on the interval [-1,32].

Ex 4: Let's go back to the scenario set in Example 1. This time, suppose that the secret got out and Dan decided to change the rental agreement. He still gave the adventurers four stakes, and they still marked off rectangles with north-south and east-west sides, but with a new constraint:
    The rectangle had to have an area of 400 square meters.
Since the adventurers had to provide their own rope for marking their plot, they had to know how long their rope should be.

    1. Did everyone use the same length of rope? Explain.

    2. (a) On the graph paper provided, draw at least five different rectangles each with an area of 400 square meters. Draw your rectangles to scale, using one grid unit to represent **two** meters.
    (b) Cut out your rectangles and tape them on the graph (that's on the back side of this page) so the lower-left corners lie on the origin. For each rectangle, only the upper-right vertex will not touch an axis.

    3. All the upper-right vertices seem to lie on a curve (that's not a line). What type of curve is it? And, what is the equation of that curve?

    4. Calculate the perimeter of each rectangle you cut out and organize your information in the table.
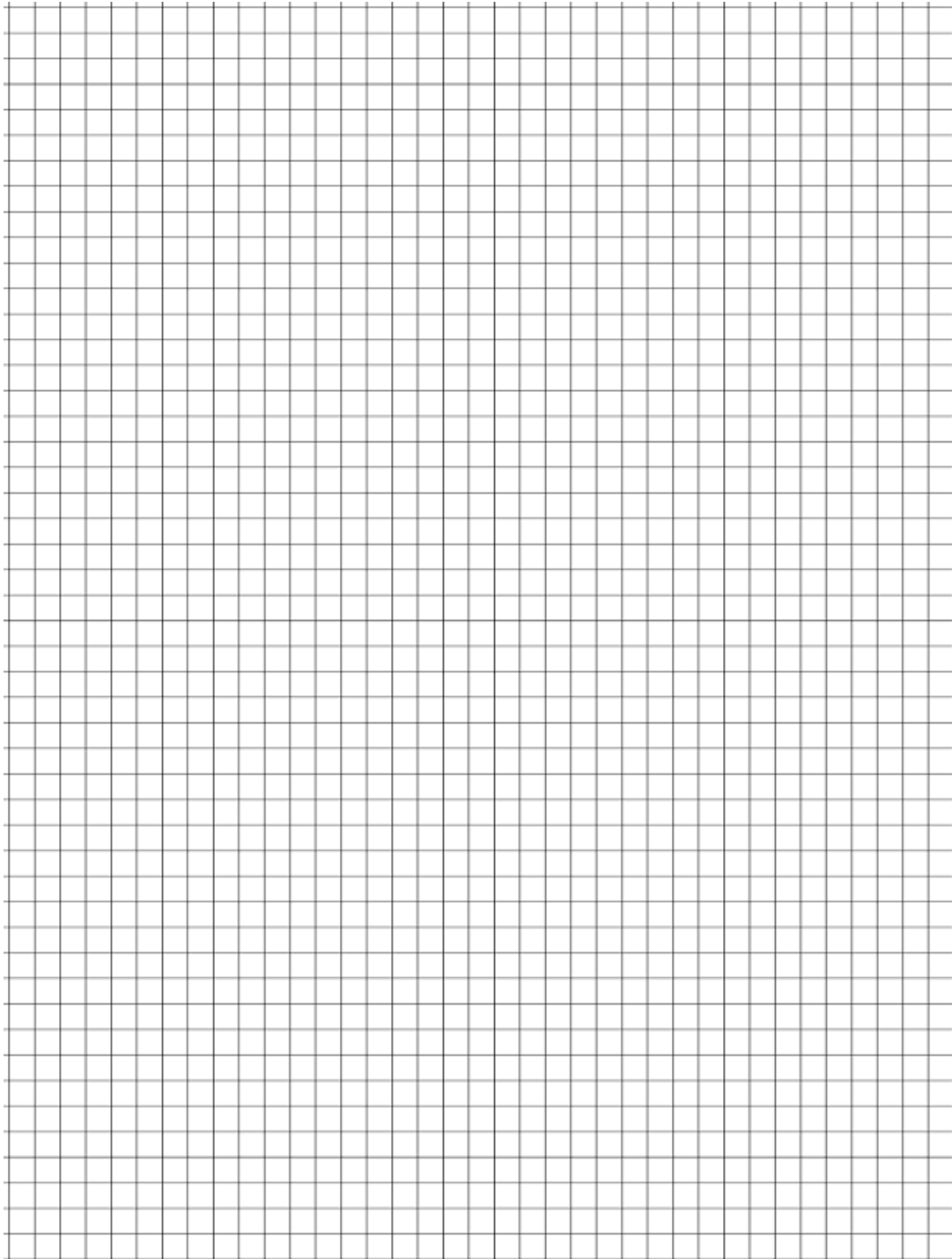
n	perimeter
1	
2	
3	
4	
5	
6	

    5. One of the diggers discovered that one kind of rectangle always used the minimal amount of rope. What is the length of rope?
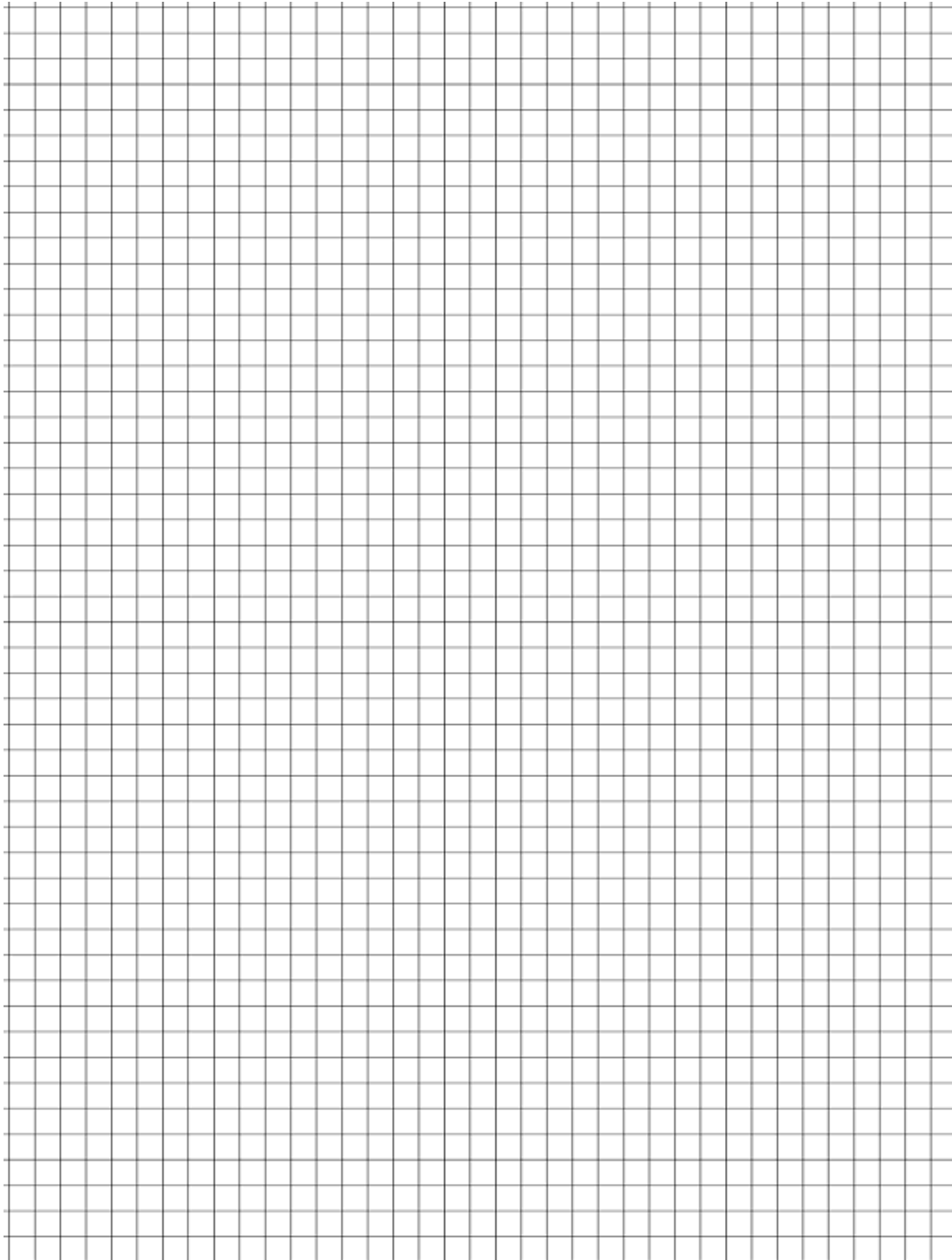
Use this for cutting.
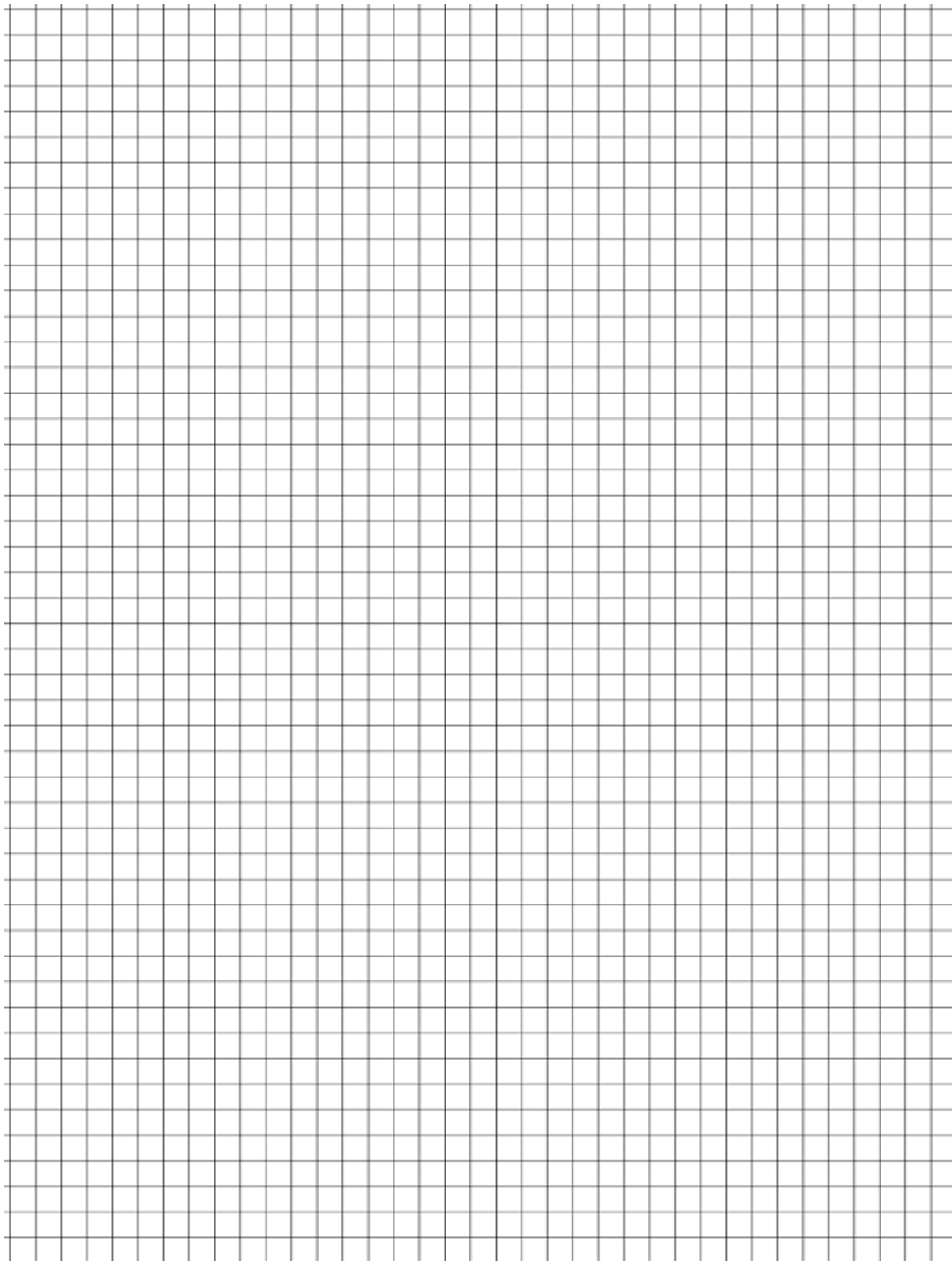
Use this for cutting.

Use this for 2(b).

Questions:
1. Does the reasoning you used in finding the rope length constitute a proof of your result? If not, how can you show that no other rectangle with an area of 400 square meters will have a perimeter smaller than the rectangle you discovered?

Let's do the proof in two different ways (one algebraic, one using calculus).

(a) (algebraic argument) Use algebra and show that $2\left(20+a+\dfrac{400}{20+a}\right)<80$ leads to a contradiction.

(b) Use calculus to prove that we found the minimum perimeter.

Ex 5: Use calculus to answer these questions for a rectangle.
(a) If we fix perimeter P, what is the maximum area? (And give dimensions of rectangle.)

(b) If we fix area A, what is the minimum perimeter? (And give dimensions of rectangle.)

Ex 6: Prove $\dfrac{P^2}{A}=(x-y)^2+4\text{A}$ algebraically and geometrically.

Ex 7: Discuss the difference between local min/max points and absolute/global min/max points.

Ex 8: For $f(x) = ax^2 + bx + c$,
      (a) Use algebra to find the min/max point.

      (b) Use calculus to find the min/max point.

Ex 9: The page of a book is to have a total area of 144 square inches, with 1-inch margins at the bottom and top and ½-inch margin at the sides. Find the dimensions of the page which will allow the largest printed area.

Ex 10: A window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 15 feet, find the dimensions which will allow the maximum amount of light to enter.

4.2 Curve Sketching

Definition: Concavity
$f(x)$ is concave up (*down*) on the interval (a,b) if for any $x_1, x_2 \in (a,b)$, the line segment joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$ lies above (*below*) the graph of $y=f(x)$ between x_1 and x_2.

Also if $f''(x)>0$ on (a,b), this means that $f'(x)$ is increasing on (a,b) (that is, the slope is getting bigger as we traverse from $x=a$ to $x=b$).
Likewise, if $f''(x)<0$ on (a,b), this means that $f'(x)$ is decreasing on (a,b) (that is, the slope is getting smaller as we traverse from $x=a$ to $x=b$)
This implies that
$f''(x)>0 \Rightarrow$ function is concave up at x, and
$f''(x)<0 \Rightarrow$ function is concave down at x.

What happens when $f''(x)=0$?

What is an inflection point?

True or False?

(a) If $f''(x)=0$, then there's an inflection point at x.

(b) If there's an inflection point at x, then $f''(x)=0$.

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Ex 1: Find these limits as x goes toward some type of infinity.

(a) $\lim\limits_{x \to \infty} \left(-5x^2 + 3x - 1 + \dfrac{1}{x} \right)$

(b) $\lim\limits_{x \to \infty} \left(x^3 + x^2 - 1{,}000{,}000\, x \right)$ and $\lim\limits_{x \to -\infty} \left(x^3 + x^2 - 1{,}000{,}000\, x \right)$

(c) $\lim\limits_{x \to \infty} \dfrac{3x^2 - 4x + 100}{-5x^2 + 7x}$

(d) $\lim\limits_{x \to \infty} \dfrac{3x^3 - 4x + 100}{-5x^2 + 7x}$

(e) $\lim\limits_{x \to \infty} \dfrac{3x^3 - 4x + 100}{-5x^4 + 7x}$
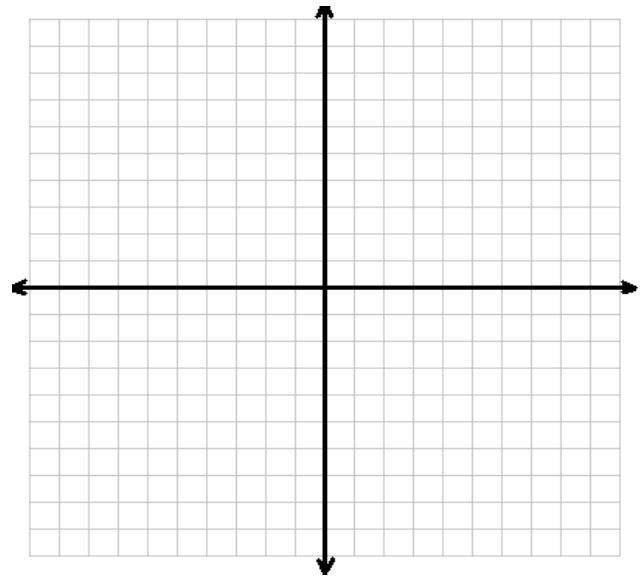
Ex 2: What do limits as x goes toward some type of infinity have to do with horizontal and/or slant asymptotes?

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Ex 3: Use first derivative and second derivative sign lines, finding asymptotes, and any other useful skills to sketch these graphs (without a calculator).
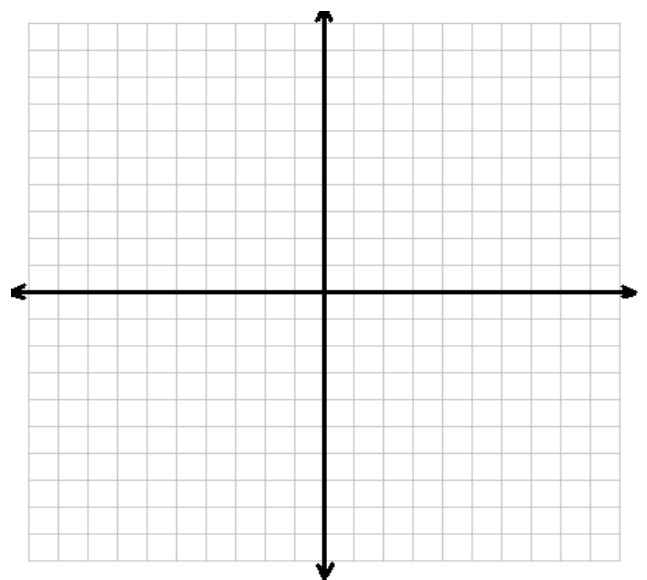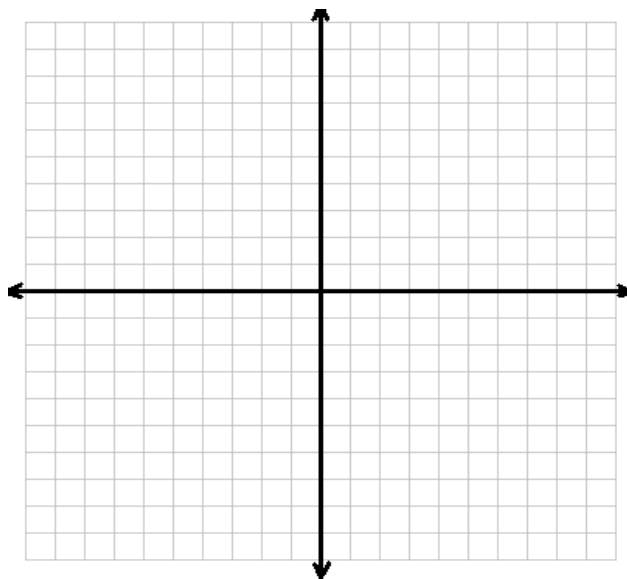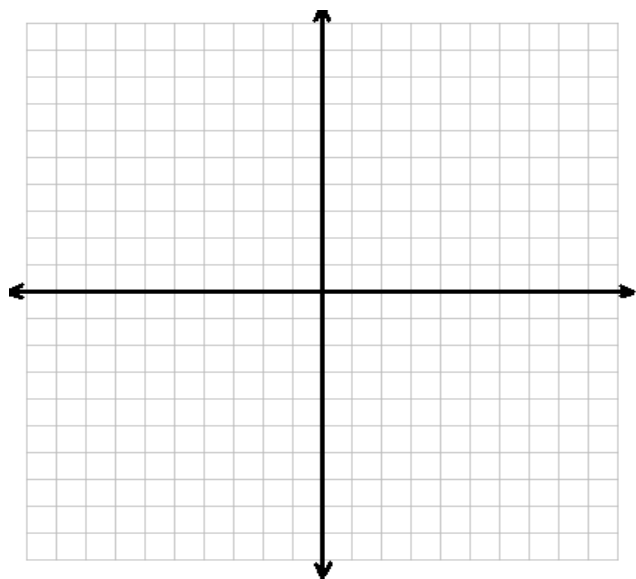
(a) $f(x)=x^4+2x^3+6x^2$
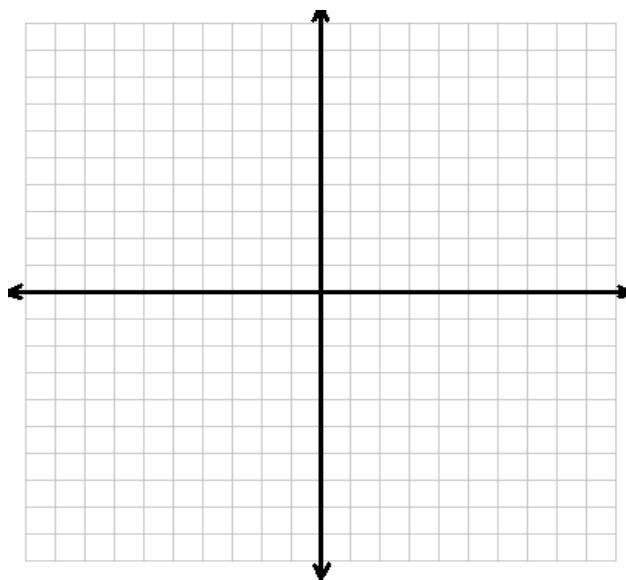
(b) $f(x)=x^5-5x^4$

(c) $f(x)=(1+x^5)^{-1}$

(d) $f(x) = \dfrac{2x^3}{(x-1)^3}$

(e) $f(x) = \dfrac{8x}{x^2+4}$

(f) $f(x)=x$, $f(x)=x^2$, $f(x)=x^3$, $f(x)=x^4$, $f(x)=x^5$, ...